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# HAZARD RATES AND SUBEXPONENTIAL DISTRIBUTIONS

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*This paper is dedicated to the memory of Tatjana Ostrogorski and also to our co-author Aleksandras Baltrunas who died during the preparation of this paper. Both were infinite dimensional mathematicians and both unfortunately died too young.*

ABSTRACT. A distribution function $F$ on the nonnegative halfline is called subexponential if $\lim_{x\to\infty}(1-F^{*n}(x))/(1-F(x)) = n$ for all $n \geqslant 2$. We obtain new sufficient conditions for subexponential distributions and related classes of distribution functions. Our results are formulated in terms of the hazard rate. We also analyze the rate of convergence in the definition and discuss the asymptotic behaviour of the remainder term $R_n(x) = 1-F^{*n}(x)-n(1-F(x))$. We use the results in studying subordinated distributions and we conclude the paper with some multivariate extensions of our results.

## 1. Introduction

Let $X, X_1, X_2, \ldots, X_n, \ldots$ denote i.i.d. random variables with distribution function (d.f.) $F(x) = P(X \leqslant x)$ and suppose that $F(0+) = 0$ and $F(x) < 1$ for all real $x$. If the mean of $X$ is finite, we shall denote it by $\mu$. The d.f. of the partial sums $S(n) = \sum_{i=1}^{n} X_i$ is given by $P(S(n) \leqslant x) = F^{*n}(x)$, where $*$ denotes Stieltjes convolution. Let $\overline{F}(x) = 1 - F(x)$ denote the tail distribution. We say that $F(x)$ belongs to the subexponential class $S$ (notation $F \in S$) if as $x \to \infty$,

$$\overline{F^{*2}}(x)/\overline{F}(x) \to 2. \tag{1}$$

Unless stated otherwise, throughout the paper we shall consider limits as $x \to \infty$. It is well known that (1) holds if and only if

$$\overline{F^{*n}}(x)/\overline{F}(x) \to n, \forall n \geqslant 2. \tag{2}$$

Subexponential distributions were first studied by Chistyakov (1964) and further analyzed by Teugels (1975) and Pitman (1980). In the past decades, the class

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$S$ has been used in a wide variety of applications in probability theory and stochastic processes. For a survey we refer to Goldie and Klüppelberg (1998), Klüppelberg (2004) and Embrechts et al. (1997).

It is well known that $F \in S$ implies that but is not equivalent to $\overline{F} \in L$, where $L$ denotes the class of positive and measurable functions $g(x)$ for which

$$g(x+y)/g(x) \to 1, \forall y \in \mathbb{R}. \tag{3}$$

Thus, apart from $\overline{F} \in L$ we need an extra condition to conclude that $F \in S$.

For densities, we define the class $SD$ of subexponential densities as the class of densities $f(x)$ for which $f \in L$ and

$$f^{\otimes 2}(x)/f(x) \to 2 \tag{4}$$

holds, where $\otimes$ denotes the Lebesgue convolution. It is well known that for $f \in L$, (4) implies that $F \in S$ and that $f^{\otimes n}(x) \sim nf(x)$ for all $n \geqslant 2$.

Related to $SD$ and $S$, we say that $F \in S^*$ iff $\mu < \infty$ and $g \in SD$, where $g(x) = \overline{F}(x)/\mu$. Klüppelberg (1988) proved that the class $S^*$ is a proper subclass of $S$.

We can easily extend the classes of functions defined above by considering $O-$statements. Recall that $A(x) = O(1)B(x)$ iff $\limsup A(x)/B(x) < \infty$. We define (as $x \to \infty$) the following classes of d.f. $F$ and densities $f$.

- $F \in OS$ if $\overline{F * F}(x) = O(1)\overline{F}(x)$;
- $g \in OL$ if $g(x+y) = O(1)g(x)$, $\forall y \in \mathbb{R}$;
- $f \in OSD$ if $f \otimes f(x) = O(1)f(x)$;
- $F \in OS^*$ if $\mu < \infty$ and if $g \in OSD$ where $g(x) = \overline{F}(x)/\mu$.

These and related classes of d.f. have been considered by several authors before. Recently Shimura and Watanabe (2005) used the class $OS$ in the context of infinite divisible d.f.. Among others they show that $F \in OS$ implies that $\overline{F} \in OL$. Klüppelberg (1990) calls a d.f. $F \in OS$ weak idempotent. Omey (1994) defined the class $D(m)$ of d.f. for which

$$\|F\|_m = \sup_{x \geqslant 0} \frac{m * F(x)}{m(x)} < \infty. \tag{5}$$

In the special case where $m(x) = \overline{F}(x)$, we find back the class $OS$. On the other hand, if $m(x) = \overline{F}^2(x)$ we obtain a class of d.f. studied by Geluk and Pakes (1991) and Geluk (1992).

Extending the class $SD$ and $OSD$ not only to densities, Baltrunas and Omey (1998, 2002) studied the following classes of functions. Let $g(x)$ denote a nonnegative and measurable function and define $G(x) = \int_0^x g(t)\,dt$; then:

- $g \in OA$ if $g \otimes g(x) = O(1)g(x)G(x)$;
- $g \in AA$ if $g \in OA$ and $g(x)G(x) = O(1)g \otimes g(x)$;
- $g \in BB$ if $\lim g \otimes g(x)/g(x)G(x) = C(g)$ exists, $0 < C(g) < \infty$.

Clearly $BB \subset AA \subset OA$. If $g \in L \cap BB$ is a density, then $g \in SD$ and $C(g) = 2$. The class $OA$ extends the class $OSD$. If $X$ has a finite mean $\mu$ then

$\overline{F} \in OA$ is equivalent to $F \in OS^*$. If $X$ has an infinite mean, then it is meaningful for example to study d.f. $F$ for which $\overline{F} \in OA$.

A useful way to find examples in the classes defined above is to consider regularly varying functions or $O$-regularly varying functions. Recall that a positive and measurable function $g(x)$ is regularly varying with (real) index $\alpha$ if it satisfies $g(xy)/g(x) \to y^\alpha, \forall y > 0$. Notation $g \in RV(\alpha)$. The function $g$ is in the class $ORV$ of $O$-regularly varying functions if it satisfies $g(xy)/g(x) = O(1), \forall y > 0$. For these classes we refer to Bingham et al. (1989) or to Seneta (1976). In what follows we shall often use the property that for $g \in ORV$ or $g \in RV(\alpha)$ the defining property holds locally uniformly in $y$.

In Lemma 1 below we provide a useful characterization of the classes $S$ and $OS$. The result goes back to Goldie (1978). Lemma 1 easily follows from the following identity:

$$\overline{F^{*2}}(x) = 2\int_0^{x/2} \overline{F}(x-u)\, dF(u) + \overline{F}^2(x/2). \tag{6}$$

Lemma 1. (i) *We have $F \in S$ if and only if as $x \to \infty$,*

$$\int_0^{x/2} \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) \to 1 \text{ and } \frac{\overline{F}^2(x/2)}{\overline{F}(x)} \to 0.$$

(ii) *L We have $F \in OS$ if and only if as $x \to \infty$,*

$$\int_0^{x/2} \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) = O(1) \text{ and } \frac{\overline{F}^2(x/2)}{\overline{F}(x)} = O(1).$$

Using this characterization, the following result gives a summary of known results.

Proposition 2. *Let $F(x)$ denote a d.f. and if they exist, let $\mu$ denote the mean and $f(x)$ the density of $F(x)$.*

(i) *If $\overline{F} \in L \cap ORV$, then $F \in S$. If also $\mu < \infty$, then $F \in S^*$.*
(ii) *If $\overline{F} \in ORV$, then $F \in OS$ and $\overline{F} \in AA$. If also $\mu < \infty$, then $F \in OS^*$.*
(iii) *If $f \in L \cap ORV$, then $f \in SD$.*
(iv) *If $f \in ORV$, then $f \in OSD$.*

Other characterization of $S$ and $S^*$ are based on the hazard function $Q(x) = -\log \overline{F}(x)$ and the hazard rate function $q(x) = f(x)/\overline{F}(x)$. As an example we mention the following result of Klüppelberg (1989a,b), Goldie and Klüppelberg (1998).

Proposition 3. (i) *If $xq(x) = O(1)$, then $F \in S$. If also $\mu < \infty$, then $F \in S^*$.*
(ii) *If $\limsup xq(x)/Q(x) < 1$, then $F \in S$.*
(iii) *If $q \in RV(\alpha)$, with $-1 < \alpha < 0$, then $F \in S^*$.*

Related characterizations can be found e.g. in Embrechts et al. (1997) or Su and Tang (2003). Teugels (1975) considered d.f. for which $Q(x)$ is asymptotically concave. Pitman (1980) used (6) and assumed that $q(x) \downarrow 0$ to prove that $F \in S$ if and only if

$$\lim_{x\to\infty} \int_0^x \exp\big(yq(x) - Q(y)\big)\, dQ(y) = 1.$$

Murphree (1989) considered d.f. for which $Q(x)/x$ decreases to 0 and replaced the limit in Pitmans result by

$$\int_0^\infty \exp\Big(\frac{1}{2}Q(2x) - Q(x)\Big)\, dx < \infty.$$

Later Murphree (1990) replaced the integral condition by a summability condition. In section 2 below, we provide more results related to $S$ and the hazard rate function.

Section 3 of this paper is devoted to rates of convergence in the definitions (1) or (2). More precisely, for $n \geqslant 2$, let $R_n(x)$ be defined as

$$R_n(x) = 1 - F^{*n}(x) - n\big(1 - F(x)\big).$$

Omey and Willekens (1986, 1987) considered the case where $F$ has a regularly varying density $f \in RV(-\alpha)$. A typical result is that for $\alpha > 2$, one has

$$R_n(x)/f(x) \to \mu n(n-1).$$

Baltrunas and Omey (1998) used the class $OA$ and obtained results of the form $R_n(x) = O(1)f(x)R_F(x)$, where $R_F(x) = \int_0^x \overline{F}(t)\, dt$. A related result was proved in Omey (1994). We say that $F \in OD(m)$ if it satisfies

$$|F(x+y) - F(x)| = O(1)m(x), \forall y.$$

If $F \in OD(m)$ with $m \in ORV$, Omey (1994, p. 128) proved that

$$R_n(x) = O(1)m(x)\int_0^x y\, dF(y) + O(1)\overline{F}^2(x).$$

Note that if $m(x) = o(1)\overline{F}(x)$ then automatically $\overline{F} \in L$.

In this paper, we examine again subclasses of $L$ and discuss extra conditions to ensure that $F \in S$ together with an asymptotic result concerning $R_n(x)$. Recall that $g \in L$ if and only if $g(\log(x)) \in RV(0)$. Using Bingham et al. (1989, Theorem 1.3.1) we have the following representation theorem:

$$g \in L \text{ iff } g(x) = c(x)\exp\Big(-\int_a^x e(u)\, du\Big), \ \ \forall x \geqslant a, \tag{7}$$

where $c(x)$ and $e(x)$ are non-negative measurable functions such that $c(x) \to c > 0$ and $e(x) \to 0$ as $x \to \infty$. Moreover, for all $\varepsilon > 0$ we have $x^{\varepsilon}g(\log(x)) \to \infty$ and $x^{-\varepsilon}g(\log(x)) \to 0$. The representation (7) also shows that $(\log g(x))/x \to 0$. If $g \in L$ we see that $g(x) \sim h(x)$, where

$$h(x) = c\exp\Big(-\int_a^x e(u)\, du\Big), \forall x \geqslant a. \tag{8}$$

Clearly $h(x)$ is differentiable and $(\log(h(x))' = -e(x) \to 0$. For d.f. with $\overline{F} \in L$, in view of (7) and (8), we shall assume that there exists a non-negative function $q(x)$ such that

$$Q(x) = -\log \overline{F}(x) = \int_0^x q(u)\,du, \quad \forall x \geqslant 0. \tag{9}$$

The function $Q(x)$ is called the hazard function and the function $q(x)$ is called the hazard rate function of $F(x)$. If $q(x) \to 0$, then automatically $\overline{F} \in L$. Note that $\overline{F} \in L$ implies that $Q(x)/x \to 0$. Our assumption also implies that $F(x)$ has a density $f(x)$ for which the relation $f(x) = q(x)\overline{F}(x)$ holds.

In section 2 below, we provide simple conditions under which $F$ belongs to one or more of the classes $S$, $OS$, $S^*$ or $OS^*$. In section 3 we analyze the rate of convergence in (2) and obtain asymptotic estimates for $R_n(x)$ and we also discuss subordination. In section 4 we briefly discuss the multivariate case.

## 2. Hazard rates and the class $S$

**2.1. Sufficient conditions for $S$ and related classes.** In what follows we shall use the following assumptions and notations. As before $F(x)$ is a d.f. for which $F(0+) = 0$ and $F(x) < 1$ for all $x$. For $F(x)$ we assume (9) holds with $q(x) \geqslant 0$ and we also define the quantities $s(x) = Q(x)/x$, $h(x)$ and $r$ where

$$h(x) = \sup_{u \geqslant x} \Big(\frac{uq(u)}{Q(u)}\Big); \quad r = \lim\sup_{x\to\infty} \frac{xq(x)}{Q(x)} \leqslant \infty.$$

In this case $X$ has a density function $f(x)$ for which $f(x) = q(x)\overline{F}(x)$ holds. Note that if $q(x)$ is nonincreasing, then $xq(x) \leqslant Q(x)$ so that $r \leqslant 1$. To extend Proposition 3, we need a preliminary result.

Lemma 4. (i) *If $r < \infty$ then $Q \in ORV$ and $s \in ORV$.*
(ii) *If $r < 1$, then $\overline{F} \in L$ and $s(x)$ is nonincreasing for $x$ large enough.*

Proof. (i) Suppose that $xq(x)/Q(x) \leqslant B$, $x \geqslant a$. It follows that $q(x)/Q(x) \leqslant B/x$ and by integrating, we find that

$$Q(xt)/Q(x) \leqslant t^B, x \geqslant a, t \geqslant 1.$$

Since $Q$ is nondecreasing, we find that $Q \in ORV$ and also that $s \in ORV$.

(ii) Since $r < 1$, we can choose $\varepsilon, a > 0$ so that $xq(x)/Q(x) \leqslant r(\varepsilon) = r + \varepsilon < 1$, for $x \geqslant a$. As in part (i) we find that

$$Q(ta) \leqslant Q(a)t^{r(\varepsilon)}, t \geqslant 1.$$

From here we find that $q(ta) \leqslant r(\varepsilon)Q(ta)/ta = O(1)t^{r(\varepsilon)-1}$ as $t \to \infty$. It follows that $q(x) \to 0$ as $x \to \infty$ and consequently also that $\overline{F} \in L$. To prove the second part, we have $(s(x))' = (xq(x) - Q(x))/x^2$ and hence also that

$$(s(x))' \leqslant (r(\varepsilon) - 1)Q(x)/x^2 < 0 \text{ for } x \geqslant a.$$

This proves the result. □

Now we can state one of the main theorems of this section. The result is similar to results of Baltrunas et al. (2004), Baltrunas (2005).

THEOREM 5. *Suppose* $r < 1$ *and choose* $\varepsilon > 0$ *so that* $0 < r(\varepsilon) = r + \varepsilon < 1$.

(i) *Then we have* $F \in S$.
(ii) *If* $\int_0^\infty \overline{F}^{1-r(\varepsilon)}(t)\,dt < \infty$ *then* $F \in S^*$.
(iii) *If* $q \in ORV$ *and* $f \in L$ *then* $f \in SD$.
(iv) *If* $q \in ORV$ *and* $f \in OL$, *then* $f \in OSD$.

PROOF. First note that the conditions of the theorem imply that $\overline{F} \in L$.

(i) To prove that $F \in S$ we use Lemma 1. First we choose $a$ as in Lemma 4 and then choose $b > a$. For $x/2 \geqslant b$ we write

$$\int_0^{x/2} \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) = \left(\int_0^b + \int_b^{x/2}\right) \frac{\overline{F}(x-u)}{\overline{F}(x)} dF(u) = I + II. \tag{10}$$

Since $\overline{F} \in L$ and $b$ is fixed, we have $I \to \int_0^b 1dF(u) = F(b)$. For the second term of (10) we write

$$II = \int_b^{x/2} \exp(Q(x) - Q(x-u))\,dF(u).$$

Using the mean value theorem we have $Q(x) - Q(x-u) = q(z)u$ where $x-u \leqslant z \leqslant x$. Now observe that

$$q(z)u \leqslant r(\varepsilon)s(z)u \leqslant r(\varepsilon)s(u)u = r(\varepsilon)Q(u)$$

because $s(\cdot)$ is nonincreasing and because $b \leqslant u \leqslant x/2 \leqslant x-u \leqslant z$. It follows that

$$II \leqslant \int_b^{x/2} \exp(r(\varepsilon)Q(u))\,dF(u) = \int_b^{x/2} \overline{F}^{-r(\varepsilon)}(u)\,dF(u)$$

and consequently also that $II \leqslant \int_b^\infty \overline{F}^{-r(\varepsilon)}(u)\,dF(u) < \infty$. By Lebesgues theorem on dominated convergence we find that $II \to \int_b^\infty 1dF(u)$. Combining the two results, we see that $I + II \to 1$. Next we consider $\overline{F}^2(x/2)/\overline{F}(x)$. We have

$$\overline{F}(x/2)/\overline{F}(x) = \exp\big(Q(x) - Q(x/2)\big).$$

As before we find that $Q(x) - Q(x/2) \leqslant r(\varepsilon)Q(x/2)$ and hence also that

$$\overline{F}^2(x/2)/\overline{F}(x) \leqslant \overline{F}^{1-r(\varepsilon)}(x/2) \to 0.$$

The proof of (i) is complete.

(ii) Clearly the integral condition implies that $\mu < \infty$. Now we have

$$\frac{\overline{F} \otimes \overline{F}(x)}{\overline{F}(x)} = 2\int_0^{x/2} \frac{\overline{F}(x-u)}{\overline{F}(x)} \overline{F}(u)\,du.$$

As in the proof of (i) we split the integral into 2 parts. As in part (i) we find that $I \to 2\int_0^b \overline{F}(u)\,du$. For the second part we use

$$II \leqslant 2\int_b^{x/2} \exp(r(\varepsilon)Q(u))\overline{F}(u)\,du = 2\int_b^{x/2} \overline{F}^{1-r(\varepsilon)}(u)\,du.$$

Again Lebesgues theorem on dominated convergence can be used and we find that $\overline{F} \otimes \overline{F}(x)/\overline{F}(x) \to 2\mu$. This proves (ii).

(iii) We have

$$\frac{f \otimes f(x)}{f(x)} = 2\int_0^{x/2} \frac{f(x-u)}{f(x)} f(u)\,du.$$

As before we split the integral into two parts. As in part (i), using $f \in L$ we find that $I \to 2\int_0^b f(u)\,du$. For the second part, observe that $f(x-u)/f(x) = q(x-u)\overline{F}(x-u)/q(x)\overline{F}(x)$. Using $q \in ORV$ it follows that $f(x-u)/f(x) = O(1)\overline{F}(x-u)/\overline{F}(x)$. Now we can proceed as in the proof of (i) and apply Lebesgues theorem on dominated convergence. This proves the result.

(iv) We proceed as in part (iii). For $I$ now we find that $I = O(1)f(x)$. For $II$, as in part (iii) we find that

$$II = O(1)\int_b^{x/2} \frac{\overline{F}(x-u)}{\overline{F}(x)} f(u)\,du$$

and hence that $II = O(1)\overline{F} * F(x)/\overline{F}(x)$. Since $F \in S$, we find that $II = O(1)$. This proves the result. □

**Remarks.** 1) Baltrunas and Omey (1998, Lemma 3.5) showed that $F \in S$ and $q \in ORV$ imply that $f^{*n}(x) = O(1)f(x)$ for all $n \geqslant 2$. If also $q \in L$, then $f \in SD$.

2) Klüppelberg (1988, 1989b) proved that $r < 1$, $q(x) \to 0$ and $xq(x) \to \infty$ imply that $F \in S^*$.

3) Suppose that $xq(x)/Q(x) \leqslant B$ for $x \geqslant a$. Now take $u$ and $x$ such that $0 \leqslant u \leqslant x/2$ and $x \geqslant 2a$. Using

$$F(x) - F(x-u) = \int_{x-u}^{x} f(z)\,dz = \int_{x-u}^{x} q(z)\overline{F}(z)\,dz$$

we see that

$$0 \leqslant F(x) - F(x-u) \leqslant B\int_{x-u}^{x} s(z)\overline{F}(z)\,dz.$$

Since $s \in ORV$ (cf. Lemma 4) it follows that, uniformly in $0 \leqslant u \leqslant x/2$,

$$0 \leqslant F(x) - F(x-u) = O(1)s(x)\int_{x-u}^{x} \overline{F}(z)\,dz = O(1)s(x)\overline{F}(x-u)u.$$

If $s(x) \to 0$, the previous analysis shows that for fixed $u$ we have

$$0 \leqslant F(x) - F(x-u) = o(1)\overline{F}(x-u)$$

so that $\overline{F} \in L$. Moreover, $\overline{F} \in OD(m)$ with $m(x) = s(x)\overline{F}(x)$ and hence $\overline{F} \in L$ with a rate of convergence determined by $s(x)$.

If we only have $s(x) = O(1)$, first observe that

$$\overline{F}(x-y)/\overline{F}(x) = \exp(Q(x) - Q(x-y)).$$

Since $Q(x) - Q(x-y) = \int_{x-y}^{x} q(z)\,dz$ and $q(x) \leqslant Bs(x) = O(1)$, we find that $Q(x) - Q(x-y) = O(1)$. It follows that $\overline{F} \in OL$.

Theorem 5(ii) shows that $F \in S^*$ under the additional conditions that $r(\varepsilon) = r + \varepsilon < 1$ and $\int_0^\infty \overline{F}^{1-r(\varepsilon)}(t)\, dt < \infty$. In the next result, we obtain other useful estimates for $\overline{F} \otimes \overline{F}(x)$. Part (ii) was obtained by Baltrunas (2005, Lemma 2.1). As before, we use the notation $R_F(x) = \int_0^x \overline{F}(t)\, dt$.

LEMMA 6. (i) *We always have* $\overline{F} \otimes \overline{F}(x) = O(1)\overline{F}(x/2)R_F(x)$.
(ii) *If* $r < 1$ *we have* $\overline{F} \otimes \overline{F}(x) = O(1)\overline{F}(x) \int_0^x \overline{F}^{1-r(\varepsilon)}(u)\, du$.
(iii) *If* $\overline{F} \in ORV$, *then* $\overline{F} \otimes \overline{F}(x) = O(1)\overline{F}(x)R_F(x)$, *i.e.* $\overline{F} \in OA$.
(iv) *If* $\overline{F}, \overline{G} \in ORV$, *then* $\overline{F} \otimes \overline{G}(x) = O(1)\overline{F}(x)R_G(x) + O(1)\overline{G}(x)R_F(x)$.

PROOF. (i) To prove (i), we use

$$\overline{F} \otimes \overline{F}(x) = 2\int_0^{x/2} \overline{F}(x-u)\overline{F}(u)\, du.$$

Since $\overline{F}(x-u) \leqslant \overline{F}(x/2)$, the result follows.

(ii) If $r < 1$, the proof of Theorem 5(ii) shows that for $x \geqslant 2b$ we have

$$II \leqslant 2\int_b^{x/2} \overline{F}^{1-r(\varepsilon)}(u)\, du \leqslant 2\int_0^x \overline{F}^{1-r(\varepsilon)}(u)\, du.$$

For $I$ we have

$$I = 2\int_0^b (\overline{F}(x-u)/\overline{F}(x))\overline{F}(u)\, du.$$

Using $\overline{F}(x-u)/\overline{F}(x) \leqslant \overline{F}(x-b)/\overline{F}(x) = \exp(Q(x) - Q(x-b))$, we obtain the estimate

$$\overline{F}(x-u)/\overline{F}(x) \leqslant \exp(r(\varepsilon)Q(b))$$

and then we see that

$$I \leqslant 2\exp(r(\varepsilon)Q(b)) \int_0^b \overline{F}(u)\, du.$$

Since $\overline{F}(u) \leqslant \overline{F}^{1-r(\varepsilon)}(u)$ we obtain that

$$I \leqslant 2\exp(r(\varepsilon)Q(b)) \int_0^b \overline{F}^{1-r(\varepsilon)}(u)\, du.$$

Now the result follows.

(iii) This follows from (i) and the definition of $ORV$.

(iv) Using

$$\overline{F} \otimes \overline{G}(x) = \int_0^{x/2} \overline{F}(x-y)\overline{G}(y)\, dy + \int_0^{x/2} \overline{F}(y)\overline{G}(x-y)\, dy$$

and $O$-variation, the result follows as in (i). □

**2.2. The asymptotic behaviour of $F(x)G(x)-F*G(x)$.** In this section we consider two d.f. $F(x)$ and $G(x)$ and consider the difference $D(x)=F(x)G(x)-F*G(x)$ between their product and their convolution product. Clearly we have $D(x)=I+II+III$ where

$$I=\int_0^{x/2}\big(F(x)-F(x-u)\big)\,dG(u)$$
$$II=\int_0^{x/2}\big(G(x)-G(x-u)\big)\,dF(u)$$
$$III=\big(F(x)-F(x/2)\big)\big(G(x)-G(x/2)\big)$$

Apart from $D(x)$, we shall also consider the difference

$$E(x)=1-F*G(x)-\overline{F}(x)-\overline{G}(x)$$

between the tail of the convolution product and the sum of the tails. Clearly we have $E(x)=D(x)-\overline{F}(x)\overline{G}(x)$.

In this section we shall obtain simple estimates for $D(x)$ and $E(x)$. In what follows we shall use the notations as before. With $F(x)$ we associate functions $Q_F(x)$, $q_F(x)$, $s_F(x)$, $h_F(x)$ and $r_F$ defined as before. We use similar notations and assumptions for the other d.f. $G(x)$. In our first result we estimate $D(x)$ and $E(x)$ in terms of $\overline{F}\otimes\overline{G}(x)$.

Theorem 7. (i) *If $Q_F\in ORV$ and $Q_G\in ORV$, then*

$$D(x)=O(1)(h_F(x/2)s_F(x)+h_G(x/2)s_G(x))\overline{F}\otimes\overline{G}(x).$$

(ii) *If $r_F+r_G<\infty$, then $D(x)=O(1)(s_F(x)+s_G(x))\overline{F}\otimes\overline{G}(x)$.*

(iii) *If $q_F\in ORV$ and $q_G\in ORV$, then $D(x)=O(1)(q_F(x)+q_G(x))\overline{F}\otimes\overline{G}(x)$.*

(iv) *If $r_F+r_G<\infty$, then $|E(x)|=O(1)(s_F(x)+s_G(x))\overline{F}\otimes\overline{G}(x)$.*

(v) *If in (iii) we have $\liminf xq_F(x)>0$, $\liminf xq_G(x)>0$, then*

$$|E(x)|=O(1)(q_F(x)+q_G(x))\overline{F}\otimes\overline{G}(x).$$

Proof. (i) and (ii). We use the decomposition $D(x)=I+II+III$. First consider $I$. Using $f(x)=q_F(x)\overline{F}(x)$, for $0\leqslant u\leqslant x/2$ we have

$$0\leqslant F(x)-F(x-u)=\int_{x-u}^{x}f(v)\,dv=\int_{x-u}^{x}q_F(v)\overline{F}(v)\,dv. \tag{11}$$

Since $x/2\leqslant x-u\leqslant v$, we can use $q_F(v)=s_F(v)vq_F(v)/Q_F(v)$ and $s_F\in ORV$ to obtain

$$q_F(v)\leqslant h_F(x/2)s_F(v)=O(1)h_F(x/2)s_F(x).$$

It follows from (11) that

$$F(x)-F(x-u)=O(1)h_F(x/2)s_F(x)\int_{x-u}^{x}\overline{F}(v)\,dv.$$

Using this expression in $I$, we see that

$$\begin{aligned} I &= O(1)h_F(x/2)s_F(x)\int_{u=0}^{x/2}\int_{v=x-u}^{x}\overline{F}(v)\,dv\,dG(u) \\ &= O(1)h_F(x/2)s_F(x)\int_{v=x/2}^{x}\int_{u=x-v}^{x/2}dG(u)\,\overline{F}(v)\,dv \\ &= O(1)h_F(x/2)s_F(x)\int_{x/2}^{x}\overline{G}(x-v)\,\overline{F}(v)\,dv. \end{aligned}$$

In a similar way it follows that

$$II = O(1)h_G(x/2)s_G(x)\int_{x/2}^{x}\overline{F}(x-v)\,\overline{G}(v)\,dv.$$

For $III$, we have

$$F(x) - F(x/2) = O(1)h_F(x/2)s_F(x)\int_{x/2}^{x}\overline{F}(v)\,dv$$

and

$$0 \leqslant G(x) - G(x/2) \leqslant \overline{G}(x/2) \leqslant \overline{G}(x-v)$$

as long as $x/2 \leqslant v$. It follows that

$$III = O(1)h_F(x/2)s_F(x)\int_{x/2}^{x}\overline{G}(x-v)\overline{F}(v)\,dv.$$

Hence we obtain that

$$III = O(1)h_F(x/2)s_F(x)\overline{F}\otimes\overline{G}(x).$$

Combining the estimates for $I$, $II$ and $III$, the result follows.

(iii) To treat $I$ again we use (11). Using $q_F \in ORV$ now we have

$$F(x) - F(x-u) = O(1)q_F(x)\int_{x-u}^{x}\overline{F}(v)\,dv, \quad 0 \leqslant u \leqslant x/2$$

and it follows that

$$I = O(1)q_F(x)\int_{x/2}^{x}\overline{G}(x-v)\overline{F}(v)\,dv.$$

Term $II$ can be treated in a similar way. For $III$ we use

$$F(x) - F(x/2) = O(1)q_F(x)\int_{x/2}^{x}\overline{F}(v)\,dv$$

and proceed as before. This proves the result.

(iv) To prove (iv) we use $|E(x)| \leqslant D(x) + \overline{F}(x)\overline{G}(x)$. In view of (ii) we have to estimate $\overline{F}(x)\overline{G}(x)$. To this end, first note that

$$\overline{F}\otimes\overline{G}(x) \geqslant \overline{F}(x)\int_0^x \overline{G}(z)\,dz \geqslant x\overline{F}(x)\overline{G}(x).$$

From here it follows that

$$\overline{F}(x)\overline{G}(x)/(s_G(x)\overline{F}\otimes\overline{G}(x)) \leqslant 1/(xs_G(x)) = 1/Q_G(x).$$

Since $Q_G(x) \to \infty$, it follows that $\overline{F}(x)\overline{G}(x) = o(1)s_G(x))\overline{F} \otimes \overline{G}(x)$.

(v) Using the approach of (iv), now we have

$$\overline{F}(x)\overline{G}(x)/(q_G(x)\overline{F} \otimes \overline{G}(x)) \leqslant 1/(xq_G(x)).$$

Since $\liminf xq_G(x) > 0$ we obtain that $\overline{F}(x)\overline{G}(x) = O(1)q_G(x)\overline{F} \otimes \overline{G}(x)$. □

In the special case where $F(x) = G(x)$, we have $E(x) = R_2(x)$ and we obtain the following corollary.

Corollary 8. (i) *If $r_F < \infty$ then $F^2(x) - F^{*2}(x) = O(1)s_F(x)\overline{F} \otimes \overline{F}(x)$.*

(ii) *If $q_F \in ORV$, then $F^2(x) - F^{*2}(x) = O(1)q_F(x)\overline{F} \otimes \overline{F}(x)$.*

(iii) *If $r_F < \infty$, then $|R_2(x)| = O(1)s_F(x)\overline{F} \otimes \overline{F}(x)$.*

(iv) *If $q_F \in ORV$ and $\liminf xq_F(x) > 0$, then $|R_2(x)| = O(1)q_F(x)\overline{F} \otimes \overline{F}(x)$.*

Part (iii) of the Corollary shows that $s_F(x)\overline{F} \otimes \overline{F}(x) = o(1)\overline{F}(x)$ and $r_F < \infty$ imply that $F \in S$. If $F \in S^*$, part (iii) shows that

$$\left|\overline{F^{*2}}(x)/\overline{F}(x) - 2\right| = O(1)s_F(x).$$

If $s_F(x) \to 0$ we do not only have $F \in S$ but we also obtain that the rate of convergence in (1) is determined by $s_F(x)$. If in part (iv) we have $F \in S^*$, then again $F \in S$ and now the rate of convergence in (1) is given by $q_F(x)$. Note that we can use Lemma 6 to simplify the expressions in Corollary 8.

## 3. Estimation of $R_n$ and subordination

In this section we use the results of the previous sections to estimate $R_n(x)$, where $R_n(x) = 1 - F^{*n}(x) - n\overline{F}(x)$. We also discuss subordinated d.f. as follows. Let $X, X_1, X_2, \ldots$ denote i.i.d. positive r.v. with d.f. $F(x)$, and, independent of the $X_i$, let $N$ denote an integer-valued random variable with $p(n) = P(N = n)$, $n \geqslant 0$. Now we consider the partial sums $S(0) = 0$ and $S(n) = X_1 + X_2 + \cdots + X_n$. The random sum $S(N)$ has d.f. $G(x)$ where

$$G(x) = \sum_{n=0}^{\infty} p(n)F^{*n}(x).$$

We say that $G$ is subordinated to $F$ with subordinator $N$. Clearly $\overline{G}(x)$ is given by

$$\overline{G}(x) = \sum_{n=1}^{\infty} p(n)\overline{F^{*n}}(x)$$

and using $S$ it should be possible to relate $\overline{G}(x)$ and $\overline{F}(x)$. If $p(0) = 0$ and $F$ has a density $f$, then also $G$ has a density $g$ and we find that

$$g(x) = \sum_{n=1}^{\infty} p(n)f^{\otimes n}(x).$$

The following results are well known, see e.g. Embrechts et al. (1979, 1982), Chover et al. (1973). Result (b) is a result of Stam (1973).

LEMMA 9. (a) *Suppose that* $\Psi(z) = E(z^N)$ *is analytic at* $z = 1$.

(i) *If* $F \in S$, *then* $G \in S$ *and* $\overline{G}(x) \sim E(N)\overline{F}(x)$.

(ii) *If* $f \in SD$, *then* $g \in SD$ *and* $g(x) \sim E(N)f(x)$.

(b) *If* $\overline{F}(x) \in RV(-\alpha)$, $\alpha > 1$, *and* $E(N^{\alpha+1+\varepsilon}) < \infty$, *then* $\overline{G}(x) \sim E(N)\overline{F}(x)$.

Result (b) shows that under weaker assumptions about $N$ we have to assume more about $\overline{F}$.

Shimura and Watanabe (2005) provide an $O$-type of result in the case where $F \in OS$. If we use $\|F\|_m$ (cf. 5), then we have $\|G\|_m \leqslant \Psi(\|F\|_m)$, where $\Psi(z) = E(z^N)$. In the special case where $F \in OS$ we can take $m(x) = \overline{F}(x)$ and we find that

$$\sup_{x\geqslant 0} \frac{\overline{G}(x)}{\overline{F}(x)} \leqslant \sum_{n=1}^{\infty} p(n) \sum_{k=0}^{n-1} \|F\|_{\overline{F}}^{k}.$$

Starting with a density $f$ with $\|f\|_m = \sup_{x\geqslant 0} m \otimes f(x)/m(x) < \infty$, we find that $\|g\|_m \leqslant \Psi(\|f\|_m)$. In the special case where $f \in OSD$, we can take $m(x) = f(x)$ and we find that

$$\sup_{x\geqslant 0} \frac{g(x)}{f(x)} \leqslant \sum_{n=1}^{\infty} p(n) \|f\|_f^{n-1}.$$

The purpose of this section is to obtain some new estimates for the differences $R_n(x)$ and for $R_N(x)$ defined as $R_N(x) = \overline{G}(x) - E(N)\overline{F}(x)$. Note that $R_N(x) = \sum_{n=2}^{\infty} p(n)R_n(x)$. To treat $R_n(x)$ and $R_N(x)$ we use the following identities, cf. Omey and Willekens (1987):

$$R_{n+1}(x) = nR_2(x) + R_n \otimes f(x) \tag{12}$$

$$R_N(x) = a(0)R_2(x) + H * R_2(x). \tag{13}$$

In (13) the function $H(x)$ and the sequence $\{a(n)\}$ are given by

$$H(x) = \sum_{n=1}^{\infty} a(n)F^{*n}(x) \text{ and } a(n) = \sum_{k=n+2}^{\infty} p(k)(k-1-n).$$

Clearly $H(x)$ has the same form as $G(x)$. If $EN^2 < \infty$ then there is a constant $c > 0$ such that $\{ca(n)\}$ is a probability distribution. If $F$ has a density $f$, then $H$ has a derivative given by $h(x) = \sum_{n=1}^{\infty} a(n)f^{\otimes n}(x)$. Under the conditions of Lemma 9(a)(ii) we obtain that, cf. Omey and Willekens (1986, 1987),

$$\lim_{x\to\infty} \overline{H}(x)/\overline{F}(x) = \lim_{x\to\infty} h(x)/f(x) = \sum_{n=1}^{\infty} na(n).$$

In the next Theorem we prove a result for $R_n(x)$ and $R_N(x)$. In the result we use the notation $R(x, r(\varepsilon)) = \int_0^x \overline{F}^{1-r(\varepsilon)}(u)\,du$. In view of Lemma 6, a similar result can be proved for the class $OA$. In view of Corollary 8, we can also formulate conditions under which we can replace $s(x)$ by $q(x)$.

THEOREM 10. (i) *If* $r < 1$ *then* $|R_n(x)| = O(1)s(x)R(x, r(\varepsilon))\overline{F}(x)$, $\forall n \geqslant 2$.

(ii) *If* $r < 1$ *and* $h(x) = O(1)f(x)$, *then* $|R_N(x)| = O(1)s(x)R(x, r(\varepsilon))\overline{F}(x)$.

PROOF. (i) First note that the conditions imply that $F \in S$, cf. Theorem 5. In Corollary 8(iii) we proved that

$$|R_2(x)| = O(1)s(x)\overline{F} \otimes \overline{F}(x).$$

Using Lemma 6(ii) we obtain that $|R_2(x)| = O(1)s(x)\overline{F}(x)R(x, r(\varepsilon))$. Now we use (12) and proceed by induction on $n$.

We choose $C$ and $x^\circ$ so that $|R_n(x)| \leqslant Cs(x)\overline{F}(x)R(x, r(\varepsilon))$ for $x \geqslant x^\circ$. For $x \geqslant 2x^\circ$ we write

$$R_n \otimes f(x) = I + II + III$$

where

$$I = \int_0^{x^\circ} R_n(x-u)f(u)\,du;$$

$$II = \int_{x^\circ}^{x/2} R_n(x-u)f(u)\,du;$$

$$III = \int_{x/2}^{x} R_n(x-u)f(u)\,du.$$

First consider $I$. We have $|I| \leqslant \int_0^{x^\circ} |R_n(x-u)|\,f(u)\,du$. Since $x-u \geqslant x-x^\circ \geqslant x^\circ$, we see that

$$|I| \leqslant C\int_0^{x^\circ} s(x-u)\overline{F}(x-u)R(x-u, r(\varepsilon))f(u)\,du.$$

Using $s \in ORV$, $\overline{F} \in L$ and the monotonicity of $R(x, r(\varepsilon))$, we see that

$$|I| = O(1)s(x)\overline{F}(x)R(x, r(\varepsilon))F(x^\circ) = O(1)s(x)\overline{F}(x)R(x, r(\varepsilon)).$$

In $II$ we have

$$|II| \leqslant C\int_{x^\circ}^{x/2} s(x-u)\overline{F}(x-u)R(x-u, r(\varepsilon))f(u)\,du$$

and consequently also that

$$|II| = O(1)s(x)R(x, r(\varepsilon))\int_{x^\circ}^{x/2} \overline{F}(x-u)f(u)\,du.$$

Using $F \in S$ we have $\int_{x^\circ}^{x/2} \overline{F}(x-u)f(u)\,du = O(1)\overline{F}(x)$ and it follows that

$$|II| = O(1)s(x)R(x, r(\varepsilon))\overline{F}(x).$$

In $III$ we choose a fixed number $b$ and write

$$III = \int_{x/2}^{x-b} R_n(x-u)f(u)\,du + \int_{x-b}^{x} R_n(x-u)f(u)\,du.$$

Since $F \in S$ we have $R_n(x) = O(1)\overline{F}(x)$ and we can choose $b$ such that $|R_n(x)| \leqslant C\overline{F}(x)$ for $x \geqslant b$. In the other case, we have $|R_n(x)| \leqslant 1+n$. Using these inequalities, we see that

$$|III| \leqslant C\int_{x/2}^{x-b} \overline{F}(x-u)f(u)\,du + (1+n)\int_{x-b}^{x} f(u)\,du = (A) + (B).$$

As to $(A)$ we have

$$(A) \leqslant C \int_{x/2}^{x} \overline{F}(x-u)q(u)\overline{F}(u)\,du$$

so that

$$(A) = O(1)s(x) \int_{x/2}^{x} \overline{F}(x-u)\overline{F}(u)\,du = O(1)s(x)\overline{F} \otimes \overline{F}(x).$$

It follows that $(A) = O(1)s(x)R(x, r(\varepsilon))\overline{F}(x)$. As to $(B)$ we use $f(x) = q(x)\overline{F}(x) = O(1)s(x)\overline{F}(x)$ to see that $(B) = O(1)s(x)\overline{F}(x)$ and consequently also that $(B) = O(1)s(x)R(x, r(\varepsilon)))\overline{F}(x)$. Using (12) we conclude that

$$|R_{n+1}(x)| = O(1)s(x)R(x, r(\varepsilon)))\overline{F}(x).$$

(ii) Now we use (13). Since we have a density, we can rewrite (13) as

$$R_N(x) = a(0)R_2(x) + h \otimes R_2(x).$$

Since $h(x) = O(1)f(x)$, the proof is similar to that of part (i) and therefore omitted. $\square$

## 4. Multivariate results

In this section we briefly discuss some multivariate analogues of our results. Suppose that $F(\mathbf{x})$ and $G(\mathbf{x})$ are d.f. of positive $d$-dimensional random vectors and suppose that the marginal d.f. are given by $F_i$ and $G_i$ respectively.

Now consider the following differences (with $N$ as in section 3)

$$\begin{aligned} D(\mathbf{x}) &= F(\mathbf{x})G(\mathbf{x}) - F * G(\mathbf{x}); & K_N(\mathbf{x}) &= \sum_{2}^{\infty} p(n)K_n(\mathbf{x}); \\ K_n(\mathbf{x}) &= F^n(\mathbf{x}) - F^{*n}(\mathbf{x}); & & \\ R_n(\mathbf{x}) &= 1 - F^{*n}(\mathbf{x}) - n(1 - F(\mathbf{x})); & R_N(\mathbf{x}) &= \sum_{2}^{\infty} p(n)R_n(\mathbf{x}). \end{aligned}$$

First we consider $R_n(\mathbf{x})$. We prove the following result.

Proposition 11. *If the marginals $F_i$ of $F$ satisfy $F_i \in S$, then $|R_n(\mathbf{x})| = o(1)\overline{F}(\mathbf{x})$, as $\min(x_i) \to \infty$.*

Proof. To prove the result first note that $R_n(\mathbf{x}) = K_n(\mathbf{x}) + O(1)\overline{F}^2(\mathbf{x})$ and that for each marginal $i = 1, 2, \dots, d$ we have $R_{n,i}(x_i) = K_{n,i}(x_i) + O(1)\overline{F_i}^2(x_i)$. Since $K_n(\mathbf{x}) \leqslant \sum_{i=1}^{d} K_{n,i}(x_i)$, it follows that

$$|R_n(\mathbf{x})| \leqslant \sum_{i=1}^{d} |R_{n,i}(x_i)| + O(1)\overline{F}^2(\mathbf{x}).$$

Since by assumption we have $F_i \in S$, we have $|R_{n,i}(x_i)| = o(1)\overline{F_i}(x_i)$ as $x_i \to \infty$. It follows that as $\min(x_i) \to \infty$,

$$|R_n(\mathbf{x})| = o(1) \sum_{i=1}^{d} \overline{F_i}(x_i) + O(1)\overline{F}^2(\mathbf{x}).$$

Since $\overline{F_i}(x_i) \leqslant \overline{F}(\mathbf{x})$, we obtain that $|R_n(\mathbf{x})| = o(1)\overline{F}(\mathbf{x}) + O(1)\overline{F}^2(\mathbf{x})$. This proves the result. $\square$

Proposition 11 shows that as $\min(x_i) \to \infty$, we have

$$\frac{\overline{F^{*n}}(\mathbf{x})}{\overline{F}(\mathbf{x})} \to n. \tag{14}$$

This means that, starting only from subexponential marginals, $F$ satisfies a form of multivariate subexponential behaviour. Using the conditions of Lemma 9 a(i) and the approach of Proposition 11 it is easy and straightforward to prove also that $|R_N(\mathbf{x})| = o(1)\overline{F}(\mathbf{x})$.

From (14) it follows that for each fixed positive $\mathbf{x}$ we have

$$\frac{\overline{F^{*n}}(t\mathbf{x})}{\overline{F}(t\mathbf{x})} \to n, \text{ as } t \to \infty \tag{15}$$

Subexponential behaviour of the form (15) has been studied by Omey (2003) and Omey et al. (2006). In the multivariate regularly varying case, (15) appeared in Omey (1990). Cline and Resnick (1992) studied a relation of the form (15) by using vague convergence.

Next we consider $D(\mathbf{x})$. It is easy to see that $D(\mathbf{x}) \leqslant \sum_{i=1}^{d} D_i(x_i)$, where the quantities $D_i(x) = F_i(x)G_i(x) - F_i * G_i(x)$ denote the differences for the marginals. We can use for example Theorem 7 to obtain estimates of the form

$$0 \leqslant D(\mathbf{x}) = O(1)\sum_{i=1}^{d}(s_{i,F}(x_i) + s_{i,G}(x_i))\overline{F_i} \otimes \overline{G_i}(x_i). \tag{16}$$

If $F = G$ we can simplify and we prove the following result.

Lemma 12. *Suppose that the marginals $F_i$ of $F$ satisfy $F_i \in S^*$ and $r_i < \infty$. Then*

$$K_2(\mathbf{x}) = O(1)\overline{F}(\mathbf{x})\sum_{i=1}^{d} s_{i,F}(x_i); \quad |R_2(\mathbf{x})| = O(1)\overline{F}(\mathbf{x})\sum_{i=1}^{d} s_{i,F}(x_i) + \overline{F}^2(\mathbf{x}).$$

Proof. If $F = G$ and all the marginals satisfy $F_i \in S^*$, from (16) we find that

$$0 \leqslant K_2(\mathbf{x}) = F^2(\mathbf{x}) - F * F(\mathbf{x}) = O(1)\sum_{i=1}^{d} s_{i,F}(x_i)\overline{F_i}(x_i).$$

Since $\overline{F_i}(x_i) \leqslant \overline{F}(\mathbf{x})$, we obtain that $0 \leqslant F^2(\mathbf{x}) - F * F(\mathbf{x}) = O(1)\overline{F}(\mathbf{x}) \sum_{i=1}^{d} s_{i,F}(x_i)$.

The second result follows from the identity $R_2(\mathbf{x}) = K_2(\mathbf{x}) - \overline{F}^2(\mathbf{x})$. $\square$

Note that since $\overline{F_i} \in L$, it follows that $s_i(x) \to 0$. Under the assumptions of Lemma 12 we have relation (14) for $n = 2$ together with a rate of convergence result. In our final result, we discuss the behaviour of $R_N(\mathbf{x})$.

Theorem 13. *Suppose the marginals $F_i$ of $F$ satisfy the conditions of Theorem 10 and assume that $E(N^2) < \infty$. Then, as $\min(x_i) \to \infty$ we have*

$$|R_N(\mathbf{x})| = O(1)\overline{F}(\mathbf{x}) \sum_{i=1}^{d} s_i(x_i) R_i(x_i, r_i(\varepsilon)) + O(1)\overline{F}^2(\mathbf{x}).$$

Proof. First observe that

$$0 \leqslant K_n(\mathbf{x}) - R_n(\mathbf{x}) = n(1 - F(\mathbf{x})) - (1 - F^n(\mathbf{x})) \leqslant \binom{n}{2} \overline{F}^2(\mathbf{x})$$

and similarly that $0 \leqslant K_N(\mathbf{x}) - R_N(\mathbf{x}) \leqslant E\binom{N}{2}\overline{F}^2(\mathbf{x})$. For the marginals we have similar expressions. Next observe that

$$0 \leqslant K_n(\mathbf{x}) \leqslant \sum_{i=1}^{d} K_{n,i}(x_i) \quad \text{and} \quad 0 \leqslant K_N(\mathbf{x}) \leqslant \sum_{i=1}^{d} K_{N,i}(x_i).$$

Using $\overline{F_i}(x_i) \leqslant \overline{F}(\mathbf{x})$, these observations show that if $EN^2 < \infty$, we have

$$|R_N(\mathbf{x})| = O(1) \sum_{i=1}^{d} |R_{N,i}(x_i)| + O(1)\overline{F}^2(\mathbf{x}). \tag{17}$$

Under the conditions of Theorem 10 we obtain

$$|R_N(\mathbf{x})| = O(1) \sum_{i=1}^{d} s_i(x_i) R_i(x_i, r_i(\varepsilon)) \overline{F_i}(x_i) + O(1)\overline{F}^2(\mathbf{x}).$$

Finally, using $\overline{F_i}(x_i) \leqslant \overline{F}(\mathbf{x})$, we obtain the desired result. □

**Remarks.** 1) It depends on the interplay between $\overline{F_i}(x)$ and $s_i(x)R_i(x, r(\varepsilon))$ to see which term is dominant here.

2) If $N = n$, one can use (17) and the one-dimensional results (cf. the discussion following Proposition 3) to obtain other types of estimates.

## 5. Concluding remarks

1) From Lemma 9(a)(ii), by integration it follows that

$$G(x+h) - G(x) \sim E(N)(F(x+h) - F(x) \sim E(N)f(x)h.$$

It could be interesting to study rates of convergence in this Blackwell type of result.

2) In the case where $r = 1$, Theorem 5 is not applicable and we should assume more about the hazard function. If $q(x)$ is nonincreasing, we see that for $0 \leqslant u \leqslant x/2$ we have

$$Q(x) - Q(x-u) = \int_{x-u}^{x} q(z)\, dz \leqslant uq(u).$$

If we set $k(x) = Q(x) - xq(x)$ we find that $Q(x) - Q(x-u) - Q(u) \leqslant -k(u)$. The following result can be proved.

Proposition 14. *Suppose that $q(x) \downarrow 0$.*

(i) *If $\int_0^\infty q(u)\exp(-k(u))\, du < \infty$ and $k(x) \to \infty$, then $F \in S$.*

(ii) *If* $\mu < \infty$ *and* $\int_0^\infty \exp(-k(u))\,du < \infty$, *then* $F \in S^*$.

More research is needed in this case.

3) In the two-dimensional case, (14) shows that as $\min(x, y) \to \infty$,

$$1 - P(S^1(n) \leqslant x, S^2(n) \leqslant y) \sim n\overline{F}(x, y)),$$

where $(S^1(n), S^2(n)) = \sum_{i=1}^n (X_i^1, X_i^2)$ are partial sums obtained from $F$. It is not clear how to obtain information about the asymptotic behaviour of the tail $1 - P(S^1(n) \leqslant x, S^2(m) \leqslant y)$, when $n \neq m$.

4) It is not clear how to define nor how to treat multivariate subexponential densities yet. To define a multidimensional class $OSD$ one could assume that $\|f\|_m < \infty$, where

$$\|f\|_m = \sup_{\mathbf{x} \geqslant \mathbf{0}} \frac{m \otimes f(\mathbf{x})}{m(\mathbf{x})}$$

for a suitable function $m$. In this case it readily follows that $\|f^{\otimes n}\|_m \leqslant \|f\|_m^n$. As a special case one can take $m = f$. In view of (14) it also makes sense to study d.f. $F(\mathbf{x})$ satisfying a relation of the form, cf.(5),

$$\|F\|_m = \sup_{\mathbf{x} \geqslant \mathbf{0}} \frac{m * F(\mathbf{x})}{m(\mathbf{x})} < \infty.$$

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