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POSITIVITY ZONES AND NORMS OF n-FOLD CONVOLUTIONS - I

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ABSTRACT. A general class of sequences $a = \{a_k : -\infty < k < \infty\}$ of real numbers is described which has the property that there exist numbers c_1, c_2, N such that $a_{nk} > 0$ for $n > N$, $c_1 n \leqslant k \leqslant c_2 n$, where $\{a_{nk} : -\infty < k < \infty\}$ is defined as the n-fold convolution $a * a * \cdots * a$ of a.

Introduction. In the first part of the paper we prove our main result—Theorem 1. In the second part, which will be forthcoming, we shall present some corollaries, open problems, and relations of Theorem 1 with the central limit theorem and related results of probability theory, with the additive number theory and with the estimates of the remainder in the asymptotic formulae for norms of N-fold convolutions. (For these asymptotic formulae see the expository article [1]).

Definitions and Result. Given an infinite array $\mathcal{M} = \{a_{nk} : n \in Z^+, k \in Z\}$ with entries real numbers, we say that the pair (C_1, C_2) of extended reals, $-\infty \leqslant C_1 < C_2 \leqslant \infty$, defines a *positivity angle* in $\mathcal{M}$ if for any pair (c_1, c_2) or reals satisfying $C_1 < c_1 < c_2 < C_2$ there exists N such that $a_{nk} > 0$ for $n \geqslant N$, $c_1 n \leqslant k \leqslant c_2 n$. (Positivity angles are special case of positivity zones, in this part of the paper we do not need the more general concept).

Let $a = (a_k)_{-\infty}^{\infty}$ be a sequence of real numbers such that $\sum |a_k| < \infty$. $\mathcal{M}(a)$ will denote the array which has as n^{th} row the n-fold convolution $\underbrace{a * a * \cdots * a}_{n \text{ times}}$.

Equivalently, if we set

$$f(e^{ti}) = \sum_{k=-\infty}^{\infty} a_k \exp(kti) \tag{1}$$

we can say that the n^{th} row of $\mathcal{M}(a)$ is formed by the Fourier coefficients of $f^n(e^{ti})$.

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THEOREM 1. *If the sequence* $(a_k)_{-\infty}^{\infty}$ *of real numbers satisfies the conditions:*

(i) $a_k \to 0$ *exponentially fast as* $|k| \to \infty$

(ii) $\sum a_k > 0$

(iii) $|\sum a_k e^{kti}| < \sum a_k$ *for* $0 < |t| \leqslant \pi$

(iv) $\sum a_k \sum k^2 a_k \neq (\sum k a_k)^2$

then the array $\mathcal{M}(a)$ *has a positivity angle* (C_1, C_2). *Moreover,* $C_1 < \dfrac{\sum k a_k}{\sum a_k} < C_2$.

The importance of the result is that here we have not just a zone of strict positivity the width of which increases to infinity as $n \to \infty$, but that the width increases at least linearly with n, that here we have a positivity *angle* (and that the ray $k = n(\sum j a_j)/(\sum a_j)$ lies inside that angle).

The result is interesting even in case when $a_k \geqslant 0$ for every k. In that case the assumptions (iii) and (iv) are equivalent, respectively, to

(iii – non-negative case) the elements of the set $S = \{k : a_k > 0\}$ are not multiples of the same integer $m, m \geqslant 2$, and

(iv – non-negative case) at least two terms of the sequences (a_k) are distinct from zero.

Proof. First we have to express conditions (i)–(iv) in the dual form, i.e., in terms of the function f, defined by (1). It is easy to see that conditions (i)–(iii) are equivalent to

(i′) f is analytic on the unit circle (i.e., on an open annulus containing the unit circle),

(ii′) $f(1) > 0$,

(iii′) $|f(e^{ti})| < f(1)$ for $0 < |t| \leqslant \pi$.

It requires some computation to check that (iv) is equivalent with

(iv′) $\frac{d^2}{dt^2}|f(e^{it})| \neq 0$ at $t = 0$,

which means that the maximum that $|f(e^{it})|$ attains at $t = 0$ is generic (not too flat).

From conditions (i′)–(iv′) it is obvious that there exists the largest open annulus $\mathcal{A} = \{z : R_1 < |z| < R_2\}$, $0 \leqslant R_1 < 1 < R_2 \leqslant \infty$, with the following properties: f is analytic on $\mathcal{A}$, $f(r) > 0$ for $R_1 < r < R_2$,

$$|f(re^{ti})| < f(r) \text{ for } 0 < |t| \leqslant \pi, R_1 < r < R_2, \tag{2}$$

$$\frac{d^2 |f(re^{ti})|}{dt^2} \neq 0 \text{ at } t = 0, R_1 < r < R_2. \tag{3}$$

Once the annulus $\mathcal{A}$, i.e., the numbers R_1, R_2 are defined, C_1 and C_2 will be determined by

$$C_1 = \inf\{rf'(r)/f(r) : R_1 < r < R_2\}, C_2 = \sup\{rf'(r)/f(r) : R_1 < r < R_2\}. \tag{4}$$

Therefore $C_1 < f'(1)/f(1) < C_2$, and since $f'(1)/f(1) = (\sum k a_k)/(\sum a_k)$, we obtain that the first moment of the normalized sequence $a_k/(\sum a_j)$ lies in (C_1, C_2).

Since $\log \mathcal{M}(r)$ is a convex function of $\log r$ for $R_1 < r < R_2$, and by (2), $\mathcal{M}(r) = f(r)$, we get that $\log f(e^x)$ is convex function of x, and therefore its derivative $\frac{rf'(r)}{f(r)}$ is strictly increasing on (R_1, R_2). This implies $\frac{rf'(r)}{f(r)}$ has an inverse function $r = r(c)$, defined on the interval (C_1, C_2). If c_1 and c_2 $(c_1 < c_2)$ belong to that interval, we let $r_1 = r(c_1), r_2 = r(c_2)$ so that $R_1 < r_1 < r_2 < R_2$ and from now on we restrict our considerations to the compact annulus $\{z : r_1 \leqslant |z| \leqslant r_2\}$.

To show that the coefficient a_{nk} is strictly positive for large n if $c_1 n \leqslant k \leqslant c_2 n$, we set $k/n = c$, and express a_{nk} by the Cauchy integral formula, where for the contour of integration Γ we choose the circle $|z| = r(c)$.

Because of compactness and (2) and (3), we can choose $\delta > 0$ such that $f(re^{ti})$ has no zeroes in $|t| \leqslant \delta, r_1 \leqslant r \leqslant r_2$ and that for some $A > 0$

$$\frac{1}{2}\frac{\partial^2 \log|f(re^{ti})|}{\partial t^2} \leqslant -A \quad \text{for } |t| \leqslant \delta,\ r_1 \leqslant r \leqslant r_2. \tag{5}$$

We divide the contour Γ into three parts: $|\arg z| \leqslant \epsilon_n$, $\epsilon_n \leqslant |\arg z| \leqslant \delta$, $\delta \leqslant |\arg z| \leqslant \pi$, where $\epsilon_n = n^{-\alpha}, \alpha$ to be selected later. This way

$$a_{nk} = \frac{1}{2\pi i}\int_\Gamma f(z)^n z^{-k-1} dz = I_1 + I_2 + I_3. \tag{6}$$

For I_3 we use the simple estimate $|I_3| \leqslant m^n(r(c))r^{-k}(c)$, where $m(r) = \sup\{|f(re^{ti})| : \delta \leqslant |t| \leqslant \pi\}$. Since there exists γ, $0 < \gamma < 1$, such that $m(r) \leqslant \gamma f(r)$ for every r, $r_1 \leqslant r \leqslant r_2$, we obtain that

$$|I_3| \leqslant \gamma^n g^n(c), \tag{7}$$

where

$$g(c) = f(r(c))/r^c(c). \tag{8}$$

To estimate I_2, we observe that, because of (5), $\varphi(t) = \log|f(r(c)e^{ti})|$ is concave on $[-\delta, \delta]$. Therefore the graph of φ lies below any of its tangents, and since $\varphi(\epsilon_n) + \varphi'(\epsilon_n)(t - \epsilon_n)$ is a tangent, we get that $\varphi(t) \leqslant \varphi(\epsilon_n) + \varphi'(\epsilon_n)(t - \epsilon_n)$. Since $\varphi(\epsilon_n) \leqslant \varphi(0) = f(r(c))$, we obtain

$$\left|\int_{\epsilon_n}^{\delta}\right| \leqslant g^n(c)\int_{\epsilon_n}^{\delta} \exp(n\varphi'(\epsilon_n)(t - \epsilon_n))dt.$$

Since $\varphi'(\epsilon_n) = \int_0^{\epsilon_n} \varphi''(u)du \leqslant \int_0^{\epsilon_n} -Adu = -A\epsilon_n$, we obtain that

$$|I_2| \leqslant n^{\alpha-1} g_n(c)/A. \tag{9}$$

We pass now to I_1:

$$\begin{aligned} I_1 &= \frac{1}{2\pi}\int_{-\epsilon_n}^{\epsilon_n} f^n(r(c)e^{ti})r^{-k}(c)e^{-kti}dt \\ &= \frac{1}{2\pi}g^n(c)\int_{-\epsilon_n}^{\epsilon_n} \exp n(\varphi(t) + i\psi(t))dt, \end{aligned} \tag{10}$$

where we used the shorthand notation

$$\varphi(t) = \log|f(r(c)e^{ti})| - \log|f(r(c))|,$$
$$\psi(t) = \arg f(r(c)e^{ti}) - ct,$$

instead of the correct $\varphi(t,c)$ and $\psi(t,c)$. The point $r(c)$ is the saddle point of the function $\log f(z) - c\log z$, so that both $\varphi'(0) = 0$, $\psi'(0) = 0$. Obviously $\varphi(0) = \psi(0) = 0$. It is not so obvious that $\psi''(0) = 0$. That is due to the fact that the coefficients a_k are real, so the function $f(z)$ has the property $f(\overline{z}) = \overline{f(z)}$, which implies that the function ψ is odd, so the second derivative is also odd, and thus $= 0$ at $t = 0$.

Therefore, taking second degree Taylor polynomials of φ and ψ, and combining the remainders, we obtain that $\varphi(t) + i\psi(t) = -A(c)t^2 + B(t,c)t^3$, where $-A(c) = \frac{1}{2}\frac{\partial^2}{\partial t^2}\log|f(r(c)e^{ti})|\Big|_{t=0}$, $B(t,c)$ is a complex-valued function, and, for some $B > 0$, $|B(t,c)| \leqslant B$ for $|t| \leqslant \delta$, $c_1 \leqslant c \leqslant c_2$. We obtain

$$\int_{-\epsilon_n}^{\epsilon_n} \exp n(\varphi(t) + i\psi(t))dt = \int_{-\epsilon_n}^{\epsilon_n} \exp(-A(c)nt^2 + B(t,c)nt^3)dt = J_1 - J_2 + J_3, \tag{11}$$

where $J_1 = \int_{-\infty}^{\infty} \exp(-A(c)nt^2)dt = \sqrt{\frac{\pi}{A(c)n}}$,

$$J_2 = \int_{|t|>\epsilon_n} \exp(-A(c)nt^2)dt \leqslant \int_{|t|>\epsilon_n} \frac{1}{A(c)nt^2}dt \leqslant \frac{n^{\alpha-1}}{A}$$

and

$$J_3 = \int_{-\epsilon_n}^{\epsilon_n} \exp(-A(c)nt^2)\cdot(\exp(B(t,c)nt^3) - 1)dt.$$

Since $|B(t,c)nt^3| \leqslant Bn\epsilon_n^3 = Bn^{1-3\alpha} < 1$ for large n, provided α is chosen $> 1/3$, and since $|e^z - 1| \leqslant e^{|z|} - 1 < 2|z|$ for $|z| < 1$, we get that $|J_3| \leqslant \int_{-\epsilon_n}^{\epsilon_n} 2Bnt^3dt = Bn^{1-4\alpha}$.

Set $b_n(c) = \frac{1}{2\sqrt{A(c)\pi n}}g^n(c)$. It follows from (10), (11) and just obtained estimates, that

$$|I_1 - b_n(c)| \leqslant Mg^n(c)(n^{\alpha-1} + n^{1-4\alpha}),$$

where M is some constant.

It follows from (6) that

$$|a_{n,cn} - b_n(c)| \leqslant |I_1 - b_n(c)| + |I_2| + |I_3|.$$

So, using (7) and (9) we obtain

$$|a_{n,cn} - b_n(c)| \leqslant Mg^n(c)(n^{\alpha-1} + n^{1-4\alpha} + \gamma^n).$$

Choosing $\alpha = 2/5$, we obtain, with possibly different constant,

$$|a_{n,cn} - b_n(c)| \leqslant Mg^n(c)n^{-3/5}.$$

Since $a_{n,cn}$ is real, $b_n(c)$ positive, and, for large n, $Mg^n(c)n^{-3/5} < b_n(c)$, it follows that, for such n, $a_{n,cn} > 0$.

References

[1] B. Baishanski, *Norms of powers and a central limit theorem for complex-valued probabilities*, Analysis of Divergence (1999), 523–543.

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