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MARKOVIAN BLACK AND SCHOLES

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Abstract. We generalize the classical binomial approach of the model of Black and Scholes to a Markov binomial approach. This leads to a new formula for the cost of an option.

1. Introduction

Consider a call option with strike price X and exercise time t. We divide the time t into the time points $t/n, 2t/n, \ldots, nt/n$. During each time unit the price goes up by a factor u or down by a factor d. The value after n time units is given by

$$S(t) = S(0)u^{S_n}d^{n-S_n}$$

where $S(0)$ is the price at time $t = 0$ and where S_n denotes the number of ups during n time periods. The cost of the option that does not give rise to an arbitrage is given by

$$K = r_0^{-n}E\big(\max(S(0)u^{S_n}d^{n-S_n} - X, 0)\big) \tag{1}$$

where $r_0 = 1 + rt/n$ is the nominal interest rate. In the usual Black–Scholes approach, cf. Ross [1999, Chapter 7], Cox et al. [1979], one assumes that S_n has a binomial distribution given by $S_n \sim BIN(n, p)$ and one takes

$$u = \exp\big(a\sqrt{t/n}\big), \quad d = \exp\big(-b\sqrt{t/n}\big) \tag{2}$$

and

$$p = \frac{1 + rt/n - d}{u - d}$$

or

$$p = \frac{r^* - d}{u - d}$$

where $r^* = \exp(rt/n)$. The basic assumption in this binomial model is that the ups and the downs appear independently from each other, i.e., the sum S_n is made up of independent $0-1$ variables. In the present paper we assume that the ups

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and the downs are governed by a Markov Chain and provide a Black and Scholes formule for this case.

2. Markovian approach

We shall consider the case where the ups and downs are governed by a Markov chain as follows. Let $Y_i = 1$ if the price goes up at the $i-th$ time unit and let $Y_i = 0$ otherwise. Assume that $P(Y_1 = 1) = p$ and that the transition probabilities are given by

$$P = \begin{pmatrix} p_{0,0} & p_{0,1} \\ p_{1,0} & p_{1,1} \end{pmatrix}$$

In the paper we assume that the transition probabilities are strictly between 0 and 1. The number of ups is given by $S_n = \sum_{i=1}^{n} Y_i$ and formula (1) holds.

It is useful to note that the Markov chain has a unique stationary vector given by (x, y) where

$$y = \frac{p_{0,1}}{p_{0,1} + p_{1,0}} \quad \text{and} \quad x = 1 - y \tag{3}$$

The eigenvalues of P are given by 1 and $\lambda = 1 - p_{0,1} - p_{1,0} = p_{1,1} - p_{0,1}$. Note that $|\lambda| < 1$. Properties of S_n can be found in e.g. Omey, Santos and Van Gulck [2006]. As a special case we also consider correlated Bernoulli trials studied by Dimitrov and Kolev [1999], see also Edwards [1960] or Wang [1981]. In this case the transition matrix is given by

$$P(p, \rho) = \begin{pmatrix} q + \rho p & p(1-\rho) \\ q(1-\rho) & p + \rho q \end{pmatrix}$$

and now we have $P(Y_i = 1) = p = y$, for all i and $\lambda = \rho = \rho(Y_i, Y_{i+1}) \neq 0$. In the Markov chain setting we have the following result concerning moments of S_n.

PROPOSITION 1 (Omey et al. [2006]). (i) *We have*

$$E(S_n) = ny - (y - p)\frac{1 - \lambda^n}{1 - \lambda}$$

and

$$\mathrm{Var}(S_n) = n\frac{1+\lambda}{1-\lambda}xy + \sum_{k=0}^{n-1}\left(C(1)\lambda^k + C(2)\lambda^{2k} + C(3)k\lambda^k\right)$$

where $C(1), C(2), C(3)$ are given in the remark below. As $n \to \infty$ we have

$$E(S_n) \sim ny \text{ and } \mathrm{Var}(S_n) \sim nxy\frac{1+\lambda}{1-\lambda}$$

(ii) *If $P = P(p, \rho)$ we have $E(S_n) = np$ and*

$$\mathrm{Var}(S_n) = \frac{pq}{1-\rho}\left(n(1+\rho) - 2\rho\frac{1-\rho^n}{1-\rho}\right)$$

REMARK. Using $a(1) = (y-p)(y-x)$ and $a(2) = (y-p)^2$ the constants are given by

$$C(1) = \big(a(1)(1-\lambda) - 2xy\lambda - 2a(2)\big)/(1-\lambda)$$
$$C(2) = a(2)(1+\lambda)/(1-\lambda)$$
$$C(3) = 2a(1)$$

For large values of n we can approximate the distribution of S_n by a normal distribution. We have the following central limit theorem.

THEOREM 2 (Omey et al. [2006]). *As $n \to \infty$ we have*

$$\frac{S_n - ny}{\sqrt{n\theta}} \overset{d}{\Rightarrow} Z \sim N(0,1),$$

where $\theta = xy(1+\lambda)/(1-\lambda)$

3. Markovian Black and Scholes

In view of (1) we define W by the following relation:

$$u^{S_n} d^{n-S_n} = \exp(W)$$

Assuming that (2) holds, we have

$$W = (a+b)\sqrt{t/n}S_n - b\sqrt{tn} \tag{4}$$

Using (4) and Proposition 1(i) we find that

$$E(W) = \sqrt{nt}\big((a+b)y - b\big) - (a+b)(y-p)\sqrt{\frac{t}{n}}\frac{1-\lambda^n}{1-\lambda} \tag{5}$$

and

$$\operatorname{Var}(W) \sim (a+b)^2 txy\frac{1+\lambda}{1-\lambda} \tag{6}$$

In order to obtain useful estimates in (5) and (6), we make the following assumptions about the transition probabilities. First we introduce some extra notations. Let α and β denote real parameters and let

$$r_u = \exp(\alpha t/n) \quad \text{and} \quad r_d = \exp(\beta t/n) \tag{7}$$

For the transition probabilities we assume that there are constants A, B, C, D such that

$$p_{0,1} = A + B\frac{r_u - d}{u-d} \quad \text{and} \quad p_{1,1} = C + D\frac{r_d - d}{u-d} \tag{8}$$

Later we shall reduce the number of parameters in (8). With model (8) we want to take into account the difference between going from a 'down' to an 'up' and from an 'up' to another 'up'.

PROPOSITION 3. *We have*

$$p_{0,1} = A + B\left(\frac{b}{a+b} + \sqrt{\frac{t}{n}}\left(\frac{\alpha - ab/2}{a+b} + o(1)\right)\right) \tag{9}$$

and

$$p_{1,1} = C + D\left(\frac{b}{a+b} + \sqrt{\frac{t}{n}}\left(\frac{\beta - ab/2}{a+b} + o(1)\right)\right) \tag{10}$$

PROOF. Let us consider $p_{0,1}$. Using (2) and (7), a Taylor expansion shows that

$$r_u - d = \alpha\frac{t}{n} + b\sqrt{\frac{t}{n}} - b^2\frac{t}{2n} + O(1)n^{-3/2}$$

and

$$u - d = (a+b)\left(\sqrt{\frac{t}{n}} + (a-b)\frac{t}{2n}\right) + O(1)n^{-3/2}$$

Now observe that

$$(a+b)\frac{r_u - d}{u-d} - b = \frac{(a+b)(r_u - d) - b(u-d)}{u-d}$$

Using the Taylor expansions, we readily obtain that

$$(a+b)\frac{r_u - d}{u-d} - b \sim \left(\alpha - \frac{ab}{2}\right)\sqrt{\frac{t}{n}}$$

From this and (8) we obtain (9). In a similar way also (10) follows. □

Note that as $n \to \infty$ we have

$$p_{0,1} \to A + B\frac{b}{a+b}$$

$$p_{1,1} \to C + D\frac{b}{a+b}$$

Since $0 < p_{i,j} < 1$, these expressions show that the parameters A, B, C, D should satisfy some restrictions. Using (3) we also obtain that $y \to y^*$ and that $\lambda \to \gamma$ where

$$y^* = \frac{A(a+b) + Bb}{(A+1-C)(a+b) + b(B-D)}$$

and

$$\gamma = C - A + \frac{b(D-B)}{a+b}$$

Note that

$$y^* = \frac{A + B\frac{b}{a+b}}{1-\gamma}$$

In view of (5) we choose A, B, C, D in such a way that

$$y^* = \frac{b}{a+b}$$

If, for example $A = 0$ and $B = D = 1 - C$, then we obtain that $y^* = b/(a+b)$ and in this case we have $\gamma = C$.

Now we can proceed in studying W, cf. (5), (6).

THEOREM 4. (i) *As* $n \to \infty$, *we have* $W \overset{d}{\Rightarrow} W^*$, *where* $W^* = \mu + \sigma Z$, *with* $Z \sim N(0,1)$ *as in Theorem* 2 *and with*

$$\mu = t\frac{1}{1-\gamma}\left(B\Big(\alpha - \frac{ab}{2}\Big) + bD\Big(\frac{\beta - ab/2}{a+b}\Big) - bB\Big(\frac{\alpha - ab/2}{a+b}\Big)\right)$$

and

$$\sigma^2 = tab\frac{1+\gamma}{1-\gamma}$$

(ii) *As* $n \to \infty$, *we have*

$$P\big(S(0)\exp(W) > X\big) \to P\Big(Z > \frac{\log(X/S(0)) - \mu}{\sigma}\Big)$$

PROOF. (i) Since by Theorem 2, S_n is asymptotically normal, also W is. We have to determine $E(W)$ and $\mathrm{Var}(W)$ as $n \to \infty$. First consider $E(W)$ and observe that

$$(a+b)y - b = \frac{I}{II}$$

where $I = (a+b)p_{0,1} - b(p_{0,1} + p_{1,0})$ and $II = p_{1,0} + p_{0,1}$. Using $II = 1 - \lambda$ we have

$$II \to 1 - \gamma$$

As to I we have $I = ap_{0,1} - b + bp_{1,1}$. Using (9) and (10) we readily obtain that

$$I = K(1)\sqrt{\frac{t}{n}}\big(1 + o(1)\big)$$

where

$$K(1) = B\Big(\alpha - \frac{ab}{2}\Big) + bD\Big(\frac{\beta - ab/2}{a+b}\Big) - bB\Big(\frac{\alpha - ab/2}{a+b}\Big)$$

We conclude that

$$(a+b)y - b = \sqrt{\frac{t}{n}}\Big(\frac{K(1)}{1-\gamma} + o(1)\Big)$$

Using this result, we find that

$$E(W) \to t\frac{K(1)}{1-\gamma}$$

For the variance we find that $y \to y^*$ and $x \to 1 - y^*$. It follows that

$$\mathrm{Var}(W) \to tab\frac{1+\gamma}{1-\gamma}$$

This proves the result.

(ii) This follows from (i) □

REMARK. With the choice $A = 0$ and $B = D = 1 - C$, we find the following simpler expressions: we have $\gamma = C$ and

$$\mu = t\left(\Big(\alpha - \frac{ab}{2}\Big) + b\Big(\frac{\beta - \alpha}{a+b}\Big)\right) \quad \text{and} \quad \sigma^2 = tab\frac{1+\gamma}{1-\gamma}$$

Taking also $\alpha = \beta = r$, we can simplify more and find that

$$\mu = t\Big(r - \frac{ab}{2}\Big) \quad \text{and} \quad \sigma^2 = tab\frac{1+\gamma}{1-\gamma}$$

If we take $a = b = \sigma_p$ where σ_p represents the volatility of the underlying security, then we find

$$\mu = t\Big(r - \frac{\sigma_p^2}{2}\Big) \quad \text{and} \quad \sigma^2 = t\sigma_p^2\frac{1+\gamma}{1-\gamma}$$

The case where $\gamma = 0$ corresponds to the usual Black and Scholes model. Here we have the extra parameter γ. Using $p_{0,1} \to (1-\gamma)/2$ and $p_{1,1} \to (1+\gamma)/2$ we see that γ is closely connected with the probability of arriving at an 'up' starting from a 'down' or an 'up'. The parameter γ heavily influences σ^2 (and hence also K, see below). Taking $\gamma = -0.5$, $\gamma = 0$ and $\gamma = 0.5$ we see that σ^2 varies from $\sigma^2 = \frac{1}{3}t\sigma_p^2$ to $\sigma^2 = t\sigma_p^2$ and $\sigma^2 = 3t\sigma_p^2$ respectively.

Returning to the option cost the following result follows from (1) and Theorem 4.

THEOREM 5. *As $n \to \infty$, we have*

$$K = \exp(-rt)E\big(\max(S(0)\exp W^* - X, 0)\big)$$

where $W^* \sim N(\mu, \sigma^2)$

Using standard formulas for the normal distribution, we find that

$$K = S(0)\exp(-rt + \mu + \sigma^2/2)\Phi(w) - \exp(-rt)X\Phi(w-\sigma)$$

where

$$w = \frac{\sigma^2 + \mu - \log(X/S(0))}{\sigma}$$

and where $\Phi(w)$ is the standard normal distribution function.

REMARKS. 1) If the parameters are chosen in such a way that $rt = \mu + \sigma^2/2$, we find that

$$K = S(0)\Phi(w) - \exp(-rt)X\Phi(w-\sigma)$$

which is similar to the classical Black and Scholes formula.

2) As a special case we consider the case where $P = P(p, \rho)$. Now we assume that

$$p = A + B\frac{r_* - d}{u - d}$$

where $r_* = \exp(\alpha t/n)$. Using $p_{0,1} = p(1-\rho)$ and $p_{1,1} = \rho + (1-\rho)p$ and the previous analysis can be used. In this case we have $\alpha = \beta$ and

$$y^* = A + B\frac{b}{a+b}$$

We have to assume that $y^* = b/(a+b)$. Using the notations as in the proof of Theorem 4, we find that $K(1) = B(1-\rho)(\alpha - ab/2)$. Now we find that $\mu = tB(\alpha - ab/2)$ and that $\sigma^2 = tab(1+\rho)/(1-\rho)$. A convenient choice seems to be $A = 0$ and $B = 1$.

Corollary 6. *If $P = P(p, \rho)$ and $p = (r_* - u)/(u - d)$, then Theorem 4 holds with $\mu = t(\alpha - ab/2)$ and σ^2 as before.*

4. Final remarks

1) A correlated binomial distribution has been introduced and studied by Madsen [1993], Altham [1978], Kupper and Haseman [1978], Mingoti [2003]. Examples and applications can be found e.g., in quality control, Lai et al. [1998]. See also Edwards [1960], Wang [1981].

2) Many stochastic processes are based on a counting process $\{N(t), t \geqslant 0\}$, where $N(t)$ denotes the number of times a certain event occurs in the time interval $(0, t]$. In many processes one models $N(t)$ with a Poisson, binomial or negative binomial distributions. In Minkova [1999, 2001], Dimitrov and Kolev [1999], the authors study inflated processes by introducing an additional parameter ρ. We introduce this process by using another approach as follows. For fixed n let $S_n \sim BIN(n, p)$ and for fixed ρ let $W(\rho)$ denote a geometric distribution. The generating function of S_n is given by $(1 - p + pz)^n$ and the generating function of $W(\rho)$ is given by $K(z) = (1 - \rho)z/(1 - \rho z)$. We define a new random variable N by defining its generating functions: $E(z^N) = (1 - p + pK(z))^n$. The r.v. N is said to have an inflated-binomial distribution with parameters p, n and ρ; notation $N \sim IBIN(n, p, \rho)$. In the context of stochastic processes, Minkova [2001] studied $N(t)$ where $N(t) \sim IBIN(n, t/\alpha, \rho)$. It could be of interest to use this type of inflated-binomial in the context of the formula of Black and Scholes.

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