

## UNIVALENT HARMONIC MAPPINGS OF ANNULI

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*Dedicated to the memory of Professor Walter Hengartner*

ABSTRACT. This paper is a survey of the author's recent results on univalent harmonic mappings of annuli.

### 1. Introduction

A *harmonic mapping*  $f$  of a region  $\mathcal{D}$  is a complex-valued function of the form  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic functions in  $\mathcal{D}$ , unique up to an additive constant, that are single-valued if  $\mathcal{D}$  is simply connected and possibly multiple-valued otherwise. We call  $h$  and  $g$  the *analytic* and *co-analytic* parts of  $f$ , respectively. If  $f$  is (locally) injective, then  $f$  is called *(locally) univalent*. Note that every conformal and anti-conformal function is a univalent harmonic mapping. The *Jacobian* and *second complex dilatation* of  $f$  are given by the functions  $J(z) = |h'(z)|^2 - |g'(z)|^2$  and  $\omega(z) = g'(z)/h'(z)$ ,  $z \in \mathcal{D}$ , respectively. Note that  $\omega$  is either a nonconstant meromorphic function or a (possibly infinite) constant. A result of Lewy [10] states that if  $f$  is a locally univalent mapping, then its Jacobian  $J$  is never zero; namely, for  $z \in \mathcal{D}$ , either  $J(z) > 0$  or  $J(z) < 0$ . In the first case  $|\omega(z)| < 1$  and  $f$  is sense-preserving, and in the second  $|\omega(z)| > 1$  and  $f$  is sense-reversing.

Throughout the paper we shall use the following notation:  $\mathbb{C}$  for the complex plane,  $\widehat{\mathbb{C}}$  for the extended complex plane,  $\mathbb{D}$  for the open unit disc  $\{z \in \mathbb{C} : |z| < 1\}$ ,  $\mathbb{T}$  for the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ ,  $0 < \rho < 1$ ,  $\mathbb{T}_\rho$  for the circle  $\{z \in \mathbb{C} : |z| = \rho\}$ ,  $\mathbb{A}(\rho, 1)$  for the annulus  $\{z \in \mathbb{C} : \rho < |z| < 1\}$ ,  $\mathbb{G}$  for a bounded convex domain unless otherwise is specified, and  $\partial S$  and  $\bar{S}$ ,  $S \subset \mathbb{C}$ , for the boundary and closure of  $S$  respectively. We shall call the *diameter* of  $S$  the least upper bound of the Euclidean distances between any two points of  $S$ , a Jordan curve *convex* if it is the boundary of a bounded convex domain, and a *ring domain* is a doubly-connected open subset of the plane. We shall need the notion of the *module* of a ring domain [16]. It is known that a ring domain  $R$  is conformally equivalent to a unique annulus

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$A(\rho, 1)$ . The module of  $R$ , denoted by  $M(R)$ , is defined by  $\log(1/\rho)$  if  $\rho \neq 0$  and by  $\infty$  if  $\rho = 0$ . It is known that  $M$  is a conformal invariant and that if  $R \subset R'$ , where  $R'$  is also a ring domain, then  $M(R) \leq M(R')$  with equality if and only if  $R = R'$ . The *Grötzsch ring domain*,  $B(s)$ ,  $0 < s < 1$ , is the ring domain whose boundary components are  $\mathbb{T}$  and the segment  $\{x : 0 \leq x \leq s\}$ . Observe that  $B(s)$  is unique. The module of  $B(s)$  is usually denoted by  $\mu(s)$ . Thus  $\mu(s) = \log(1/\rho)$ . It is known that  $\mu$  is a strictly decreasing function of  $[0, 1)$ .

The purpose of this article is to survey the author's recent results on harmonic univalent mappings of annuli.

## 2. Boundary Functions

The boundary functions of univalent harmonic mappings onto punctured convex domains are characterized by the following notion introduced by Bshouty, Hengartner and Naghibi-Beidokhti [3].

DEFINITION 2.1. Let  $f$  be a function of  $\mathbb{T}$  into a Jordan curve  $C$  of  $\mathbb{C}$ . We say  $f$  is a sense-preserving quasihomomorphism of  $\mathbb{T}$  into  $C$  if it is a pointwise limit of a sequence of sense-preserving homeomorphisms of  $\mathbb{T}$  onto  $C$ . If in addition  $f$  is a continuous function onto  $C$ , then  $f$  is called a *sense-preserving weak homeomorphism*.

Sense-preserving quasihomomorphisms and sense-preserving weak homeomorphisms are characterized as follows [12].

PROPOSITION 2.1. Let  $f$  be a function of  $\mathbb{T}$  into a Jordan curve  $C$ , and let  $F$  be a sense-preserving homeomorphism of  $\mathbb{T}$  onto  $C$ .

- (i) If  $f$  is a sense-preserving quasihomomorphism of  $\mathbb{T}$  onto  $C$ , then there is a real-valued nondecreasing function  $\varphi$  on  $\mathbb{R}$  such that  $\varphi(t+2\pi) = \varphi(t) + 2\pi$  and  $f(e^{it}) = F(e^{i\varphi(t)})$ .
- (ii) If  $f(e^{it}) = F(e^{i\varphi(t)})$ , where  $\varphi$  is a real-valued nondecreasing function on  $\mathbb{R}$  such that  $\varphi(t+2\pi) = \varphi(t) + 2\pi$ , and if  $E$  is the countable set of points  $e^{i\varphi(t)}$  where  $\varphi$  is discontinuous, then  $f$  coincides on  $\mathbb{T} \setminus E$  with a sense-preserving quasihomomorphism of  $\mathbb{T}$ . In this case,  $f$  is the pointwise limit in  $\mathbb{T} \setminus E$  of a sequence of sense-preserving homeomorphisms  $f_n(e^{it}) = F(e^{i\varphi_n(t)})$  of  $\mathbb{T}$  onto  $C$ , where each  $\varphi_n$  is a real-valued infinite differentiable function on  $\mathbb{R}$  such that  $\varphi_n(t+2\pi) = \varphi_n(t) + 2\pi$  and  $\varphi_n'(t)$  is always positive.
- (iii)  $f$  is a sense-preserving weak homeomorphism of  $\mathbb{T}$  onto  $C$  if and only if there is a real-valued continuous nondecreasing function  $\varphi$  on  $\mathbb{R}$  such that  $\varphi(t+2\pi) = \varphi(t) + 2\pi$  and  $f(e^{it}) = F(e^{i\varphi(t)})$ . In this case,  $f$  is the uniform limit of a sequence of sense-preserving homeomorphisms  $f_n(e^{it}) = F(e^{i\varphi_n(t)})$  of  $\mathbb{T}$  onto  $C$ , where each  $\{\varphi_n\}$  is a real-valued infinite differentiable function on  $\mathbb{R}$  such that  $\varphi_n(t+2\pi) = \varphi_n(t) + 2\pi$  and  $\varphi_n'(t)$  is always positive.

Let  $f$  be a function of  $\mathbb{A}(\rho, 1)$  into  $\widehat{\mathbb{C}}$ , and let  $\xi \in \mathbb{T}$ . We say that  $f$  has the *unrestricted limit*  $a \in \widehat{\mathbb{C}}$  at  $\xi$  if

$$f(z) \rightarrow a \quad z \rightarrow \xi, \quad z \in \mathbb{A}(\rho, 1);$$

by defining  $f(\xi) = a$  the function  $f$  becomes continuous at  $\xi$  as a function in  $\mathbb{A}(\rho, 1) \cup \{\xi\}$ . We shall use  $f(\xi)$  to denote the unrestricted limit whenever it exists, and call the resulting function, on its domain of definition in  $\mathbb{T}$ , the *unrestricted limit function*  $f$ . We also define the *cluster set*  $C(f, \xi)$  of  $f$  at  $\xi$  as the set of all  $b \in \widehat{\mathbb{C}}$  for which there are sequences  $\{z_n\}$  such that

$$z_n \in \mathbb{A}(\rho, 1), \quad z_n \rightarrow \xi, \quad f(z_n) \rightarrow b \quad \text{as } n \rightarrow \infty.$$

Moreover, if  $F$  is a subset of  $\mathbb{T}$ , then we define the *cluster set*  $C(f, F)$  of  $f$  at  $F$  as the set-union of the cluster sets  $C(f, \xi)$  for  $\xi \in F$ .

Sense-preserving quasihomeomorphisms are essential for describing the boundary behavior of univalent harmonic mappings of ring domains onto bounded convex domains. Suppose  $f$  is a univalent harmonic mapping of  $\mathbb{A}(\rho, 1)$  onto a ring domain  $\mathbb{G} \setminus \{\zeta\}$ ,  $\zeta \in \mathbb{G}$ . Then either  $\lim_{|z| \uparrow 1} f(z) = \zeta$  and  $C(f, \mathbb{T}_\rho) = \partial\mathbb{G}$ , or  $\lim_{|z| \downarrow \rho} f(z) = \zeta$  and  $C(f, \mathbb{T}) = \partial\mathbb{G}$ . In the first case,  $f(\rho/z)$  becomes a univalent harmonic mapping of  $\mathbb{A}(\rho, 1)$  onto  $\mathbb{G} \setminus \{\zeta\}$  with  $\lim_{|z| \downarrow \rho} f(\rho/z) = \zeta$  and  $C(f(\rho/z), \mathbb{T}) = \partial\mathbb{G}$ . This leads us to consider, without loss of generality, only univalent harmonic mappings of  $\mathbb{A}(\rho, 1)$  onto ring domains  $\mathbb{G} \setminus \{\zeta\}$ ,  $\zeta \in \mathbb{G}$ , with  $\lim_{|z| \downarrow \rho} f(z) = \zeta$ .

DEFINITION 2.2. Let  $\mathcal{H}_u(\rho, \mathbb{G})$  be the class of all univalent harmonic mappings  $f$  of  $\mathbb{A}(\rho, 1)$  onto a ring domain  $\mathbb{G} \setminus \{\zeta\}$ ,  $\zeta \in \mathbb{G}$ , with  $f(\mathbb{T}_\rho) = \zeta$ .

The boundary behavior of functions  $f \in \mathcal{H}_u(\rho, \mathbb{G})$  is given as follows [12].

THEOREM 2.1. Let  $f \in \mathcal{H}_u(\rho, \mathbb{G})$ . Then there is a countable set  $E \subset \mathbb{T}$  such that the following hold:

- (i) For each  $e^{i\theta} \in \mathbb{T} \setminus E$ , the unrestricted limit  $f(e^{i\theta})$  exists and belongs to  $\partial\mathbb{G}$ . Furthermore,  $f$  is continuous in  $\overline{\mathbb{A}(\rho, 1)} \setminus E$ .
- (ii) For each  $e^{i\theta_0} \in E$ , the side-limits  $\lim_{\theta \uparrow \theta_0} f(e^{i\theta})$  and  $\lim_{\theta \downarrow \theta_0} f(e^{i\theta})$  exist in  $\partial\mathbb{G}$  and are distinct.
- (iii) For each  $e^{i\theta_0} \in E$ , the cluster set  $C(f, e^{i\theta_0})$  lies in  $\partial\mathbb{G}$  and is the straight-line segment joining the side-limits  $\lim_{\theta \uparrow \theta_0} f(e^{i\theta})$  and  $\lim_{\theta \downarrow \theta_0} f(e^{i\theta})$ .
- (iv)  $\overline{\text{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$ ;  $\overline{\text{co}}(-)$  is the closed convex hull of  $-$ .
- (v) There is a sense-preserving quasihomeomorphism of  $\mathbb{T}$  into  $\partial\mathbb{G}$  that coincides with the unrestricted limit function  $f$  on  $\mathbb{T} \setminus E$ .
- (vi)  $f$  is the Dirichlet solution in  $\mathbb{A}(\rho, 1)$  of the function  $f^*$  defined by the unrestricted limit function of  $f$  on  $\mathbb{T}$  and the value of  $f$  on  $\mathbb{T}_\rho$ .

The fact that  $f^*$  is not defined on  $E$  in (vi) is insignificant. Indeed, Dirichlet solutions in multiply connected domains coincide whenever their boundary functions coincide almost everywhere.

### 3. A Representation Theorem and Univalence Criteria

Hengartner and Szynal [7] and Bshouty and Hengartner [1] gave the following useful representation for harmonic mappings  $f$  defined on an annulus  $\mathbb{A}(\rho, 1)$  and constant on the inner circle.

**THEOREM 3.1.** *Let  $f$  be a harmonic mapping of  $\mathbb{A}(\rho, 1)$  that extends continuously across  $\mathbb{T}_\rho$  with  $f$  identically  $\zeta$  there. Then there exist a constant  $c$  and a function  $h$  analytic in  $\mathbb{A}(\rho^2, 1)$  such that*

$$(3.1) \quad f(z) = h(z) - h(\rho^2/\bar{z}) + \zeta + 2c \log(|z|/\rho).$$

*Further, if  $f$  extends continuously across  $\mathbb{T}$  and  $f^*$  is the restriction of  $f$  on  $\mathbb{T}$ , then  $c = 0$  if and only if  $\zeta$  equals*

$$(3.2) \quad \zeta_0 = \frac{1}{2\pi} \int_0^{2\pi} f^*(e^{it}) dt.$$

Using Theorem 3.1, Bshouty and Hengartner [1] obtained the following result.

**THEOREM 3.2.** *Let  $f^*$  be a sense-preserving homeomorphism between  $\mathbb{T}$  and  $\partial\mathbb{G}$  that assumes on  $\mathbb{T}_\rho$  the constant  $\zeta_0 \in \mathbb{G}$  given by (3.2), and let  $f$  be the Dirichlet solution of  $f^*$  in  $\mathbb{A}(\rho, 1)$ . Then  $f \in \mathcal{H}_u(\rho, \mathbb{G})$ .*

Theorem 3.2 was extended by the author [12] as follows.

**THEOREM 3.3.** *Let  $f^*$  be a sense-preserving quasihomomorphism of  $\mathbb{T}$  into  $\partial\mathbb{G}$  such that  $\overline{\text{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$ , and let  $f^*$  be defined on  $\mathbb{T}_\rho$  by the constant  $\zeta_0$  given in (3.2). Also, let  $f$  be the Dirichlet solution of  $f^*$  in  $\mathbb{A}(\rho, 1)$ . Then  $\zeta_0 \in \mathbb{G}$  and  $f \in \mathcal{H}_u(\rho, \mathbb{G})$ .*

Further, the author [11, Theorem 2] showed that, without using Theorem 3.1, Theorem 3.2 remains true under the weaker condition  $f(\mathbb{A}(\rho, 1)) \subset \mathbb{G}$  rather than the convexity of  $\mathbb{G}$ . In fact, in view of Theorem 2.1 and the proof of the previous theorem, the following more general result can be obtained.

**THEOREM 3.4.** *Let  $f^*$  be a sense-preserving quasihomomorphism of  $\mathbb{T}$  into the boundary of a bounded Jordan domain  $\mathbb{G}$  such that  $\overline{\text{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$ , and let  $f^*$  be defined on  $\mathbb{T}_\rho$  by the constant  $\zeta_0$  given in (3.2). Also, let  $f$  be the Dirichlet solution of  $f^*$  in  $\mathbb{A}(\rho, 1)$ . Then  $\zeta_0 \in \mathbb{G}$  and  $f \in \mathcal{H}_u(\rho, \mathbb{G})$ .*

We introduce here the following subclass of  $\mathcal{H}_u(\rho, \mathbb{G})$ .

**DEFINITION 3.1.** Denote by  $\mathcal{H}_0(\rho, \mathbb{G})$  the class of all Dirichlet solutions  $f$  satisfying the hypotheses of Theorem 3.3.

The classes  $\mathcal{H}_u(\rho, \mathbb{G})$  and  $\mathcal{H}_0(\rho, \mathbb{G})$  are related as follows [12].

**PROPOSITION 3.1.** *Suppose that the following are true:*

- (i)  $f^*$  is a sense-preserving quasihomomorphism of  $\mathbb{T}$  into  $\partial\mathbb{G}$  such that  $\overline{\text{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$ .
- (ii)  $f$  is the Dirichlet solution in  $\mathbb{A}(\rho, 1)$  of the function defined on  $\mathbb{T}$  by  $f^*$  and on  $\mathbb{T}_\rho$  by a constant  $\zeta \in \mathbb{G}$ .

- (iii)  $f_0 \in \mathcal{H}_0(\rho, \mathbb{G})$  is the Dirichlet solution of the function defined on  $\mathbb{T}$  by  $f^*$  and on  $\mathbb{T}$  by the average  $\zeta_0$  of  $f^*$ .

Then there is an analytic function  $h$  in  $\mathbb{A}(\rho^2, 1)$  such that  $f$  has form (3.1) or the equivalent form

$$(3.3) \quad f(z) = f_0(z) + 2c_\zeta \log |z|$$

where

$$(3.4) \quad f_0(z) = h(z) - h(\rho^2/\bar{z}) + \zeta_0, \quad (z \in \mathbb{A}(\rho, 1)),$$

and

$$(3.5) \quad c_\zeta = \frac{\zeta - \zeta_0}{2 \log \rho}.$$

The function  $f_0$  is called the *average associate* of  $f$ .

In what follows we use  $f'(e^{i\theta})$  for  $df(e^{i\theta})/d\theta$ . Note that, according to Proposition 3.1,  $f$  may not belong to  $\mathcal{H}_u(\rho, \mathbb{G})$  even though it shares with its average associate  $f_0$  the same analytic and co-analytic part  $h$ . Thus only the functions  $f \in \mathcal{H}_0(\rho, \mathbb{G})$  of the form (3.4) are used in stating the following two results [12].

**THEOREM 3.5.** *Let  $f \in \mathcal{H}_0(\rho, \mathbb{G})$  be of form (3.4). Then*

- (a)  $h'$  is nonvanishing on  $\mathbb{T}_\rho$  and  $h$  maps  $\mathbb{T}_\rho$  homeomorphically onto a convex curve whose diameter is bounded above by

$$D = (4d/\pi) \tanh^{-1} (\mu^{-1}(\log(1/\rho))).$$

- (b) If  $h(z) = \sum_{-\infty}^{\infty} a_n z^n$ ,  $z \in \mathbb{A}(\rho^2, 1)$ , then

$$\sum_{n=1}^{\infty} n|a_{-n}|^2 \rho^{-2n} < \sum_{n=1}^{\infty} n|a_n|^2 \rho^{2n} \leq D^2/4 + \sum_{n=1}^{\infty} n|a_{-n}|^2 \rho^{-2n}.$$

**THEOREM 3.6.** *Let  $f \in \mathcal{H}_0(\rho, \mathbb{G})$  be of form (3.4). Then there is a univalent close-to-convex function  $H$  of the unit disc  $\mathbb{D}$  and a homeomorphism  $\phi$  of  $\mathbb{A}(\rho, 1) \cup \mathbb{T}$  into  $\overline{\mathbb{D}}$  with  $\phi(\mathbb{T}) = \mathbb{T}$  such that  $h = H \circ \phi$ .*

The author conjectures that the function  $H$  is convex.

#### 4. A Hengartner's Problem Regarding Univalent Harmonic Mappings

Let  $f$  be the Dirichlet solution in  $\mathbb{A}(\rho, 1)$  of a function  $f^*$  of  $\partial\mathbb{A}(\rho, 1)$  defined by a sense-preserving quasihomomorphism of  $\mathbb{T}$  into  $\partial\mathbb{G}$  satisfying  $\overline{\text{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$ , and by a constant  $\zeta \in \mathbb{G}$  on  $\mathbb{T}_\rho$ . Theorem 3.3 asserts that  $f$  belongs to  $\mathcal{H}_u(\rho, \mathbb{G})$  if  $\zeta = \zeta_0$ , where  $\zeta_0$  is the average of  $f^*$  on  $\mathbb{T}_\rho$  given by (3.2). Hengartner and Schober [6] showed that this condition is not necessary, and recently Duren and Hengartner [4, Example 1] gave the harmonic mapping

$$F(z) = (z - \rho^2/\bar{z})/(1 - \rho^2) + 2c \log |z|, \quad (z \in \mathbb{A}(\rho, 1)),$$

which belongs to  $\mathcal{H}_u(\rho, \mathbb{D})$ , with  $\zeta_0 = 0$ , whenever  $|c| < \rho/(1 - \rho^2)$ ; note that  $F(e^{it}) = e^{it}$  and  $F(\mathbb{T}_\rho) = 2c \log \rho$ . This concludes a negative answer to the following question of Nitsche [15, §879]:

QUESTION (Nitsche). *Are all univalent harmonic mappings of  $\mathbb{A}(\rho, 1)$  onto  $\mathbb{A}(0, 1)$ , up to a rotation, of the form*

$$(4.1) \quad f(z) = (z - \rho^2/\bar{z})/(1 - \rho^2)?$$

In this connection, Hengartner [2, Problem 15] raised the following problem:

PROBLEM (Hengartner). *Let  $f^*$  be a sense-preserving quasihomomorphism of  $\mathbb{T}$  into  $\partial\mathbb{G}$  with  $\overline{\text{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$ , and let  $f$  be the harmonic extension in  $\mathbb{A}(\rho, 1)$  of the function defined by  $f^*$  on  $\mathbb{T}$  and by a constant  $\zeta \in \mathbb{G}$  on  $\mathbb{T}_\rho$ . Find the set of all  $\zeta$  for which  $f : \mathbb{A}(\rho, 1) \rightarrow \mathbb{G} \setminus \{\zeta\}$  a homeomorphism.*

Denote by  $\mathcal{H}(\rho, f^*)$  the class of Dirichlet solutions in  $\mathbb{A}(\rho, 1)$  of functions of  $\partial\mathbb{A}(\rho, 1)$  defined on  $\mathbb{T}$  by  $f^*$  and on  $\mathbb{T}_\rho$  by some constant  $\zeta \in \mathbb{G}$ , by  $\mathcal{H}_u(\rho, f^*)$  the subclass of  $\mathcal{H}(\rho, f^*)$  of univalent mappings, and by  $K(\rho, f^*)$  the set of values  $\zeta \in \mathbb{G}$  for which a function  $f \in \mathcal{H}(\rho, f^*)$  belongs to  $\mathcal{H}_u(\rho, f^*)$ .

The results of this section were obtained in an attempt by the author to give a satisfactory answer to Hengartner’s problem. The first result [12] states as follows.

THEOREM 4.1.  *$K(\rho, f^*)$  is a nonempty compact subset of  $\mathbb{G}$ .*

By Proposition 3.1, the class  $\mathcal{H}(\rho, f^*)$  yields an analytic function  $h$  in  $\mathbb{A}(\rho^2, 1)$ , unique up to an additive constant, such that every  $f \in \mathcal{H}_u(\rho, f^*)$  is of form (3.3). The second result [12] characterizes the boundary points of  $K(\rho, f^*)$  in terms of  $h$  and  $f^*$  in a manner leading to a univalence criterion for functions  $f \in \mathcal{H}(\rho, f^*)$ . The result states as follows.

THEOREM 4.2. *Let  $f \in \mathcal{H}_u(\rho, f^*)$  be of form (3.3), where  $f^* : \mathbb{T} \rightarrow \partial\mathbb{G}$  is a twice-differentiable function with nonvanishing derivative and absolutely continuous second derivative. Then the dilatation  $\omega$  of  $f$  and  $zh'(z) + c_\zeta$  extend continuously to  $\mathbb{A}(\rho, 1) \cup \mathbb{T}$  such that  $e^{i\theta}h'(e^{i\theta}) + c_\zeta \neq 0$  for all  $\theta$ . Moreover, we have:*

- (a) *If  $\zeta \in \partial K(\rho, f^*)$ , then either  $\rho e^{i\theta_1}h'(\rho e^{i\theta_1}) + c_\zeta = 0$  for some  $\theta_1$ , or  $|\omega(e^{i\theta_2})| = 1$  for some  $\theta_2$ .*
- (b) *If  $|\omega(e^{i\theta})| = 1$  for some  $\theta$ , then  $\zeta \in \partial K(\rho, f^*)$ .*
- (c) *If in (a) and (b) the function  $|\omega(e^{i\theta})|$  is replaced by the function*

$$2 \operatorname{Re} \left\{ \frac{e^{i\theta}h'(e^{i\theta}) + c_\zeta}{e^{i\theta}f'(e^{i\theta})} \right\},$$

*then (a) and (b) continue to hold.*

Regarding (a), Hengartner and Szynal [7, Theorem 3.1] asserted that if  $\zeta \in \partial K(\rho, f^*)$ , then  $\rho e^{i\theta_1}h'(\rho e^{i\theta_1}) + c_\zeta$  has at most one zero and that this zero is of order one.

Applying Theorem 4.2 to functions  $f \in \mathcal{H}_u(\rho, f^*)$  of form (3.4), where  $\zeta$  is the average  $\zeta_0$  of  $f^*$  on  $\mathbb{T}$ ,  $c_\zeta = 0$ ,  $\rho e^{i\theta}h'(\rho e^{i\theta}) \neq 0$  for all  $\theta$  by Theorem 3.5(a), and  $|\omega(e^{i\theta})| = 1$  for some  $\theta$  if and only if  $\rho^2|h'(\rho^2 e^{i\theta})| = |h'(e^{i\theta})|$ , the following result is obtained [12].

COROLLARY 4.1. *Let  $f \in \mathcal{H}_u(\rho, f^*)$  be of form (3.4), where  $f^*$  is as in Theorem 4.2. Then the following statements are equivalent:*

- (a)  $\zeta_0 \in \partial K(\rho, f^*)$ .
- (b)  $\rho^2 |h'(\rho^2 e^{i\theta})| = |h'(e^{i\theta})|$  for some  $\theta$ .
- (c)  $2 \operatorname{Re}\{h'(e^{i\theta})/f'(e^{i\theta})\} = 1$  for some  $\theta$ .

The next result [12] provides sufficient conditions for the univalence of functions  $f \in \mathcal{H}(\rho, f^*)$  whose  $f^*$  are as given in Theorem 4.2.

**THEOREM 4.3.** *Let  $f \in \mathcal{H}(\rho, f^*)$  be of form (3.3), where  $f^*$  be smooth as in Theorem 4.2. Then  $f \in \mathcal{H}_u(\rho, f^*)$  if  $zh'(z) + c_\zeta \neq 0$  for  $z \in \overline{\mathbb{A}(\rho, 1)}$ , and if one of the following two inequalities holds for all  $\theta$ :*

$$(a) |\omega(e^{i\theta})| \leq 1. \quad (b) 2 \operatorname{Re}\left\{\frac{e^{i\theta} h'(e^{i\theta}) + c_\zeta}{e^{i\theta} f'(e^{i\theta})}\right\} \geq 1.$$

We remark that  $f^*$  as defined in Theorem 4.2 yields  $zh'(z) \neq 0$  for  $z \in \overline{\mathbb{A}(\rho, 1)}$  which makes the above hypothesis,  $zh'(z) + c_\zeta \neq 0$  for  $z \in \overline{\mathbb{A}(\rho, 1)}$ , easily achievable for functions  $f \in \mathcal{H}(\rho, f^*)$  with sufficiently small  $c_\zeta$ .

The next result [12] asserts the existence of a large family of triplets,  $0 < \rho < 1$ ,  $\mathbb{G}_\rho, f^*$ , where  $\mathbb{G}_\rho$  is a bounded convex domain and  $f_\rho^* : \mathbb{T} \rightarrow \partial\mathbb{G}_\rho$  is a sense-preserving homeomorphism, such that  $K(\rho, f^*)$  has a nonempty interior containing the average of  $f^*$ .

**THEOREM 4.4.** *Let  $\Omega$  be a bounded convex domain, and let  $h$  be a homeomorphism of  $\overline{\mathbb{D}}$  onto  $\overline{\Omega}$  that maps  $\mathbb{D}$  conformally onto  $\Omega$ . Suppose that  $h''$  is continuous on  $\overline{\mathbb{D}}$ ,  $h''(e^{i\theta})$  is absolutely continuous, and*

$$(4.2) \quad \operatorname{Re}\left\{1 + e^{i\theta} \frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} > 0$$

for all  $\theta$ . Then there exists  $\delta > 0$  such that for each  $0 < \rho < \delta$  we can find a bounded convex domain  $\mathbb{G}_\rho$  such that the harmonic mapping

$$f_\rho(z) = h(z) - h(\rho^2/\bar{z}), \quad (z \in \overline{\mathbb{A}(\rho, 1)}),$$

satisfies the following properties:

- (i)  $f_\rho : \mathbb{T} \rightarrow \partial\mathbb{G}_\rho$  is a sense-preserving homeomorphism.
- (ii)  $f_\rho$  is continuously twice-differentiable on  $\overline{\mathbb{A}(\rho, 1)}$ .
- (iii)  $f_\rho \in \mathcal{H}_0(\rho, \mathbb{G}_\rho)$ .
- (iv) There is  $\sigma > 0$ , depending on  $\rho$ , such that for any  $|\zeta| < \sigma$  the function

$$f_\zeta(z) = h(z) - h(\rho^2/\bar{z}) + \zeta + 2c_\zeta \log(|z|/\rho)$$

belongs to  $\mathcal{H}_u(\rho, \mathbb{G}_\rho)$ .

**REMARK 4.1.** (i) Without (4.2), the hypothesis of the theorem yields the following weaker form of(4.2):

$$(4.3) \quad \operatorname{Re}\left\{1 + e^{i\theta} \frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} \geq 0.$$

To see this, observe that  $zh'(z)$  is a univalent starlike function in  $\mathbb{D}$  which gives

$$(4.4) \quad \operatorname{Re}\left\{1 + z \frac{h''(z)}{h'(z)}\right\} > 0, \quad (z = re^{i\theta} \in \mathbb{D}).$$

Now, because  $h''$  extends continuously to  $\overline{\mathbb{D}}$ , the integral

$$\int_0^z h''(\zeta) d\zeta, \quad (z \in \overline{\mathbb{D}}),$$

where the differentiable path of integration from 0 to  $z$  lies in  $\overline{\mathbb{D}}$ , yields, by Cauchy's theorem, the continuous extension of  $h'(z)$  to  $\overline{\mathbb{D}}$ . On the other hand, since  $zh'(z)$  is univalent in  $\mathbb{D}$  and maps the origin to itself,  $zh'(z) \neq 0$  for  $z \in \overline{\mathbb{D}}$ . Then (4.3) follows at once by letting  $r \rightarrow 1$  in (4.4).

(ii) Using Kellogg and Warschawski [16, Theorem 3.6, p. 49], the hypothesis that  $h''(z)$  admits a continuous extension to  $\overline{\mathbb{D}}$  with absolutely continuous  $h''(e^{i\theta})$  follows if  $\partial\mathbb{G}$  has a parameterization  $w(t)$ ,  $0 \leq t \leq 2\pi$ , whose first derivative is nonvanishing and second derivative is Lipschitz of order  $\alpha$ ,  $0 < \alpha < 1$ .

### 5. Nitsche's Question Revisited

In this section all harmonic mappings  $f \in \mathcal{H}_u(\rho, \mathbb{G})$  whose analytic parts extend analytically throughout  $\mathbb{D}$  are determined explicitly. It follows that the function  $f$  defined by (4.1) is the only harmonic mapping, up to rotation, in  $\mathcal{H}_0(\rho, \mathbb{D})$ , (here  $\mathbb{G}$  is taken as  $\mathbb{D}$ ), of  $\mathbb{A}(\rho, 1)$  onto  $\mathbb{A}(0, 1)$  whose analytic part is analytic in  $\mathbb{D}$ . This somehow justifies Nitsche's Question above. The result of this section states as follows [12].

**THEOREM 5.1.** *Let  $f \in \mathcal{H}_u(\rho, \mathbb{G})$  be of form (3.3) with  $h$  analytic in  $\mathbb{D}$ . Then*

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1 - \rho^{2n}} [z^n - (\rho^2/\bar{z})^n] + \zeta + 2c_\zeta \log(|z|/\rho) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1 - \rho^{2n}} [z^n - (\rho^2/\bar{z})^n] + \zeta_0 + 2c_\zeta \log |z|, \end{aligned}$$

where  $b_n$ ,  $n = 1, 2, \dots$ , is the  $n$ -th coefficient of the conformal map

$$F(z) = \zeta_0 + \sum_{n=1}^{\infty} b_n z^n$$

of  $\mathbb{D}$  onto  $\mathbb{G}$  satisfying  $F(0) = \zeta_0$  and  $c_\zeta$  is as given in (3.5).

As an application of Theorem 5.1, if  $\mathbb{G} = \mathbb{D}$ , then

$$F(z) = \frac{z + \zeta_0}{1 + \bar{\zeta}_0 z} = \zeta_0 + (1 - |\zeta_0|^2) \sum_{n=2}^{\infty} (-\zeta_0)^{n-1} z^n$$

and the following result holds.

**COROLLARY 5.1.** *Let  $f \in \mathcal{H}_u(\rho, \mathbb{D})$  be of the form (3.3) with  $h$  analytic in  $\mathbb{D}$ . Then there is a unimodular constant  $\lambda$  such that*

$$\begin{aligned} f(z) &= \lambda(1 - |\zeta_0|^2) \left\{ \frac{z - \rho^2/\bar{z}}{1 - \rho^2} + \sum_{n=2}^{\infty} \frac{(-\lambda\zeta_0)^{n-1}}{1 - \rho^{2n}} [z^n - (\rho^2/\bar{z})^n] \right\} \\ &\quad + \zeta + 2c_\zeta \log(|z|/\rho), \quad (z \in \mathbb{A}(\rho, 1)). \end{aligned}$$



In particular, if  $\zeta_0 = 0$ , then

$$f(z) = \lambda \frac{z - \rho^2/\bar{z}}{1 - \rho^2} + 2c_\zeta \log |z|, \quad (z \in \mathbb{A}(\rho, 1)).$$

Also, if  $\zeta_0 = 0$  and  $f(\mathbb{T}_\rho) = 0$ , then

$$f(z) = \lambda \frac{z - \rho^2/\bar{z}}{1 - \rho^2}, \quad (z \in \mathbb{A}(\rho, 1)).$$

### 6. The Modulus of the Image Annuli under Univalent Harmonic Mappings and a Conjecture of Nitsche

For harmonic mappings  $f : \mathbb{A}(\rho, 1) \rightarrow A(R, 1)$ ,  $R$  can possibly be zero as with (4.1) which maps  $\mathbb{A}(\rho, 1)$  univalently onto the punctured disc  $A(0, 1)$ . On the other hand,  $R$  admits a universal upper bound (less than 1) as was shown in 1962 by Nitsche [14]. To state this result, let  $\mathcal{K}(\rho)$  be the class of univalent harmonic mappings of the annulus  $\mathbb{A}(\rho, 1)$  onto some annulus  $A(R, 1)$ , and let  $\kappa(\rho)$  be the supremum of  $R$  as  $f$  ranges over all  $f \in \mathcal{K}(\rho)$ . Using Harnack’s inequality, Nitsche proved the following result [14]:

**THEOREM 6.1.** *The value  $\kappa(\rho)$  is less than 1.*

Consider now the class of harmonic mappings

$$f_t(z) = tz + (1 - t)/\bar{z} = [t + (1 - t)/\sigma]e^{i\theta} \quad (z = \sigma e^{i\theta}).$$

Each  $f_t$  maps concentric circles onto concentric circles, and maps  $\mathbb{A}(\rho, 1)$  univalently onto  $A(R(t), 1)$ ,  $R(t) = t\rho + (1 - t)/\rho$ , if, and only if,  $1/(1 + \rho^2) \leq t \leq 1/(1 - \rho^2)$ . Restricted to these values of  $t$ , Nitsche [14] observed that  $R(t)$  admits its maximum value  $2\rho/(1 + \rho^2)$  at  $t = 1/(1 + \rho^2)$ . This led him to suggest the following:

**CONJECTURE (Nitsche).**  $\kappa(\rho) = 2\rho/(1 + \rho^2)$ .

The conjecture was raised again in 1989 by Schober [17] as “an intriguing open problem”, and subsequently in 1994 by Bshouty and Hengartner [2] as “open problem 3.1”. Looking closer at Nitsche’s proof of the above theorem, the latter authors observed that the proof also applies to the wider class of harmonic mappings of  $\mathbb{A}(\rho, 1)$  that are not necessarily univalent and that admit a point in each of the vertical strips  $\{w : R < \operatorname{Re} w < 1\}$  and  $\{w : -1 < \operatorname{Re} w < -R\}$ . Consequently, they remarked that  $\kappa(\rho)$  is unlikely to be found by parlaying Nitsche’s proof of his Theorem 6.1.

Until recently, it was believed that no quantitative upper bound for  $\kappa(\rho)$  was found. However, in a personal communication dated December 1999, Nitsche wrote that he had “developed the estimate  $(\kappa(\rho) \leq \tanh[\pi(1 + \rho)/(1 - \rho)]) \approx 0.9926$  at the time (of his article [14])”, but refrained from publishing the “poor bound” in order “not to detract from the impact of the conjecture”.

The author, being unaware of Nitsche’s result, gave a substantial upper bound of  $\kappa(\rho)$  in terms of the Grötzsch’s ring domain  $B(s)$  of  $\mathbb{A}(\rho, 1)$  [13].

**THEOREM 6.2.** *Let  $f$  be a univalent harmonic mapping of the annulus  $\mathbb{A}(\rho, 1)$  onto the annulus  $A(R, 1)$ , and let  $B(s(\rho))$  be the Grötzsch's ring domain that is conformally equivalent to  $\mathbb{A}(\rho, 1)$ . Then  $R \leq s(\rho)$ .*

Further, it was conjectured in [13] that the inequality  $R \leq s$  is sharp, but the conjecture was subsequently disproved by Weitsman [18] in the following result.

**THEOREM 6.3.** *Let  $f$  be a univalent harmonic mapping of the annulus  $\mathbb{A}(\rho, 1)$  onto the annulus  $A(R, 1)$ . Then*

$$R \leq \sigma(\rho) = \frac{1}{1 + (\rho \log \rho)^2 / 2}.$$

Computations reveal that for a value  $\rho_0 \approx 0.36$ ,  $\sigma(r\rho) < s(\rho)$  if  $\rho_0 < \rho < 1$ ,  $\sigma(\rho)$  is substantially smaller than  $s(\rho)$  if  $\rho$  is close to 1,  $\sigma(\rho) > s(\rho)$  if  $0 < \rho < \rho_0$ , and  $\sigma(\rho)$  is of no value when  $\rho$  is small. Further, if  $\tau(\rho) = 2\rho/(1 + \rho^2)$ , which is the upper bound conjectured by Nitsche, then  $\lim_{\rho \rightarrow 1^-} (1 - \tau(\rho))/(1 - \sigma(\rho)) = 1$ .

It was noted by the referee that Kalaj [8] has recently improved Weitsman's result above as follows.

**THEOREM 6.4.** *Let  $f$  be a univalent harmonic mapping of the annulus  $\mathbb{A}(\rho, 1)$ ,  $0 < \rho < 1$ , onto the annulus  $A(R, 1)$ . Then*

$$R \leq \eta(\rho) = \frac{1}{1 + (\log \rho)^2 / 2}.$$

Obviously,  $\eta(\rho) < \sigma(\rho)$  and  $\eta(\rho)$ , like  $\sigma(\rho)$ , is of no value when  $\rho$  is small.

In conclusion, Nitsche's conjecture remains an unsettled interesting problem.

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