PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 75(89) (2004), 253–260

# MÖBIUS TRANSFORMATIONS AND MULTIPLICATIVE REPRESENTATIONS FOR SPHERICAL POTENTIALS

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Abstract. For the unit spheres $S^n \subset \mathbf{R}^{n+1}$ and $S^{2n-1} \subset \mathbf{R}^{2n} = \mathbf{C}^n$ we prove the following identities for two classical potentials

$$\int_{S^n} \frac{f(y)}{|x-y|^{n+\alpha}} d\sigma_y = \frac{1}{|1-|x|^2|^\alpha} \int_{S^n} \frac{f(T_{n,x}(y))}{|x-y|^{n-\alpha}} d\sigma_y,$$

$$\int_{S^{2n-1}} \frac{F(\zeta) d\sigma_\zeta}{|1-(z,\zeta)|^{n+\alpha}} = \frac{1}{(1-|z|^2)^\alpha} \int_{S^{2n-1}} \frac{F(\Phi_{n,z}(\zeta)) d\sigma_\zeta}{|1-(z,\zeta)|^{n-\alpha}},$$

where $x \in \mathbf{R}^{n+1}$ ($|x| \neq 0$ and $|x| \neq 1$), $z \in \mathbf{C}^n$ ($|z| < 1$), $T_{n,x}$ and $\Phi_{n,z}$ are explicit involutions of $S^n$ and $S^{2n-1}$ respectively. Some applications of these formulas are also considered.

## 1. Introduction

The aim of this paper is to present a new approach to study boundary behavior of classical potentials using Möbius transformations in two and several dimensions.

We consider two spherical potentials in the spaces $\mathbf{R}^{n+1}$ and $\mathbf{C}^n$ for $n \geqslant 1$. The first one is the Riesz potential

$$P_{n,\alpha}(x, f) = \int_{S^n} \frac{f(y)}{|x-y|^{n+\alpha}} d\sigma_y \tag{1}$$

of the sphere $S^n = \{y \in \mathbf{R}^{n+1} : |y| = 1\}$ in $\mathbf{R}^{n+1}$ for $|x| \neq 1$, and the second is the complex potential

$$Q_{n,\alpha}(z, F) = \int_{S^{2n-1}} \frac{F(\zeta)}{|1-(z,\zeta)|^{n+\alpha}} d\sigma_\zeta \tag{2}$$

of the sphere $S^{2n-1} = \{\zeta \in \mathbf{C}^n : |\zeta| = 1\}$ in $\mathbf{C}^n$ for $|z| < 1$. In (1) and (2) $d\sigma_y$ and $d\sigma_\zeta$ denote the differential elements of surface area of the spheres $S^n \subset \mathbf{R}^{n+1}$ and $S^{2n-1} \subset \mathbf{R}^{2n}$, respectively, and $(z, \zeta) = z_1\overline{\zeta}_1 + z_2\overline{\zeta}_2 + \cdots + z_n\overline{\zeta}_n$ is the scalar product in $\mathbf{C}^n$.

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2000 *Mathematics Subject Classification*: Primary 31B25; Secondary 30C65.

This research was supported by a grant of the Deutsche Forschungsgemeinschaft and by Russian Fund of Basic Research (Grant 02-01-00168).

It is very well known that the classical methods use some special additive representations of (1) and (2) to study their boundary behavior (see, for instance, [**1**], [**3**], [**5**], [**6**]). We will give new formulas to find the singularities of spherical potentials in the case, when $\alpha$ is a complex number such that $\operatorname{Re}\alpha > 0$.

Namely, for (1) and (2) we obtain multiplicative representations which explicitly give the principal singularities of these potentials near the spheres $S^n$ and $S^{2n-1}$ respectively. Moreover, we apply the multiplicative representations to find sharp estimates for the functions

$$|1-|x|^2|^\beta P_{n,\alpha}(x,f) \quad \text{and} \quad |1-|z|^2|^\beta Q_{n,\alpha}(z,F)$$

when $\beta \geqslant \operatorname{Re}\alpha$ and the densities $f$ and $F$ belong to $L^q$ with $q>1$. We also show that the multiplicative representations may be used to prove Fatou type theorems.

The paper is organized as follows. In Section 2 the Riesz potential $P_{n,\alpha}$ is considered. In Section 3 we study the complex potential (2) in some details. It is clear that $P_{1,\alpha}(x,f) \equiv Q_{1,\alpha}(z,F)$ for $f=F$ and $x := (x_1,x_2)$, $z := (x_1+ix_2)$, but $Q_{n,\alpha}(z,F)$ does not reduce to $P_{2n-1,\alpha}(x,f)$ for $n \geqslant 2$.

## 2. Riesz spherical potentials

We intend to transform integral (1) by a change of variables using Möbius transformations. Consider first the trivial case $n=0$. We can take $S^0 = \{-1,1\}$ and

$$P_{0,\alpha}(x,f) := \frac{f(-1)}{|x+1|^\alpha} + \frac{f(1)}{|x-1|^\alpha}, \quad x \in \mathbf{R} \setminus S^0,$$

for any function $f : S^0 \to \mathbf{C}$. If $T_0 : S^0 \to S^0$ is involute, i.e., $T_0(1) = -1$, $T_0(-1) = 1$, then the following identity

$$P_{0,\alpha}(x,f) = \frac{|x-1|^\alpha f(-1) + |x+1|^\alpha f(1)}{|1-x^2|^\alpha} = \frac{1}{|1-x^2|^\alpha} P_{0,-\alpha}(x, f \circ T_0)$$

is valid in $\mathbf{R} \setminus S^0$. Surprisingly, this elementary formula has a direct extension to the case $n \geqslant 1$.

For $n \geqslant 1$ and every fixed $x \in \mathbf{R}^{n+1} \setminus S^n$, $|x| \neq 0$, we will consider the following Möbius transformations of $\mathbf{R}^{n+1}$

$$T_{n,x}(y) = \begin{cases} x + \dfrac{(|x|^2-1)(y-x)}{|y-x|^2}, & \text{if } |x|>1, \\ \dfrac{x}{|x|^2} + \dfrac{(|x|^{-2}-1)(y-x/|x|^2)}{|y-x/|x|^2|^2}, & \text{if } 0<|x|<1. \end{cases} \tag{3}$$

For fixed $x$ the transformation $T_{n,x}$ is a conformal automorphism of the unit ball $B_{n+1} := \{y \in \mathbf{R}^{n+1} : |y| \leqslant 1\}$ (see [**1**]) and the restriction $T_{n,x} \mid S^n$ presents the standard inversion of $S^n$ about the sphere $S_x^{n-1} = \{y \in S^n : |y-x| = \sqrt{|1-|x|^2|}\}$.

THEOREM 1. *Suppose that $n \geqslant 1$ and $f \in L^1(S^n)$. For any $\alpha \in \mathbf{C}$ and for all $x \in \mathbf{R}^{n+1} \setminus S^n$, $|x| \neq 0$, the following identity is valid*

$$\int_{S^n} \frac{f(y)}{|x-y|^{n+\alpha}} d\sigma_y = \frac{1}{|1-|x|^2|^\alpha} \int_{S^n} \frac{f(T_{n,x}(y))}{|x-y|^{n-\alpha}} d\sigma_y \tag{4}$$

PROOF. Let $x \in \mathbf{R}^{n+1} \setminus S^n$, $|x| \neq 0$. To simplify computations it is convenient to use a new orthonormed basis $(e_1, e_2, \dots, e_{n+1})$ obtained by a rotation of $\mathbf{R}^{n+1}$ about the origine and such that $x = |x| e_1$.

Suppose that

$$y = \sum_{k=1}^{n+1} y_k e_k \qquad \text{and} \qquad u = T_{n,x}(y) = \sum_{k=1}^{n+1} u_k e_k.$$

Straightforward computations using (3) give

$$u_1 = T_{1,|x|}(y_1) := \frac{2|x| - (1+|x|^2) y_1}{1 + |x|^2 - 2|x| y_1} \tag{5}$$

and

$$u_k = \frac{|1 - |x|^2|}{1 + |x|^2 - 2|x| y_1} y_k = \sqrt{\frac{1 - u_1^2}{1 - y_1^2}} y_k, \quad 2 \leqslant k \leqslant n+1, \tag{6}$$

in both cases: $|x| > 1$ or $0 < |x| < 1$. To deduce the second equalities for $u_k$ in (6) we used the following consequence of (5):

$$1 - u_1^2 = \frac{(1 - |x|^2)^2}{(1 + |x|^2 - 2|x| y_1)^2} (1 - y_1^2). \tag{7}$$

Moreover, equality (5) implies that $y_1 = T_{1,|x|}(u_1)$, hence

$$1 - y_1^2 = \frac{(1 - |x|^2)^2}{(1 + |x|^2 - 2|x| u_1)^2} (1 - u_1^2). \tag{8}$$

Using (5) and (6) we also obtain that $u = T_{n,x}(y) \in S^n$ for any $y \in S^n$ and $T_{n,x} \,|\, S^n$ is an involution of $S^n$.

From (7) and (8) it follows that

$$(1 + |x|^2 - 2|x| y_1)(1 + |x|^2 - 2|x| u_1) = (1 - |x|^2)^2$$

which is equivalent to the equality

$$|x - u| \cdot |x - y| = |1 - |x|^2| \tag{9}$$

for any $y \in S^n$ and $u = T_{n,x}(y)$.

Thus,

$$\int_{S^n} \frac{f(u)}{|x - u|^{n+\alpha}} d\sigma_u = \frac{1}{|1 - |x|^2|^{n+\alpha}} \int_{S^n} f(T_{n,x}(y)) |x - y|^{n+\alpha} I(y) d\sigma_y, \tag{10}$$

where $I(y) = d\sigma_u / d\sigma_y$ $(u = T_{n,x}(y))$ is the Jacobian of the map $T_{n,x} \,|\, S^n$. To compute $I(y)$ we consider a diffeomorphism $K : B_{n+1} \to B_{n+1}$ defined by

$$(K \,|\, S^n)(\xi) = (T_{n,x} \,|\, S^n)(\xi) \quad \text{for} \quad \xi \in S^n$$

and

$$v = K(\xi) = \sum_{k=1}^{n+1} v_k e_k \quad \text{for} \quad |\xi| < 1,$$

where

$$v_1 = T_{1,|x|}(\xi_1), \quad v_k = \sqrt{\frac{1-v_1^2}{1-\xi_1^2}}\xi_k \quad \text{for} \quad 2 \leqslant k \leqslant n+1. \tag{11}$$

For any $\xi \in S^n$ and $v = K(\xi)$ one has

$$I(y) = \lim_{\substack{\xi\to y\\ |\xi|<1}} \frac{1-|\xi|}{1-|v|}\left|\det\left(\frac{\partial v_k}{\partial \xi_j}\right)_{1\leqslant j,k\leqslant n+1}\right| \lim_{\substack{\xi\to y\\ |\xi|<1}} \frac{1-|\xi|^2}{1-|v|^2}\left|\det\left(\frac{\partial v_k}{\partial \xi_j}\right)_{1\leqslant j,k\leqslant n+1}\right|.$$

Since

$$\frac{\partial v_1}{\partial \xi_1} = -\frac{1-v_1^2}{1-\xi_1^2}, \quad \frac{\partial v_k}{\partial \xi_k} = \sqrt{\frac{1-v_1^2}{1-\xi_1^2}} \quad \text{for} \quad k \geqslant 2$$

and

$$\frac{\partial v_k}{\partial \xi_j} = 0 \quad \text{for} \quad k \geqslant 1 \quad \text{and} \quad j > k,$$

we have

$$I(y) = \lim_{\substack{\xi\to y\\ |\xi|<1}} \frac{1-|\xi|^2}{1-|v|^2}\left(\frac{1-v_1^2}{1-\xi_1^2}\right)^{1+n/2}.$$

From (11) it follows that

$$1-|v|^2 = \frac{1-v_1^2}{1-\xi_1^2}(1-|\xi|^2).$$

Using this and the formula (8) for $v = T_{n,x}(y) = K(y) \in S^n$ we obtain

$$I(y) = \frac{|1-|x|^2|^n}{|x-y|^{2n}}, \quad y \in S^n. \tag{12}$$

Formulas (10) and (12) imply (4). Thus, the proof of Theorem 1 is complete. □

COROLLARY 1.1. *Let $F \in L^q(S^n)$, $q > 1$. If $\beta = \operatorname{Re}\alpha + n/q > 0$ then for any fixed $x \in \mathbf{R}^{n+1} \setminus S^n$*

$$\sup_{\|f\|_q=1}\left|\int_{S^n} \frac{|1-|x|^2|^\beta f(y)}{|x-y|^{n+\alpha}} d\sigma_y\right| = \left(\int_{S^n} \frac{d\sigma_y}{|x-y|^{n-\beta t}}\right)^{1/t}, \tag{13}$$

*where $t = (q-1)/q < 1$ and*

$$\|f\|_q = \left(\int_{S^n} |f(y)|^q d\sigma_y\right)^{1/q}.$$

PROOF. According to Hölder's inequality

$$\sup_{\|f\|_q=1} |P_{n,\alpha}(x,f)| = P_{n,\beta t}^{1/t}(x,1). \tag{14}$$

Applying Theorem 1 we obtain

$$P_{n,\beta t}(x,1) = \frac{1}{|1-|x|^2|^{\beta t}} P_{n,-\beta t}(x,1). \tag{15}$$

Equalities (14) and (15) imply (13). □

By virtue of well-known properties of Riesz potentials the integral $P_{n,-\beta t}(x,1)$ depends on $|x|$ only and has three critical points that are $|x|=0$, $|x|=1$ and $|x|=\infty$. Compare $P_{n,-\beta t}(0,1)$, $P_{n,-\beta t}(1,1)$ and $P_{n,-\beta t}(\infty,1)$ one may compute its maximum and minimum for $0\leqslant |x|\leqslant 1$ or $1\leqslant |x|\leqslant\infty$. In particular, if $n\geqslant 2$, $0\leqslant n-\beta t\leqslant n-1$, then $P_{n,-\beta t}(0,1)\geqslant \max\{P_{n,-\beta t}(1,1),P_{n,-\beta t}(\infty,1)\}$.

Consequently, (13) implies the sharp estimate

$$|1-|x|^2|^\beta |P_{n,\alpha}(x,f)|\leqslant \sigma_n^{1/t}\|f\|_q,\quad \forall x\in \mathbf{R}^{n+1}\setminus S^n, \tag{16}$$

where $\sigma_n=\dfrac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$ is "the surface area" of $S^n$ in $\mathbf{R}^{n+1}$. Equality in (16) occurs for $|x|=0$ and $f(y)\equiv$ const.

Using classical methods for Poisson's integrals (see, for instance, [**1**]) one may prove the following *Fatou's theorem for* $P_{n,\alpha}(x,f)$ *in the case* $\operatorname{Re}\alpha>0$ *and* $f\in L^1(S^n)$ : *for almost all* $\xi\in S^n$

$$\lim_{\substack{x\to\xi\\ |x-\xi|<M(1-|x|)}} |1-|x|^2|^\alpha P_{n,\alpha}(x,f)=2^\alpha\pi^{n/2}\frac{\Gamma(\alpha/2)}{\Gamma((\alpha+n)/2)}f(\xi,) \tag{17}$$

where $M$ is a positive constant.

In the next Corollary 1.2 we examine (17) for a particular case when (17) is a simple consequence of Theorem 1 and a property of $T_{n,x}$.

COROLLARY 1.2. *If* $\operatorname{Re}\alpha>0$, $f\in L^\infty(S^n)$ *and* $f$ *is continuons at the point* $\xi\in S^n$, *then*

$$\lim_{\substack{x\to\xi\\ |x|\neq 1}} |1-|x|^2|^\alpha P_{n,\alpha}(x,f)=2^\alpha\pi^{n/2}\frac{\Gamma(\alpha/2)}{\Gamma((\alpha+n)/2)}f(\xi).$$

PROOF. According to Theorem 1 we have to prove that

$$\lim_{\substack{x\to\xi\\ |x|\neq 1}}\int_{S^n}\frac{f(T_{n,x}(y))}{|x-y|^{n-\alpha}}d\sigma_y=f(\xi)\int_{S^n}\frac{d\sigma_y}{|\xi-y|^{n-\alpha}}=\frac{\Gamma(\alpha/2)}{\Gamma((\alpha+n)/2)}f(\xi),$$

which is equivalent to

$$A(x,\xi)=\int_{S^n}\frac{f(T_{n,x}(y))-f(\xi)}{|x-y|^{n-\alpha}}d\sigma_y\to 0\quad\text{as } x\to\xi,\ |x|\neq 1.$$

Since Hölder's inequality on can write

$$|A(x,\xi)|\leqslant C\left(\int_{S^n}|f(T_{n,x}(y))-f(\xi)|^q d\sigma_y\right)^{1/q},$$

where $C$ is a constant.

From (3) it follows that

$$\lim_{\substack{x\to\xi\\ |x|\neq 1}} T_{n,x}(y)=\xi,\quad \forall y\in S^n\setminus\{\xi\}.$$

Consequently, $f(T_{n,x}(y))\to f(\xi)$ as $x\to\xi$, $|x|\neq 1$ for any $y\in S^n\setminus\{\xi\}$ and $\|f\circ T_{n,x}-f(\xi)\|\to 0$ as $x\to\xi$, $|x|\neq 1$ by Lebesgue's theorem on the majorized convergence. This completes the proof of Corollary 1.2. □

The function $P_{1,\alpha}(r,1)$ is used in many problems related to the spaces of functions analytic in the unit disk. We add to known results (see [**2**], [**4**]) the following assertion. We will need the beta function

$$B\Big(\frac{1}{2},\frac{\alpha}{2}\Big)=\frac{\sqrt{\pi}\,\Gamma(\alpha/2)}{\Gamma((\alpha+1)/2)}.$$

COROLLARY 1.3. *If* $0\leqslant r<1$, $\alpha>0$ *and* $\alpha\neq 1$ *then*

$$\frac{2\pi}{(1-r^2)^\alpha}\leqslant\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^{1+\alpha}}<\frac{2^\alpha B(1/2,\alpha/2)}{(1-r^2)^\alpha}.$$

*Equality in the left-hand side inequality occurs if and only if* $r=0$. *The right-hand side inequality is sharp asymptotically as* $r\to 1-0$.

PROOF. By Theorem 1

$$\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^{1+\alpha}}=\frac{1}{(1-r^2)^\alpha}\int_0^{2\pi}\frac{d\theta}{|1-re^{i\theta}|^{1-\alpha}}.$$

According to Hardy's theorem $P_{1,-\alpha}(.,1)$ is an increasing function in $[0,1)$ if $\alpha\neq 1$. Consequently, for any $r\in(0,1)$, $\alpha>0$ and $\alpha\neq1$

$$(1-r^2)^\alpha P_{1,\alpha}(r,1)>P_{1,-\alpha}(0,1)=2\pi,$$

$$(1-r^2)^\alpha P_{1,\alpha}(r,1)<\lim_{r\to1-0}(1-r^2)^\alpha P_{1,\alpha}(r,1)=P_{1,-\alpha}(1,1)=2^\alpha B(1/2,\alpha/2).$$

Two last formulas complete the proof of Corollary 1.3. □

## 3. The potential of $S^{2n-1}$ in $\mathbf{C}^n$

Let $B$ be the unit ball $\{\zeta\in\mathbf{C}^n : |\zeta|<1\}$, $\partial B=S^{2n-1}$. For fixed $z\in B\setminus\{0\}$ we consider the biholomorphic map $\Phi_{n,z}$ of $B$ onto $B$ defined as follows (see [**5**]):

$$\Phi_{n,z}(\zeta)=\frac{z-p_z(\zeta)-\sqrt{1-|z|^2}(\zeta-p_z(\zeta))}{1-(\zeta,z)},\ \ |\zeta|\leqslant1,$$

where

$$p_z(\zeta)=\frac{z}{|z|^2}(\zeta,z).$$

It is known (see [**5**]) that

(i) $\Phi_{n,z}$ is an involution, i.e., $\Phi_{n,z}(\Phi_{n,z}(\zeta))=\zeta$ for any $\zeta\in\overline{B}$;

(ii) $\Phi_{n,z}$ satisfies the conditions

$$\Phi_{n,z}(z)=0,\Phi_{n,z}(z/|z|)=-z/|z|,$$

$$\Phi_{n,z}(\zeta)\in S^{2n-1}\quad\text{and}\quad\Phi_{n,z}(\zeta)\neq\zeta\quad\text{for any}\quad\zeta\in S^{2n-1};$$

(iii) $\Phi_{n,z}\,|\,S^{2n-1}:S^{2n-1}\to S^{2n-1}$ is a diffeomorphism;

(iv) there is the identity

$$1-(\Phi_{n,z}(\zeta),\Phi_{n,z}(w))=\frac{(1-|z|^2)(1-(\zeta,w))}{(1-(\zeta,z))(1-(z,w))}.$$

For $Q_{n,\alpha}(z,F)$ we have the following analog of Theorem 1. Note that the assertion of Theorem 2 is known in the case $\alpha=n$ (see [**5**, Chapter 1]).

THEOREM 2. *Suppose that* $\alpha \in \mathbf{C}$, $F \in L^1(S^{2n-1})$. *For any* $z \in B \setminus \{0\}$ *the following identity is valid:*

$$\int_{S^{2n-1}} \frac{F(\zeta)d\sigma_\zeta}{|1-(z,\zeta)|^{n+\alpha}} = \frac{1}{(1-|z|^2)^\alpha} \int_{S^{2n-1}} \frac{F(\Phi_{n,z}(\zeta))d\sigma_\zeta}{|1-(z,\zeta)|^{n-\alpha}}, \tag{18}$$

*where* $S^{2n-1} = \partial B = \{\zeta \in \mathbf{C}^n : |\zeta| = 1\}$.

PROOF. Let $z \in B \setminus \{0\}$. Taking $w = \Phi_{n,z}(\zeta)$, $\zeta \in S^{2n-1}$, we have

$$\int_{S^{2n-1}} \frac{F(w)d\sigma_w}{|1-(z,w)|^{n+\alpha}} = \int_{S^{2n-1}} \frac{F(\Phi_{n,z}(\zeta))I(\zeta)d\sigma_\zeta}{|1-(z,\Phi_{n,z}(\zeta))|^{n+\alpha}}. \tag{19}$$

From the properties (i), (ii) and (iv) we have $\zeta = \Phi_{n,z}(w)$ and

$$1-(w,\zeta) = \frac{(1-|z|^2)(1-(\zeta,w))}{(1-(\zeta,z))(1-(z,w))}, \quad (\zeta,w) \neq 1.$$

Consequently, for any $\zeta \in S^{2n-1}$ and $w = \Phi_{n,z}(\zeta)$

$$|1-(z,w)| \cdot |1-(z,\zeta)| = 1-|z|^2 \tag{20}$$

According to Theorem 3.3.8 in [**5**] the Jacobian

$$I(\zeta) = \frac{d\sigma_w}{d\sigma_\zeta} = \frac{(1-|z|^2)^n}{|1-(z,\zeta)|^{2n}}. \tag{21}$$

From (19), (20) and (21) we have (18) immediately. The proof of Theorem 2 is complete. □

In [**5**, Proposition 1.4.10], for $-\dfrac{n+\alpha}{2} \notin \mathbf{N}$ it is proved that

$$Q_{n,\alpha}(z,1) = \frac{\sigma_{2n-1}\Gamma(n)}{\Gamma^2((n+\alpha)/2)} \sum_{k=0}^{\infty} \frac{\Gamma^2(k+(n+\alpha)/2)}{\Gamma(k+1)\Gamma(k+n)} |z|^{2k} \tag{22}$$

and that

$$Q_{n,\alpha}(z,1) \approx (1-|z|^2)^{-\alpha} \quad \text{for} \quad \alpha > 0. \tag{23}$$

It is to note that

$$\sigma_{2n-1} = \int_{S^{2n-1}} d\sigma_\zeta = \frac{2\pi^n}{\Gamma(n)}$$

is "the surface area" of $S^{2n-1}$ in $\mathbf{R}^{2n}$, and in [**5**] the normalized measure

$$d\sigma(\zeta) = d\sigma_\zeta / \sigma_{2n-1}$$

is considered. Hence, $Q_{n,\alpha}(z,1)/\sigma_{2n-1}$ is $I_c(z)$ from [**5**, Chapter 1], with $c = \alpha$.

Using Theorem 2 and the series (22) we get a refined version of (23).

COROLLARY 2.1. *If* $\alpha > 0$, $z \in B_n$ *and* $F \in L^\infty(S^{2n-1})$ *then*

$$\left| \int_{S^{2n-1}} \frac{F(\zeta)d\sigma_\zeta}{|1-(z,\zeta)|^{n+\alpha}} \right| \leqslant \frac{2\pi^n\Gamma(\alpha)}{\Gamma^2((n+\alpha)/2)} \frac{\|F\|_\infty}{(1-|z|^2)^\alpha}, \tag{24}$$

*where* $\|F\|_\infty = \sup\{|F(\zeta)| : \zeta \in S^{2n-1}\}$. *If* $F(\zeta) = \text{const.} \neq 0$ *then the inequality is asymptotically sharp as* $|z| \to 1-0$.

PROOF. Using Theorem 2 and the series (22) one has

$$\sup_{\|F\|_\infty=1} |Q_{n,\alpha}(z,F)|(1-|z|^2)^\alpha = Q_{n,-\alpha}(|z|,1) = \sigma_{2n-1} F\Big(\frac{n-\alpha}{2}, \frac{n-\alpha}{2}; n, |z|^2\Big), \tag{25}$$

where $F(a,b;c;|z|^2)$ is the hypergeometric function.

Since $c-a-b=\alpha>0$, we have by Gauss' formula

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

Taking $c=n$, $a=b=(n-\alpha)/2$ and letting $|z|\to 1-0$ we obtain

$$Q_{n,-\alpha}(1,1) = \frac{2\pi^n\Gamma(\alpha)}{\Gamma^2((n+\alpha)/2)} \tag{26}$$

(see another proof of (26) in [**5**, Theorem 4.2.7]).

The equalities (25), (26) and the following consequence of (22)

$$Q_{n,-\alpha}(|z|,1) \leqslant \lim_{|z|\to 1-0} Q_{n,-\alpha}(|z|,1) = Q_{n,-\alpha}(1,1)$$

imply (24) and the asymptotic equality

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha Q_{n,\alpha}(z,1) = \frac{2\pi^n\Gamma(\alpha)}{\Gamma^2((n+\alpha)/2)}.$$

These complete the proof of Corollary 2.1. □

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(Received 16 10 2003)