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ON EXTENDED GENERALIZED ϕ-RECURRENT β-KENMOTSU MANIFOLDS

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Abstract. We extend the notion of generalized ϕ-recurrent β-Kenmotsu manifold and study its various geometric properties with the existence of such notion.

1. Introduction

In 1972 Kenmotsu [**8**] introduced a new class of almost contact Riemannian manifolds which are nowadays called Kenmotsu manifolds. It is well known that odd dimensional spheres admit Sasakian structures whereas odd dimensional hyperbolic spaces can not admit Sasakian structure, but have so-called Kenmotsu structure. Kenmotsu manifolds are normal (noncontact) almost contact Riemannian manifolds. Kenmotsu [**8**] investigated fundamental properties on the local structure of such manifolds. Kenmotsu manifolds are locally isometric to warped product spaces with one dimensional base and Kähler fiber. As a generalization of both Sasakian and Kenmotsu manifolds, Oubiña [**9**] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are respectively called the cosympletic, α-Sasakian and β-Kenmotsu manifold, α,β being scalar functions. In particular, if $\alpha=0$, $\beta=1$; and $\alpha=1$, $\beta=0$, then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively. As β is a scalar function, β-Kenmotsu manifolds provide a large varieties of Kenmotsu manifolds.

The notion of local symmetry of Riemannian manifolds has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [**14**] introduced the notion of local ϕ-symmetry on a Sasakian manifold. Generalizing the notion of local ϕ-symmetry of Takahashi [**14**],

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De et al. [**3**] introduced the notion of ϕ-recurrent Sasakian manifold. Recently De et al. [**4**] introduced the notion of ϕ-recurrent Kenmotsu manifolds. The locally ϕ-symmetric LP-Sasakian manifold is also studied by Shaikh and Baishya [**11**]. Again locally ϕ-symmetric and locally ϕ-recurrent $(LCS)_n$-manifolds are respectively studied in [**12**] and [**13**].

The notion of generalized recurrent manifolds has been introduced by Dubey [**7**] and studied by De and Guha [**5**]. Again, the notion of generalized Ricci-recurrent manifolds has been introduced and studied by De et al. [**6**]. A Riemannian manifold (M^n, g), $n > 2$, is called generalized recurrent [**5, 7**] if its curvature tensor R satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \tag{1.1}$$

where A and B are nonvanishing 1-forms defined by $A(\cdot) = g(\cdot, \rho_1)$, $B(\cdot) = g(\cdot, \rho_2)$ and the tensor G is defined by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{1.2}$$

for all X, Y, $Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric g. The 1-forms A and B are called the associated 1-forms of the manifold.

A Riemannian manifold $(M^n, g), n > 2$, is called generalized Ricci-recurrent [**6**] if its Ricci tensor S of type (0,2) satisfies the condition $\nabla S = A \otimes S + B \otimes g$, where A and B are non-vanishing 1-forms defined in (1.1).

In 2007, Özgür [**10**] studied generalized recurrent Kenmotsu manifolds. Generalizing the notion of Özgür [**10**], recently Basari and Murathan [**1**] introduced the notion of generalized ϕ-recurrent Kenmotsu manifolds. Extending the notion of Basari and Murathan [**1**], we here introduce the notion of *extended generalized ϕ-recurrent β-Kenmotsu manifolds*. The paper is organized as follows. Section 2 deals with a brief account of β-Kenmotsu manifolds. In Section 3, we study extended generalized ϕ-recurrent β-Kenmotsu manifolds and we obtain a necessary and sufficient condition for such a manifold to be a generalized Ricci-recurrent manifold. We also study extended generalized concircularly ϕ-recurrent β-Kenmotsu manifold and obtain the nature of its associated 1-forms. Finally, the last section deals with an example for the existence of extended generalized ϕ-recurrent β-Kenmotsu manifolds.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold M is said to be an almost contact metric manifold [**2**] if it admits an $(1, 1)$ tensor field ϕ, a vector field ξ, an 1-form η and a Riemannian metric g, which satisfy

$$\text{(a)}\ \phi\xi = 0, \quad \text{(b)}\ \eta(\phi X) = 0, \quad \text{(c)}\ \phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\text{(a)}\ g(\phi X, Y) = -g(X, \phi Y), \quad \text{(b)}\ \eta(X) = g(X, \xi), \quad \text{(c)}\ \eta(\xi) = 1, \tag{2.2}$$

$$(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all X, $Y \in \chi(M)$. An almost contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is said to be a β-Kenmotsu manifold if the following conditions hold [**8**]:

$$\nabla_X \xi = \beta\big[X - \eta(X)\xi\big], \tag{2.4}$$

$$(\nabla_X \phi)(Y) = \beta\big[g(\phi X, Y)\xi - \eta(Y)\phi X\big]. \tag{2.5}$$

If $\beta = 1$, then a β-Kenmotsu manifold is called a Kenmotsu manifold; and if β is constant, then it is called a homothetic Kenmotsu manifold. In a β-Kenmotsu manifold, the following relations hold [**8, 9**]:

$$(\nabla_X \eta)(Y) = \beta\big[g(X,Y) - \eta(X)\eta(Y)\big], \tag{2.6}$$

$$\begin{aligned} R(X,Y)\xi = &-\beta^2\big[\eta(Y)X - \eta(X)Y\big] \\ &+ (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\}, \end{aligned} \tag{2.7}$$

$$R(\xi, X)Y = \big[\beta^2 + (\xi\beta)\big]\big[\eta(Y)X - g(X,Y)\xi\big], \tag{2.8}$$

$$\begin{aligned} \eta(R(X,Y)Z) = &\,\beta^2\big[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\big] \\ &- (X\beta)\{g(Y,Z) - \eta(Y)\eta(Z)\} \\ &+ (Y\beta)\{g(X,Z) - \eta(Z)\eta(X)\}, \end{aligned} \tag{2.9}$$

$$S(X,\xi) = -\{2n\beta^2 + (\xi\beta)\}\eta(X) - (2n-1)(X\beta), \tag{2.10}$$

$$S(\xi,\xi) = -\{2n\beta^2 + (\xi\beta)\} \tag{2.11}$$

for all X, Y, $Z \in \chi(M)$.

We now state and prove some basic results in a β-Kenmotsu manifold which will be frequently used later on.

LEMMA 2.1. *Let $M^{2n+1}(\phi,\xi,\eta,g)$ be a β-Kenmotsu manifold. Then for any vector fields X, Y, W the following relation holds:*

$$\begin{aligned} (\nabla_W R)(X,Y)\xi = &-2\beta(W\beta)\{\eta(Y)X - \eta(X)Y\} - \beta^3\{g(Y,W)X - g(X,W)Y\} \\ &- \beta R(X,Y)W \\ &+ \beta(X\beta)\big[-g(Y,W)\xi + \eta(Y)\eta(W)\xi - \eta(Y)W + \eta(W)Y\big] \\ &- \beta(Y\beta)\big[-g(X,W)\xi + \eta(X)\eta(W)\xi - \eta(X)W + \eta(W)X\big]. \end{aligned} \tag{2.12}$$

PROOF. By virtue of (2.4), (2.6) and (2.7) we can easily get (2.12). □

LEMMA 2.2. *In a Riemannian manifold (M^n, g) the following relation holds:*

$$g\big((\nabla_W R)(X,Y)Z, U\big) = -g\big((\nabla_W R)(X,Y)U, Z\big) \tag{2.13}$$

for all vector fields $X, Y, Z, W, U \in \chi(M)$.

PROOF. . It is easy to prove (2.13) and hence we omit the proof. □

3. Extended generalized ϕ-recurrent β-Kenmotsu manifolds

DEFINITION 3.1. A β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n > 1$, is said to be an *extended generalized ϕ-recurrent β-Kenmotsu manifold* if its curvature tensor R satisfies the relation

$$\phi^2((\nabla_W R)(X,Y)Z) = A(W)\phi^2(R(X,Y)Z) + B(W)\phi^2(G(X,Y)Z) \tag{3.1}$$

for all $X, Y, Z, W \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the metric g, i.e., ∇ is the Riemannian connection; A and B are nonvanishing 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$ and G is a tensor of type $(1, 3)$ defined in (1.2). The 1-forms A and B are called the associated 1-forms of the manifold.

We consider a β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n > 1$, which is extended generalized ϕ-recurrent. Then by virtue of (2.1), (3.1) yields

$$
\begin{aligned}
(\nabla_W R)(X,Y)Z = \eta\big((\nabla_W R)(X,Y)Z\big)\xi \\
+ A(W)\big[R(X,Y)Z - \eta(R(X,Y)Z)\xi\big] \\
+ B(W)\big[G(X,Y)Z - \eta(G(X,Y)Z)\xi\big],
\end{aligned}
\tag{3.2}
$$

from which it follows that

$$
\begin{aligned}
g\big((\nabla_W R)(X,Y)Z, U\big) - \eta\big((\nabla_W R)(X,Y)Z\big)\eta(U) \\
= A(W)\big[g(R(X,Y)Z,U) - \eta(R(X,Y)Z)\eta(U)\big] \\
+ B(W)\big[g(G(X,Y)Z,U) - \eta(G(X,Y)Z)\eta(U)\big].
\end{aligned}
\tag{3.3}
$$

Let $\{e_i : i = 1, 2, \cdots, 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Setting $X = U = e_i$ in (3.3) and taking summation over i, $1 \leqslant i \leqslant 2n+1$, and then using (2.8), we obtain

$$
\begin{aligned}
(\nabla_W S)(Y,Z) - g\big((\nabla_W R)(\xi,Y)Z, \xi\big) \\
= A(W)\big[S(Y,Z) + \{\beta^2 + (\xi\beta)\}\{g(Y,Z) - \eta(Y)\eta(Z)\}\big] \\
+ B(W)\big[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)\big].
\end{aligned}
\tag{3.4}
$$

Using (2.9) and (2.13), we get

$$
\begin{aligned}
g\big((\nabla_W R)(\xi,Y)Z, \xi\big) = 2\beta(W\beta)\big[\eta(Y)\eta(Z) - g(Y,Z)\big] \\
- \beta\big[(Y\beta) - (\xi\beta)\eta(Y)\big]\big[g(W,Z) - \eta(W)\eta(Z)\big].
\end{aligned}
\tag{3.5}
$$

By virtue of (3.5), it follows from (3.4) that

$$
\begin{aligned}
(\nabla_W S)(Y,Z) = A(W)S(Y,Z) \\
+ \big[(2n-1)B(W) - 2\beta(W\beta) + A(W)\{\beta^2 + (\xi\beta)\}\big]g(Y,Z) \\
+ \big[2\beta(W\beta) - A(W)\{\beta^2 + (\xi\beta)\} + B(W)\big]\eta(Y)\eta(Z) \\
- \beta\{(Y\beta) - \eta(Y)(\xi\beta)\}\big[g(W,Z) - \eta(W)\eta(Z)\big].
\end{aligned}
\tag{3.6}
$$

From (3.6), it follows that an extended generalized ϕ-recurrent β-Kenmotsu manifold is a generalized Ricci-recurrent manifold if and only if

$$
\begin{aligned}
\big[2\beta(W\beta) - A(W)\{\beta^2 + (\xi\beta)\} + B(W)\big]\eta(Y)\eta(Z) \\
-\beta\{(Y\beta) - \eta(Y)(\xi\beta)\}\big[g(W,Z) - \eta(W)\eta(Z)\big] = 0.
\end{aligned}
\tag{3.7}
$$

This leads to the following:

THEOREM 3.1. *An extended generalized ϕ-recurrent β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n > 1$, is generalized Ricci-recurrent if and only if the relation* (3.7) *holds.*

Setting $Z = \xi$ in (3.2), we obtain

$$(\nabla_W R)(X, Y)\xi = A(W)R(X, Y)\xi + B(W)G(X, Y)\xi. \tag{3.8}$$

By virtue of (2.7) and (1.2), it follows from (3.8) that

$$\begin{aligned}(\nabla_W R)(X, Y)\xi = &\big[B(W) - \beta^2 A(W)\big]\{\eta(Y)X - \eta(X)Y\} \\ &+ A(W)\big[(X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\}\big].\end{aligned} \tag{3.9}$$

From (2.12) and (3.9), we obtain

$$\begin{aligned}\beta R(X, Y)W = &-\beta^3\{g(Y, W)X - g(X, W)Y\} \\ &+ \beta(X\beta)\big[-g(Y, W)\xi + \eta(Y)\eta(W)\xi - \eta(Y)W + \eta(W)Y\big] \\ &- \beta(Y\beta)\big[-g(X, W)\xi + \eta(X)\eta(W)\xi - \eta(X)W + \eta(W)X\big] \\ &+ \{\beta^2 A(W) - B(W) - 2\beta(W\beta)\}\{\eta(Y)X - \eta(X)Y\} \\ &- A(W)\big[(X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\}\big].\end{aligned} \tag{3.10}$$

This leads to the following:

THEOREM 3.2. *In an extended generalized ϕ-recurrent β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n > 1$, the curvature tensor is of the form of* (3.10).

From (3.10), we have

$$\begin{aligned}\beta\tilde{R}(X, Y, W, U) = &-\beta^3\{g(Y, W)g(X, U) - g(X, W)g(Y, U)\} \\ &+ \beta(X\beta)\big[\{-g(Y, W) + \eta(Y)\eta(W)\}\eta(U) - \eta(Y)g(W, U) + \eta(W)g(Y, U)\big] \\ &- \beta(Y\beta)\big[\{-g(X, W) + \eta(X)\eta(W)\}\eta(U) - \eta(X)g(W, U) + \eta(W)g(X, U)\big] \\ &+ \big\{\beta^2 A(W) - B(W) - 2\beta(W\beta)\big\}\big\{\eta(Y)g(X, U) - \eta(X)g(Y, U)\big\} \\ &- A(W)\big[(X\beta)\{g(Y, U) - \eta(Y)\eta(U)\} - (Y\beta)\{g(X, U) - \eta(X)\eta(U)\}\big],\end{aligned} \tag{3.11}$$

where $\tilde{R}(X, Y, W, U) = g(R(X, Y)W, U)$. Setting $X = U = e_i$ in (3.11) and taking summation over i, $1 \leqslant i \leqslant 2n + 1$, we get

$$\begin{aligned}\beta S(Y, W) = &-\beta\{2n\beta^2 + (\xi\beta)\}g(Y, W) \\ &- (4n + 1)\beta(W\beta)\eta(Y) - (2n - 1)\beta\eta(W)(Y\beta) \\ &+ A(W)\big[\{2n\beta^2 + (\xi\beta)\}\eta(Y) + (2n - 1)(Y\beta)\big] \\ &- 2nB(W)\eta(Y) + \beta(\xi\beta)\eta(Y)\eta(W).\end{aligned} \tag{3.12}$$

This leads to the following:

THEOREM 3.3. *In an extended generalized ϕ-recurrent β-Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n > 1$, the Ricci tensor is of the form of* (3.12).

Replacing $Y = \xi$ in (3.12) and then using (2.10), we get

$$(n + 1)\beta(W\beta) + (n - 1)\beta(\xi\beta)\eta(W) - n\{\beta^2 + (\xi\beta)\}A(W) + nB(W) = 0. \tag{3.13}$$

If $\beta = 1$, then from (3.13), we can state the following:

COROLLARY 3.1. *In an extended generalized ϕ-recurrent Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$, $n > 1$, the* 1*-forms A and B are equal.*

Again by virtue of Corollary 3.1, it follows from (3.12) that

$$S(Y,W) = -2ng(Y,W). \tag{3.14}$$

Thus we can state the following:

Corollary 3.2. *Every extended generalized ϕ-recurrent Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is an Einstein manifold.*

Also, if $\beta = 1$, then from (3.10), we get

$$\begin{aligned} R(X,Y)W = &\{A(W) - B(W)\}\{\eta(Y)X - \eta(X)Y\} \\ &- \{g(Y,W)X - g(X,W)Y\}. \end{aligned} \tag{3.15}$$

So, by virtue of Corollary 3.1, it follows from (3.15) that

$$R(X,Y)W = -\{g(Y,W)X - g(X,W)Y\}.$$

This leads to the following:

Corollary 3.3. *An extended generalized ϕ-recurrent Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is of a constant curvature -1.*

Changing W, X, Y cyclically in (3.2) and adding them, we get by virtue of the Bianchi identity that

(3.16)

$$\begin{aligned} &A(W)\big[R(X,Y)Z - \eta(R(X,Y)Z)\xi\big] + B(W)\big[G(X,Y)Z - \eta(G(X,Y)Z)\xi\big] \\ &+A(X)\big[R(Y,W)Z - \eta(R(Y,W)Z)\xi\big] + B(X)\big[G(Y,W)Z - \eta(G(Y,W)Z)\xi\big] \\ &+A(Y)\big[R(W,X)Z - \eta(R(W,X)Z)\xi\big] + B(Y)\big[G(W,X)Z - \eta(G(W,X)Z)\xi\big] = 0. \end{aligned}$$

Taking the inner product on both sides of (3.16) by U and then contracting over Y and Z, we obtain

$$\begin{aligned} &A(W)\big[S(X,U) + \{2n\beta^2 + (\xi\beta)\}\eta(X)\eta(U) + (2n-1)(X\beta)\eta(U)\big] \\ &\qquad +2nB(W)\big[g(X,U) - \eta(X)\eta(U)\big] \\ &-A(X)\big[S(W,U) + \{2n\beta^2 + (\xi\beta)\}\eta(W)\eta(U) + (2n-1)(W\beta)\eta(U)\big] \\ &\qquad -2nB(X)\big[g(W,U) - \eta(W)\eta(U)\big] - A(R(W,X)U) \\ &-\beta^2\big\{\eta(X)A(W) - \eta(W)A(X)\big\}\eta(U) + (W\beta)\big\{A(X) - \eta(X)A(\xi)\big\}\eta(U) \\ &-(X\beta)\big\{A(W) - \eta(W)A(\xi)\big\}\eta(U) + B(X)\big[g(W,U) - \eta(W)\eta(U)\big] \\ &\qquad -B(W)\big[g(X,U) - \eta(X)\eta(U)\big] = 0. \end{aligned} \tag{3.17}$$

By virtue of (3.12), it follows from (3.17) that

$$\begin{aligned} &A(W)\big[-\tfrac{1}{\beta}\{4n\beta(U\beta) + (U(\xi\beta))\}\eta(X) - \{2n\beta^2 + (\xi\beta)\}g(X,U) \\ &\qquad +\tfrac{1}{\beta}A(U)\{2n\beta^2 + (\xi\beta)\}\eta(X) + \tfrac{1}{\beta}(2n-1)A(U)(X\beta) \\ &\qquad -\tfrac{1}{\beta}2nB(U)\eta(X) + \{2n\beta^2 + (\xi\beta)\}\eta(X)\eta(U)\big] \\ &\qquad +2nB(W)\big[g(X,U) - \eta(X)\eta(U)\big] \\ &-A(X)\big[-\tfrac{1}{\beta}\{4n\beta(U\beta) + (U(\xi\beta))\}\eta(W) - \{2n\beta^2 + (\xi\beta)\}g(W,U) \end{aligned} \tag{3.18}$$

$$+\tfrac{1}{\beta}A(U)\{2n\beta^2+(\xi\beta)\}\eta(W)+\tfrac{1}{\beta}(2n-1)A(U)(W\beta)$$
$$-\tfrac{1}{\beta}2nB(U)\eta(W)+\{2n\beta^2+(\xi\beta)\}\eta(W)\eta(U)\big]$$
$$-2nB(X)\big[g(W,U)-\eta(W)\eta(U)\big]-A(R(W,X)U)$$
$$-\beta^2\{\eta(X)A(W)-\eta(W)A(X)\}\eta(U)+(W\beta)\{A(X)-\eta(X)A(\xi)\}\eta(U)$$
$$-(X\beta)\{A(W)-\eta(W)A(\xi)\}\eta(U)+B(X)\big[g(W,U)-\eta(W)\eta(U)\big]$$
$$-B(W)\big[g(X,U)-\eta(X)\eta(U)\big]=0.$$

Setting $X=U=\xi$ in (3.18), we get by virtue of (2.2) and (2.7) that

$$\big[4n\beta(\xi\beta)+(\xi(\xi\beta))-\{2n\beta^2+(\xi\beta)\}A(\xi)+2nB(\xi)\big]\big[A(W)-\eta(W)A(\xi)\big] = (2n-1)A(\xi)\big[A(W)(\xi\beta)-A(\xi)(W\beta)\big]. \tag{3.19}$$

If the vector fields ξ and ρ_1 are co-directional, then we have

$$A(W)=A(\xi)\eta(W). \tag{3.20}$$

From (3.19) and (3.20) it follows that

$$A(W)(\xi\beta)-A(\xi)(W\beta)=0. \tag{3.21}$$

Conversely, if the relation (3.21) holds, then (3.19) yields (3.20) provided that

$$4n\beta(\xi\beta)+(\xi(\xi\beta))-\{2n\beta^2+(\xi\beta)\}A(\xi)+2nB(\xi)\neq 0. \tag{3.22}$$

Thus we can state the following:

THEOREM 3.4. *Let $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, be an extended generalized ϕ-recurrent β-Kenmotsu manifold satisfying the condition* (3.22). *Then ξ and ρ_1 are co-directional if and only if* (3.21) *holds.*

DEFINITION 3.2. A β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, is said to be an *extended generalized concircularly ϕ-recurrent β-Kenmotsu manifold* if its concircular curvature tensor $\tilde{C}$ satisfies the relation

$$\phi^2((\nabla_W\tilde{C})(X,Y)Z)=A(W)\,\phi^2(\tilde{C}(X,Y)Z)+B(W)\,\phi^2(G(X,Y)Z), \tag{3.23}$$

where A and B are nonvanishing 1-forms defined in (3.1), ∇ denotes the operator of covariant differentiation with respect to the metric g i.e., ∇ is the Riemannian connection, and the concircular curvature tensor $\tilde{C}$ of type $(1,3)$ is given by [**15**]

$$\tilde{C}(X,Y)Z=R(X,Y)Z-\frac{r}{2n(2n+1)}\,G(X,Y)Z, \tag{3.24}$$

where r is the scalar curvature of the manifold.

Let us consider an extended generalized concircularly ϕ-recurrent β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$. Then by virtue of (2.1), it follows from (3.23) that

$$\begin{aligned}(\nabla_W\tilde{C})(X,Y)Z&=\eta\big((\nabla_W\tilde{C})(X,Y)Z\big)\xi\\&\quad+A(W)\big[\tilde{C}(X,Y)Z-\eta(\tilde{C}(X,Y)Z)\xi\big]\\&\quad+B(W)\big[G(X,Y)Z-\eta(G(X,Y)Z)\xi\big],\end{aligned} \tag{3.25}$$

from which it follows that

$$g\big((\nabla_W\tilde{C})(X,Y)Z,U\big) - \eta\big((\nabla_W\tilde{C})(X,Y)Z\big)\eta(U) = A(W)\big[g(\tilde{C}(X,Y)Z,U) - \eta(\tilde{C}(X,Y)Z)\eta(U)\big] + B(W)\big[g(G(X,Y)Z,U) - \eta(G(X,Y)Z)\eta(U)\big]. \tag{3.26}$$

Taking contraction of (3.26) over X and U, we get

$$(\nabla_W S)(Y,Z) - \frac{dr(W)}{2n+1}g(Y,Z) - g((\nabla_W\tilde{C})(\xi,Y)Z,\xi) = A(W)\Big[S(Y,Z) - \frac{r}{2n+1}g(Y,Z) - \eta(\tilde{C}(\xi,Y)Z)\Big] + B(W)\big[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)\big]. \tag{3.27}$$

In view of (3.24) and (3.5), we get

$$g\big((\nabla_W\tilde{C})(\xi,Y)Z,\xi\big) = \Big[2\beta(W\beta) + \frac{dr(W)}{2n(2n+1)}\Big]\big[\eta(Y)\eta(Z) - g(Y,Z)\big] - \beta\big[(Y\beta) - (\xi\beta)\eta(Y)\big]\big[g(W,Z) - \eta(W)\eta(Z)\big]. \tag{3.28}$$

Also from (3.24) and (2.9), we get

$$\eta(\tilde{C}(\xi,Y)Z) = \Big[\beta^2 + (\xi\beta) + \frac{r}{2n(2n+1)}\Big]\big[\eta(Y)\eta(Z) - g(Y,Z)\big]. \tag{3.29}$$

Using (3.28) and (3.29) in (3.27), we obtain

$$\begin{aligned}(\nabla_W S)(Y,Z) = {} & A(W)S(Y,Z) \\ & + \Big[(2n-1)B(W) - 2\beta(W\beta) \\ & \quad + \frac{2n-1}{2n(2n+1)}\{dr(W) - rA(W)\} + A(W)\{\beta^2 + (\xi\beta)\}\Big]g(Y,Z) \\ & + \Big[2\beta(W\beta) + \frac{dr(W)}{2n(2n+1)} + B(W) \\ & \quad - A(W)\Big\{\beta^2 + (\xi\beta) + \frac{r}{2n(2n+1)}\Big\}\Big]\eta(Y)\eta(Z) \\ & - \beta\big[(Y\beta) - (\xi\beta)\eta(Y)\big]\big[g(W,Z) - \eta(W)\eta(Z)\big].\end{aligned} \tag{3.30}$$

From (3.30), we can state the following:

THEOREM 3.5. *An extended generalized concircularly ϕ-recurrent β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n > 1$, is generalized Ricci-recurrent if and only if the following relation holds:*

$$\Big[2\beta(W\beta) + \frac{dr(W)}{2n(2n+1)} - A(W)\Big\{\beta^2 + (\xi\beta) + \frac{r}{2n(2n+1)}\Big\} + B(W)\Big]\eta(Y)\eta(Z) - \beta\big[(Y\beta) - (\xi\beta)\eta(Y)\big]\big[g(W,Z) - \eta(W)\eta(Z)\big] = 0. \tag{3.31}$$

Setting $Y = Z = \xi$ in (3.30) and using (2.11), we get

$$\Big[2n\beta^2 + (\xi\beta) + \frac{r}{2n+1}\Big]A(W) - 2nB(W) = \frac{dr(W)}{2n+1} + 4n\beta(W\beta) + (W(\xi\beta)). \tag{3.32}$$

This leads to the following:

THEOREM 3.6. *In an extended generalized concircularly ϕ-recurrent β-Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, the 1-forms A and B are related by the relation* (3.32).

COROLLARY 3.4. *In an extended generalized concircularly ϕ-recurrent Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, the 1-forms A and B are related by the relation*

$$\Big[2n+\frac{r}{2n+1}\Big]A(W)-2nB(W)=\frac{dr(W)}{2n+1}.$$

COROLLARY 3.5. *In an extended generalized concircularly ϕ-recurrent Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$, $n>1$, with constant scalar curvature, the associated 1-forms A and B are related by $A=kB$, where k is a nonzero constant.*

4. Example of extended generalized ϕ-recurrent β-Kenmotsu manifolds

We consider a 3-dimensional manifold $M=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where (x,y,z) are the standard coordinates of $\mathbb{R}^3$. Let $\{E_1,E_2,E_3\}$ be a linearly independent global frame on M given by

$$E_1=z^2\frac{\partial}{\partial x},\quad E_2=z^2\frac{\partial}{\partial y},\quad E_3=\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by $g(E_1,E_3)=g(E_2,E_3)=g(E_1,E_2)=0$, $g(E_1,E_1)=g(E_2,E_2)=g(E_3,E_3)=1$. Let η be the 1-form defined by $\eta(U)=g(U,E_3)$ for any $U\in\chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi E_1=-E_2$, $\phi E_2=E_1$ and $\phi E_3=0$. Then using the linearity of ϕ and g we have $\eta(E_3)=1$, $\phi^2U=-U+\eta(U)E_3$ and $g(\phi U,\phi W)=g(U,W)-\eta(U)\eta(W)$ for any $U,W\in\chi(M)$. Thus for $E_3=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M.

Let ∇ be the Riemannian connection of g. Then we have

$$\big[E_1,E_2\big]=0,\quad \big[E_1,E_3\big]=-\frac{2}{z}E_1,\quad \big[E_2,E_3\big]=-\frac{2}{z}E_2.$$

Using the Koszul formula for the Riemannian metric g, we can easily calculate

$$\begin{array}{lll}\nabla_{E_1}E_1=\frac{2}{z}E_3, & \nabla_{E_1}E_2=0, & \nabla_{E_1}E_3=-\frac{2}{z}E_1,\\ \nabla_{E_2}E_1=0, & \nabla_{E_2}E_2=\frac{2}{z}E_3, & \nabla_{E_2}E_3=-\frac{2}{z}E_2,\\ \nabla_{E_3}E_1=0, & \nabla_{E_3}E_2=0, & \nabla_{E_3}E_3=0.\end{array}$$

From the above it can be easily seen that (ϕ,ξ,η,g) is a β-Kenmotsu structure on M. Consequently $M^3(\phi,\xi,\eta,g)$ is a β-Kenmotsu manifold with $\beta=-\frac{2}{z}$. Using the above relations, we can easily calculate the nonvanishing components of the curvature tensor as follows:

$$\begin{array}{lll}R(E_1,E_2)E_1=\dfrac{4}{z^2}E_2, & R(E_1,E_2)E_2=-\dfrac{4}{z^2}E_1, & R(E_1,E_3)E_1=\dfrac{6}{z^2}E_3,\\ R(E_1,E_3)E_3=-\dfrac{6}{z^2}E_1, & R(E_2,E_3)E_2=\dfrac{6}{z^2}E_3, & R(E_2,E_3)E_3=-\dfrac{6}{z^2}E_2.\end{array}$$

and the components which can be obtained from these by the symmetry properties. Since $\{E_1, E_2, E_3\}$ forms a basis of the β-Kenmotsu manifold, any vector field X, Y, $Z \in \chi(M)$ can be written as

$$X = a_1E_1 + b_1E_2 + c_1E_3,$$
$$Y = a_2E_1 + b_2E_2 + c_2E_3,$$
$$Z = a_3E_1 + b_3E_2 + c_3E_3,$$

where a_i, b_i, $c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2, 3$. Then

$$\begin{aligned} R(X,Y)Z = &-\frac{2}{z^2}\big[2(a_1b_2 - a_2b_1)b_3 + 3(a_1c_2 - a_2c_1)c_3\big]E_1 \\ &+ \frac{2}{z^2}\big[2(a_1b_2 - a_2b_1)a_3 - 3(b_1c_2 - b_2c_1)c_3\big]E_2 \\ &+ \frac{6}{z^2}\big[(a_1c_2 - a_2c_1)a_3 + (b_1c_2 - b_2c_1)b_3\big]E_3, \end{aligned} \tag{4.1}$$

$$\begin{aligned} G(X,Y)Z = &(a_2a_3 + b_2b_3 + c_2c_3)(a_1E_1 + b_1E_2 + c_1E_3) \\ &- (a_1a_3 + b_1b_3 + c_1c_3)(a_2E_1 + b_2E_2 + c_2E_3). \end{aligned} \tag{4.2}$$

By virtue of (4.1) we have the following:

$$\begin{aligned} (\nabla_{E_1}R)(X,Y)Z = &\frac{4}{z^3}(5b_1c_2 - b_2c_1)b_3E_1 + \frac{20}{z^3}(a_1b_2 - a_2b_1)b_3E_3 \\ &- \frac{4}{z^3}\big[5(a_1b_2 - a_2b_1)c_3 + (5b_1c_2 - b_2c_1)a_3\big]E_2, \end{aligned} \tag{4.3}$$

$$\begin{aligned} (\nabla_{E_2}R)(X,Y)Z = &\frac{20}{z^3}\big[(a_1b_2 - a_2b_1)c_3 - (a_1c_2 - a_2c_1)b_3\big]E_1 \\ &+ \frac{20}{z^3}(a_1c_2 - a_2c_1)a_3E_2 - \frac{20}{z^3}(a_1b_2 - a_2b_1)a_3E_3, \end{aligned} \tag{4.4}$$

$$\begin{aligned} (\nabla_{E_3}R)(X,Y)Z = &\frac{4}{z^3}\big[2(a_1b_2 - a_2b_1)b_3 + 3(a_1c_2 - a_2c_1)c_3\big]E_1 \\ &- \frac{4}{z^3}\big[2(a_1b_2 - a_2b_1)a_3 - 3(b_1c_2 - b_2c_1)c_3\big]E_2 \\ &- \frac{12}{z^3}\big[(a_1c_2 - a_2c_1)a_3 + (b_1c_2 - b_2c_1)b_3\big]E_3. \end{aligned} \tag{4.5}$$

From (4.1) and (4.2), we get

$$\phi^2(R(X,Y)Z) = pE_1 + qE_2, \quad \phi^2(G(X,Y)Z) = mE_1 + sE_2, \tag{4.6}$$

where

$$p = \frac{2}{z^2}\big[2(a_1b_2 - a_2b_1)b_3 + 3(a_1c_2 - a_2c_1)c_3\big],$$
$$q = -\frac{2}{z^2}\big[2(a_1b_2 - a_2b_1)a_3 - 3(b_1c_2 - b_2c_1)c_3\big],$$
$$m = a_2(b_1b_3 + c_1c_3) - a_1(b_2b_3 + c_2c_3),$$
$$s = b_2(a_1a_3 + c_1c_3) - b_1(a_2a_3 + c_2c_3).$$

Also from (4.3)–(4.5), we obtain

$$\phi^2((\nabla_{E_i}R)(X,Y)Z) = u_iE_1 + v_iE_2 \quad \text{for } i = 1,2,3, \tag{4.7}$$

where

$$u_1 = -\frac{4}{z^3}(5b_1c_2 - b_2c_1)b_3, \quad v_1 = \frac{4}{z^3}\big[5(a_1b_2 - a_2b_1)c_3 + (5b_1c_2 - b_2c_1)a_3\big],$$
$$u_2 = -\frac{20}{z^3}\big[(a_1b_2 - a_2b_1)c_3 - (a_1c_2 - a_2c_1)b_3\big], \quad v_2 = -\frac{20}{z^3}(a_1c_2 - a_2c_1)a_3,$$
$$u_3 = -\frac{4}{z^3}\big[2(a_1b_2 - a_2b_1)b_3 + 3(a_1c_2 - a_2c_1)c_3\big],$$
$$v_3 = \frac{4}{z^3}\big[2(a_1b_2 - a_2b_1)a_3 - 3(b_1c_2 - b_2c_1)c_3\big].$$

Let us now consider the 1-forms as

$$A(E_i) = \frac{su_i - mv_i}{ps - qm} \quad \text{and} \quad B(E_i) = \frac{pv_i - qu_i}{ps - qm} \quad \text{for } i = 1,2,3 \tag{4.8}$$

such that $ps - qm \neq 0$, $su_i - mv_i \neq 0$ and $pv_i - qu_i \neq 0$, $i = 1,2,3$. From (3.1), we have

$$\phi^2((\nabla_{E_i}R)(X,Y)Z) = A(E_i)\phi^2(R(X,Y)Z) + B(E_i)\phi^2(G(X,Y)Z), \quad i = 1,2,3. \tag{4.9}$$

By virtue of (4.6)–(4.8), it can be easily shown that the manifold satisfies relation (4.9). Hence the manifold under consideration is an extended generalized ϕ-recurrent β-Kenmotsu manifold, which is neither ϕ-recurrent nor generalized ϕ-recurrent. This leads to the following:

THEOREM 4.1. *There exists an extended generalized ϕ-recurrent β-Kenmotsu manifold $M^3(\phi,\xi,\eta,g)$, which is neither ϕ-recurrent nor generalized ϕ-recurrent.*

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