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A CLASSIFICATION OF 3-TYPE CURVES IN MINKOWSKI 3-SPACE E_1^3, II

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ABSTRACT. We give a complete classification of all non-planar spacelike and timelike curves in Minkowski 3–space $\mathbb{E}_1^3$, which are of 3–type.

1. Introduction

The notion of submanifolds of *finite type* was introduced by Chen in [2]. A submanifold M in the Euclidean space E^n is said to be of *finite type* if each component of its position vector field x can be written as a finite sum of eigenfunctions of the Laplacian Δ of M. This means that

$$\mathrm{x} = \mathrm{x}_0 + \sum_{t=1}^{k} \mathrm{x}_t, \quad \Delta \mathrm{x}_t = \lambda_t \mathrm{x}_t, \tag{1.1}$$

where $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_k$ are mutually different eigenvalues of Δ. When M is compact, the component x_0 in (1.1) is a constant vector. However, when M is non-compact, the component x_0 is not necessary a constant vector. In particular, a submanifold M is said to be of *k-type* if all eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ are different from zero. If one of the λ_i's is equal to zero ($i = 1, 2, \dots, k$), M is said to be of *null k-type.*

Finite type curves in Euclidean space E^n were studied intensively in [2], [3] and [4]. The classification of all 2-type curves in E^n is given in [6].

2. Preliminaries

Let α be a curve in E_1^n parameterized by a pseudo-arclength parameter s. Then the Laplace operator Δ of α is given by $\Delta = \pm \frac{d^2}{ds^2}$. Its eigenfunctions are s, $\cos(as)$, $\sin(as)$, $\cosh(as)$ and $\sinh(as)$. Following the definition of Chen, every *finite type* curve α in E_1^n can be written as

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$$\alpha(s) = a_0 + b_0 s + \sum_{t=1}^{k_1} (a_t \cos(p_t s) + b_t \sin(p_t s)) + \sum_{t=1}^{k_2} (c_t \cosh(q_t s) + d_t \sinh(q_t s)),$$

where $a_0, b_0, a_i, b_i, c_j, d_j \in R^n$ are constants, $i = 1, \dots, k_1$, $j = 1, \dots, k_2$ and $0 < p_1 < \cdots < p_{k_1}$, $0 < q_1 < \cdots < q_{k_2}$ are positive integers (*frequency numbers* of the curve). For a finite type curve α, *frequency ratio* is the ratio of its frequency numbers.

In particular, a curve α in E_1^n is said to be of *k-type* if there are k mutually different eigenvalues $\lambda_1, \dots, \lambda_k$ of Δ and they are all different from zero. If one of the λ_i's ($i = 1, \dots, k$) is equal to zero, α is said to be of *null k-type.*

Recall that an arbitrary vector v in E_1^n can have one of three causal characters: it can be *spacelike* if $g(v,v) > 0$ or $v = 0$, *timelike* if $g(v,v) < 0$ and *null* if $g(v,v) = 0$ and $v \neq 0$. The norm of a vector v is given by $\|v\| = \sqrt{|g(v,v)|}$.

The unit vectors, orthogonality and orthonormality are defined as in the Euclidean spaces. An arbitrary curve $\alpha(s)$ in E_1^3 can localy be *spacelike, timelike* or *null*, if respectively all of its velocity vectors $\dot{\alpha}(s)$ are spacelike, timelike or null.

Curves of finite type in Minkowski space–time have been investigated in [5] and [7]. The following classification theorem is obtained in [7].

THEOREM 2.1. *Every curve of finite type in Minkowski plane E_1^2 is of* 1*-type and hence an open part of an orthogonal hyperbola or an open part of a straight line.*

THEOREM 2.2. *A planar* 2*-type curve, lying in an isotropic plane of E_1^3 is a null* 2*-type spacelike curve.*

THEOREM 2.3. *Up to rigid motions of E_1^3, a non-planar curve α in E_1^3 is a null* 2*-type curve if and only if α is a part of one of the following curves:*

(i) $\alpha(s) = (as, b\cos s, b\sin s)$, $\quad a, b \in R_0, |a| \neq |b|$;
(ii) $\alpha(s) = (a\cosh s, a\sinh s, bs)$, $\quad a, b \in R_0, |a| \neq |b|$;
(iii) $\alpha(s) = (a\sinh s, a\cosh s, bs)$, $\quad a, b \in R_0, |a| \neq |b|$;

THEOREM 2.4. *Up to rigid motions of E_1^3, a non-planar curve α in E_1^3 is a* 2*-type curve with both eigenvalues different from zero if and only if α is a part of one of the following curves:*

(i) $\alpha(s) = (\rho\sin s, \epsilon\cos s + a\cos 3s, \epsilon\sin s + a\sin 3s)$, $\rho^2 - 12a\epsilon = 0$,
(ii) $\alpha(s) = (a\cosh s + \lambda b\sinh s - 4ce^{3\lambda s}, -b\cosh s - \lambda a\sinh s + 4ce^{3\lambda s}, 2de^{\lambda s})$, $d^2 - 6(a-b)c = 0$, $\lambda \in \{-1, 1\}$,
(iii) $\alpha(s) = (ae^s + b\cosh 3s, ae^s + b\sinh 3s, ce^{-s})$, $c^2 + 6ab = 0$,
(iv) $\alpha(s) = (\epsilon\cosh s + a\cosh 3s, \epsilon\sinh s + a\sinh 3s, \rho\cosh s)$, $\rho^2 + 12a\epsilon = 0$,
(v) $\alpha(s) = (\epsilon\cosh s + a\cosh 3s, \epsilon\sinh s + a\sinh 3s, \rho\sinh s)$, $\rho^2 + 12a\epsilon = 0$,
(vi) $\alpha(s) = (ae^s + b\sinh 3s, ae^s + b\cosh 3s, ce^{-s})$, $c^2 - 6ab = 0$,

(vii) $\alpha(s) = (\epsilon \sinh s + a \sinh 3s, \epsilon \cosh s + a \cosh 3s, \rho \cosh s)$, $\rho^2 - 12a\epsilon = 0$,
(viii) $\alpha(s) = (\epsilon \sinh s + a \sinh 3s, \epsilon \cosh s + a \cosh 3s, \rho \sinh s)$, $\rho^2 - 12a\epsilon = 0$,
where $a, b, c, d, \epsilon, \rho \in R_0$.

All closed 3-type curves in Euclidean 3-space E^3 were classified by Blair in [1]. He obtained the following classification theorem.

THEOREM 2.5. *A closed 3-type curve in E^3 is either a curve which lies on a quadric of revolution or a curve whose frequency ratio is $1:3:7$ and the curve belongs to a 3-parameter family of such curves, or the frequency ratio is $1:3:5$ and the curve belongs to a 5-parameter family of such curves. Some curves with frequency ratio $1:3:5$ or $1:3:7$ also lie on quadrics of revolution.*

3. A classification of all non-planar 3-type curves in $\mathbf{E}_1^3$

All planar 3-type curves in E_1^3 have been classified in the part I of this paper ([8]). Now we shall classify all non-planar spacelike and timelike 3-type curves in this space. For a non-planar 3-type curve α in E_1^3 all three eigenvalues of its Laplacian can be different from zero, or two of them can be different from zero and one of them equal to zero. In the second case, α is said to be of a null 3-type.

THEOREM 3.1. *A non-planar spacelike or timelike curve α in E_1^3 is a null 3-type curve if and only if its frequency ratio is $1:2$ and the curve belongs to one of three a 3-parameter families of such curves, or to one of three a 4-parameter families of such curves.*

Proof. Let $\alpha(s)$ be a non-planar null 3-type spacelike or timelike curve in E_1^3, parameterized by a pseudo-arclength parameter s. Then α can be written as:
(i) $\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cosh(ts) + f\sinh(ts)$,
(ii) $\alpha(s) = a + bs + c\cos(ps) + d\sin(ps) + e\cos(ts) + f\sin(ts)$,
(iii) $\alpha(s) = a + bs + c\cosh(ps) + d\sinh(ps) + e\cosh(ts) + f\sinh(ts)$,
where $0 < p < t$ and $a, b, c, d, e, f \in R^3$. Let $b, c, d, e, f \in R^3$ be of the form $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$, and so on. We may take up to a translation that $a = (0,0,0)$. In the sequel, we shall consider the cases (i), (ii) and (iii) separately. In all of them, we may take $p = 1$.

CASE (i). Since the functions $\sin x$, $\cos x$, $\sinh x$, $\cosh x$ are linearly independent, from the condition $g(\dot\alpha, \dot\alpha) = \pm 1$, we get the following system of equations:

$$g(b,b) + \tfrac{p^2}{2}(g(c,c) + g(d,d))7 + \tfrac{t^2}{2}(g(f,f) - g(e,e)) = \pm 1, \tag{1}$$
$$g(d,d) - g(c,c) = 0, \tag{2}$$
$$g(e,e) + g(f,f) = 0, \tag{3}$$
$$g(b,c) = g(b,d) = g(b,e) = g(b,f) = 0, \tag{4}$$
$$g(c,d) = g(c,e) = g(c,f) = 0, \tag{5}$$
$$g(d,e) = g(d,f) = 0, \tag{6}$$
$$g(e,f) = 0. \tag{7}$$

If vectors c, d, e and f are different from zero and not null vectors, then equations (2),(3),(5) and (6) imply that there are three mutually orthogonal spacelike vectors in E_1^3, which is impossible. So, there is a vector, say e, with the property $g(e,e)=0$. Now, equations (3) and (7) imply $f=\lambda e$, for some $\lambda \in R$. Taking $e=(e_1,e_1,0)$, $e_1 \neq 0$, equations (4), (5) and (6) imply $b=(b_1,b_1,b_3)$, $c=(c_1,c_1,c_3)$, $d=(d_1,d_1,d_3)$, while equations (2) and (5) imply $c=(c_1,c_1,0)$, $d=(d_1,d_1,0)$. Next (1) implies $b=(b_1,b_1,\pm 1)$ and consequently α lies in the plane $\mathrm{x}_1=\mathrm{x}_2$, which is a contradiction. Therefore, curve α of the form (i) does not exist.

CASE (ii). Since $g(\dot\alpha,\dot\alpha)=\pm 1$ and $0<p<t$, we distinguish the subcases:
(ii.1) $2p=t-p$; (ii.2) $2p=t$; (ii.3) $2p\neq t-p,t$.
In subcases (ii.1) and (ii.3), we obtain a contradiction.

(ii.2) $2p=t$. Then the corresponding system reads:

$$\begin{aligned}
&(1) && g(b,b)+\tfrac{p^2}{2}(g(c,c)+g(d,d))+\tfrac{t^2}{2}(g(e,e)+g(f,f))=\pm 1,\\
&(2) && \tfrac{p^2}{2}(g(d,d)-g(c,c))+2tg(b,f)=0,\\
&(3) && g(f,f)-g(e,e)=0,\\
&(4) && -2pg(b,c)+pt(g(c,f)-g(d,e))=0,\\
&(5) && 2pg(b,d)+pt(g(c,e)+g(d,f))=0,\\
&(6) && -2tg(b,e)-p^2g(c,d)=0,\\
&(7) && g(d,f)-g(c,e)=0,\\
&(8) && g(c,f)+g(d,e)=0,\\
&(9) && g(e,f)=0.
\end{aligned}$$

Now the equations (3) and (9) imply two possibilities: (ii.2.1) $g(e,e)=g(f,f)=0$, $g(e,f)=0$; (ii.2.2) $g(e,e)=g(f,f)>0$, $g(e,f)=0$. Again, we shall discuss these subcases separately.

(ii.2.1) In this subcase, we may take $e=(e_1,e_1,0)$, $e_1\neq 0$, so it follows that $f=\lambda e$, $\lambda\in R$. We may take $f=(f_1,f_1,0)$, $f_1\in R$. The equations (7) and (8) now imply $g(d,e)=g(c,e)=0$, so that $d=(d_1,d_1,d_3)$, $c=(c_1,c_1,c_3)$. Then the equation of the curve α reads:

$$\begin{aligned}
\alpha(s)=(&b_1s+c_1\cos(ps)+d_1\sin(ps)+e_1\cos(2ps)+f_1\sin(2ps),\\
&b_2s+c_1\cos(ps)+d_1\sin(ps)+e_1\cos(2ps)+f_1\sin(2ps),\\
&b_3s+c_3\cos(ps)+d_3\sin(ps)),
\end{aligned}$$

where $b_1\neq b_2$ and b_1, b_2, b_3, c_1, c_3, d_1, d_3, e_1, f_1 satisfy the conditions:

$$\begin{aligned}
&(1) && -b_1^2+b_2^2+b_3^2+\tfrac{1}{2}(c_3^2+d_3^2)=\pm 1,\\
&(2) && d_3^2-c_3^2+8f_1(b_2-b_1)=0,\\
&(4) && -b_1c_1+b_2c_1+b_3c_3=0,\\
&(5) && -b_1d_1+b_2d_1+b_3d_3=0,\\
&(6) && c_3d_3+4e_1(b_2-b_1)=0.
\end{aligned}$$

Next, we shall regard the numbers b_1, b_2, e_1 and f_1 as the parameters of a 4-parameter family of curves. The equation (6) implies that

$$c_3 = \frac{4e_1(b_1 - b_2)}{d_3}, \quad d_3 \neq 0,$$

which together with the equation (2) gives

$$d_3^4 + 8f_1(b_2 - b_1)d_3^2 - 16e_1^2(b_1 - b_2)^2 = 0.$$

If we put $d_3^2 = t$, we get that

$$t = 4(b_1 - b_2)(f_1 \pm \sqrt{f_1^2 + e_1^2}).$$

Thus we obtain that

$$d_3 = \pm 2\sqrt{(b_1 - b_2)(f_1 + \sqrt{f_1^2 + e_1^2})}, \quad (b_1 > b_2),$$

or else

$$d_3 = \pm 2\sqrt{(b_1 - b_2)(f_1 - \sqrt{f_1^2 + e_1^2})}, \quad (b_1 < b_2).$$

Therefore,

$$c_3 = \frac{2e_1(b_1 - b_2)}{\pm\sqrt{(b_1 - b_2)(f_1 + \sqrt{f_1^2 + e_1^2})}}, \quad (b_1 > b_2),$$

or else

$$c_3 = \frac{2e_1(b_1 - b_2)}{\pm\sqrt{(b_1 - b_2)(f_1 - \sqrt{f_1^2 + e_1^2})}}, \quad (b_1 < b_2).$$

The equation (1) implies that

$$b_3^2 = b_1^2 - b_2^2 - \tfrac{1}{2}(c_3^2 + d_3^2) \pm 1,$$

and the equations (4) and (5) give

$$c_1 = \frac{b_3 c_3}{b_1 - b_2}, \qquad d_1 = \frac{b_3 d_3}{b_1 - b_2}.$$

Therefore, we have expressed the solution b_3, c_1, d_1, c_3, d_3 of the above system of equations as the function of the parameters b_1, b_2, e_1 and f_1. Consequently, α belongs to a 4-parameter family of curves with frequency ratio $p : t = 1 : 2$.

(ii.2.2) In this subcase, we may take $e = (0, e_2, 0)$, $f = (0, 0, e_2)$, $e_2 \neq 0$. Equations (7) and (8) imply $c = (c_1, c_2, c_3)$ and $d = (d_1, -c_3, c_2)$, so that the curve α has the form

$$\begin{aligned}\alpha(s) = (&b_1 s + c_1 \cos(ps) + d_1 \sin(ps), b_2 s + c_2 \cos(ps) - c_3 \sin(ps) + e_2 \cos(2ps),\\ &b_3 s + c_3 \cos(ps) + c_2 \sin(ps) + e_2 \sin(2ps)),\end{aligned}$$

where b_1, b_2, b_3, c_1, c_2, c_3, d_1, e_2 satisfy the following equations:

$$(1)\qquad -b_1^2+b_2^2+b_3^2+\tfrac{1}{2}(2c_2^2+2c_3^2-c_1^2-d_1^2)+4e_2^2=\pm 1,$$
$$(2)\qquad c_1^2-d_1^2+8b_3e_2=0,$$
$$(4)\qquad b_1c_1-b_2c_2-b_3c_3+2c_3e_2=0,$$
$$(5)\qquad -b_1d_1-b_2c_3+b_3c_2+2c_2e_2=0,$$
$$(6)\qquad c_1d_1-4b_2e_2=0.$$

Therefore, α belongs to a 3-parameter family of curves with frequency ratio $p:t=1:2$.

CASE (iii) Since $g(\dot\alpha,\dot\alpha)=\pm 1$ and $0<p<t$, we shall distinguish the subcases: (iii.1) $2p=t-p$; (iii.2) $2p=t$; (iii.3) $2p\neq t-p,t$. It is easy to see that in subcases (iii.1) and (iii.3) we get a contradiction.

(iii.2) $2p=t$. Then the corresponding system reads:

$$(1)\qquad g(b,b)+\tfrac{p^2}{2}(g(d,d)-g(c,c))+\tfrac{t^2}{2}(g(f,f)-g(e,e))=\pm 1,$$
$$(2)\qquad \tfrac{p^2}{2}(g(c,c)+g(d,d))+2tg(b,f)=0,$$
$$(3)\qquad g(e,e)+g(f,f)=0,$$
$$(4)\qquad 2pg(b,c)+pt(g(d,e)-g(c,f))=0,$$
$$(5)\qquad 2pg(b,d)+pt(g(d,f)-g(c,e))=0,$$
$$(6)\qquad 2tg(b,e)+p^2g(c,d)=0,$$
$$(7)\qquad g(c,e)+g(d,f)=0,$$
$$(8)\qquad g(c,f)+g(d,e)=0,$$
$$(9)\qquad g(e,f)=0.$$

Equations (3) and (9) imply three possibilities:

(iii.2.1) $g(e,e)=g(f,f)=0,\quad g(e,f)=0$;
(iii.2.2) $g(e,e)=-g(f,f)>0$, $g(e,f)=0$;
(iii.2.3) $g(e,e)=-g(f,f)<0$, $g(e,f)=0$;

We shall again discuss all these subcases separately.

(iii.2.1) In this subcase, we may take $e=(e_1,e_1,0)$, $e_1\neq 0$, $f=\lambda e$, $\lambda\in R$. Equations (7) and (8) imply $(1-\lambda^2)g(d,e)=0$ and we shall distinguish the subcases: (iii.2.1.1) $g(d,e)=0$, $\lambda^2\neq 1$; (iii.2.1.2) $\lambda^2=1$, $g(d,e)\neq 0$.

(iii.2.1.1) From $g(d,e)=0$, it follows that $d=(d_1,d_1,d_3)$, while (8) implies that $c=(c_1,c_1,c_3)$. Therefore, α has the form:

$$\alpha(s)=(b_1,b_2,b_3)s+(c_1,c_1,c_3)\cosh(s)+(d_1,d_1,d_3)\sinh(s)$$
$$+(e_1,e_1,0)\cosh(2s)+\lambda(e_1,e_1,0)\sinh(2s),$$

where $b_1 \neq b_2$ and b_1, b_2, b_3, c_1, c_3, d_1, d_3, e_1, λ satisfy

$$\begin{aligned}
&(1) && -b_1^2 + b_2^2 + b_3^2 + \tfrac{1}{2}(d_3^2 - c_3^2) = \pm 1,\\
&(2) && c_3^2 + d_3^2 + 8\lambda e_1(b_2 - b_1) = 0,\\
&(4) && -b_1c_1 + b_2c_1 + b_3c_3 = 0,\\
&(5) && -b_1d_1 + b_2d_1 + b_3d_3 = 0,\\
&(6) && c_3d_3 + 4e_1(b_2 - b_1) = 0.
\end{aligned}$$

Consequently, α belongs to a 4-parameter family of curves with frequency ratio $p : t = 1 : 2$.

(iii.2.1.2) Then the equations (2) and (6) imply $g(c - \lambda d, c - \lambda d) = 0$. The vectors e and $c - \lambda d$ are linear independent null vectors, so we may take $e = (-e_1, e_1, 0)$, $e_1 \neq 0$, $c - \lambda d = (n_1, n_1, 0)$, $n_1 \neq 0$. Equation (7) now implies that $c + \lambda d = (m_1, -m_1, m_3)$, whence $c = \frac{1}{2}(m_1 + n_1, n_1 - m_1, m_3)$ and $d = \frac{\lambda}{2}(m_1 - n_1, -m_1 - n_1, m_3)$. Next equations (4) and (5) imply $b = (b_1, b_1, b_3)$, so that the curve α has the form:

$$\begin{aligned}
\alpha(s) = (b_1, b_1, b_3)s &+ \tfrac{1}{2}(m_1 + n_1, n_1 - m_1, m_3)\cosh(s)\\
&+ \tfrac{\lambda}{2}(m_1 - n_1, -m_1 - n_1, m_3)\sinh(s)\\
&+ (-e_1, e_1, 0)\cosh(2s) + \lambda(-e_1, e_1, 0)\sinh(2s),
\end{aligned}$$

where $\lambda^2 = 1$ and b_1, b_3, m_1, n_1, m_3, e_1 satisfy the equations

$$\begin{aligned}
&(1) && b_3^2 + m_1n_1 = \pm 1,\\
&(5) && -2b_1m_1 + b_3m_3 - 4\lambda n_1e_1 = 0,\\
&(6) && m_3^2 + 32\lambda b_1e_1 = 0.
\end{aligned}$$

Consequently, α belongs to a 3-parameter family of curves with frequency ratio $p : t = 1 : 2$.

(iii.2.2) In this subcase, we may take $e = (0, e_2, 0)$, $f = (e_2, 0, 0)$, $e_2 \neq 0$. Now the equations (7) and (8) imply $c = (c_1, c_2, c_3)$ and $d = (c_2, c_1, d_3)$, so that α has the form:

$$\begin{aligned}
\alpha(s) = (b_1, b_2, b_3)s &+ (c_1, c_2, c_3)\cosh(s) + (c_2, c_1, d_3)\sinh(s)\\
&+ (0, e_2, 0)\cosh(2s) + (e_2, 0, 0)\sinh(2s),
\end{aligned}$$

where b_1, b_2, b_3, c_1, c_2, c_3, d_3, e_2 satisfy

$$\begin{aligned}
&(1) && -b_1^2 + b_2^2 + b_3^2 + \tfrac{1}{2}(2c_1^2 - 2c_2^2 + d_3^2 - c_3^2) - 4e_2^2 = \pm 1,\\
&(2) && c_3^2 + d_3^2 - 8b_1e_2 = 0,\\
&(5) && -b_1c_2 + b_2c_1 + b_3d_3 - 2c_2e_2 = 0,\\
&(6) && c_3d_3 + 4b_2e_2 = 0.
\end{aligned}$$

Hence the curve α belongs to a 4-parameter family of curves with frequency ratio $p : t = 1 : 2$.

(iii.2.3) In this subcase, we may take $e = (e_1, 0, 0)$, $f = (0, e_1, 0)$, $e_1 \neq 0$. Equations (7) and (8) imply $c = (c_1, c_2, c_3)$, $d = (c_2, c_1, d_3)$, so that α has the form:

$$\alpha(s) = (b_1, b_2, b_3)s + (c_1, c_2, c_3)\cosh(s) + (c_2, c_1, d_3)\sinh(s) + (e_1, 0, 0)\cosh(2s) + (0, e_1, 0)\sinh(2s),$$

where b_1, b_2, b_3, c_1, c_2, c_3, d_3, e_1 satisfy the relations

$$-b_1^2 + b_2^2 + b_3^2 + \tfrac{1}{2}(2c_1^2 - 2c_2^2 + d_3^2 - c_3^2) + 4e_1^2 = \pm 1, \tag{1}$$

$$c_3^2 + d_3^2 + 8b_2e_1 = 0, \tag{2}$$

$$-b_1c_1 + b_2c_2 + b_3c_3 - 2c_2e_1 = 0, \tag{4}$$

$$-b_1c_2 + b_2c_1 + b_3d_3 + 2c_1e_1 = 0, \tag{5}$$

$$c_3d_3 - 4b_1e_1 = 0. \tag{6}$$

It follows that α belongs to a 3-parameter family of curves with frequency ratio $p : t = 1 : 2$. This completes the proof of Theorem 3.1. $\square$

In the sequel, let $\alpha(s)$ be a 3-type curve in E_1^3 of the form

(A) $\alpha(s) =$

$$a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts) + f\cos(qs) + h\sin(qs),$$

or of the form

(B) $\alpha(s) =$

$$a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts) + f\cosh(qs) + h\sinh(qs).$$

Then it is easy to prove that the following two Lemmas hold.

LEMMA 3.1. *For a non-planar 3-type spacelike or timelike curve α in E_1^3, of the form (A) or (B), we have $q \neq 3t$.*

LEMMA 3.2. *For a non-planar 3 type spacelike or timelike curve α in E_1^3, of the form (A) or (B), at least one of the following possibilities holds:*
(I) $2t = q - p$, (II) $2t = p + q$, (III) $2p = q - t$.

A non-planar curves in E_1^3 with all three eigenvalues different from zero, are characterized by the following theorem.

THEOREM 3.2.. *A non-planar 3-type spacelike or timelike curve α in E_1^3 with all three eigenvalues of its Laplacian Δ different from zero, is either a curve which lies on a quadric of revolution in E_1^3, or it belongs to one a 4-parameter or to one of two a 2-parameter families of curves with frequency ratio $1:3:7$, or it belongs to one of three a 4-parameter or to one of two a 5-parameter families of curves with* *frequency ratio* $1:3:5$, *or it belongs to one of three a 2-parameter or to one of two a 3-parameter families of curves with frequency ratio* $1:2:3$.

Proof. Let $\alpha(s)$ be a non-planar 3-type spacelike or timelike curve in E_1^3, parameterized by a pseudo-arclength parameter s. Suppose that all three eigenvalues of its Laplacian Δ are different from zero. Then α can be written as:

$$\alpha(s) = a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts) + f\cosh(qs) + h\sinh(qs), \tag{i}$$

$$\alpha(s) = a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts) + f\cos(qs) + h\sin(qs), \tag{ii}$$

$$\alpha(s) = a + b\cos(ps) + c\sin(ps) + d\cos(ts) + e\sin(ts) + f\cos(qs) + h\sin(qs), \tag{iii}$$

$$\alpha(s) = a + b\cosh(ps) + c\sinh(ps) + d\cosh(ts) + e\sinh(ts) + f\cosh(qs) + h\sinh(qs), \tag{iv}$$

where $0 < p < t < q$ and $a, b, c, d, e, f, h \in R^3$. Let $b, c, d, e, f, h \in R^3$ be of the form $b = (b_1, b_2, b_3)$, $c = (c_1, c_2, c_3)$, and so on. We may take up to a translation that $a = (0, 0, 0)$. In the sequel, we shall distinguish the cases (i), (ii), (iii) and (iv). In all these cases, we may take $p = 1$.

CASES (i) AND (ii). Using the same methods as in Theorem 3.1 and distinguishing the subcases $t - p = 2p$, $t - p \neq 2p$, we find that a curve α in E_1^3 of such forms does not exist.

CASE (iii). The corresponding proof follows the cases of Lemma 3.2 and the same methods as in Theorem 3.1, so we distinguish the subcases: (iii.1) $2t = q - p$; (iii.2) $2t = p + q$; (iii.3) $2p = q - t$.

(iii.1) $2t = q - p$. It follows that $q - t = t + p$. Then we shall also distinguish the subcases: (iii.1.1) $t - p = 2p$; (iii.1.2) $t - p \neq 2p$.

(iii.1.1) $2p = t - p$. It follows that $p : t : q = 1 : 3 : 7$. Assuming that $g(f, f) = g(h, h) > 0$, we find that the curve α has the form

$$\begin{aligned}\alpha(s) = (&b_1\cos(s) + c_1\sin(s) + d_1\cos(3s) + e_1\sin(3s),\\ &b_2\cos(s) - b_3\sin(s) + e_3\cos(3s) + e_2\sin(3s) + f_2\cos(7s),\\ &b_3\cos(s) + b_2\sin(s) - e_2\cos(3s) + e_3\sin(3s) + f_2\sin(7s)),\end{aligned}$$

where b_1, b_2, b_3, c_1, d_1, e_1, e_2, e_3, f_2 satisfy the relations

$$\tfrac{1}{2}(-b_1^2 - c_1^2 + 2b_2^2 + 2b_3^2) + \tfrac{9}{2}(-e_1^2 - d_1^2 + 2(e_2^2 + e_3^2)) + 49f_2^2 = \pm 1, \tag{1}$$

$$b_1^2 - c_1^2 + 6(-b_1d_1 - c_1e_1 + 2(b_2e_3 - b_3e_2)) = 0, \tag{2}$$

$$9(d_1^2 - e_1^2) + 28b_2f_2 = 0, \tag{3}$$

$$-b_1c_1 + 3(b_1e_1 - c_1d_1 - 2(b_3e_3 + b_2e_2)) = 0, \tag{4}$$

$$b_1d_1 - c_1e_1 + 14f_2e_3 = 0, \tag{5}$$

$$(6) \qquad -b_1e_1 - c_1d_1 + 14f_2e_2 = 0,$$
$$(7) \qquad 14b_3f_2 + 9d_1e_1 = 0.$$

Hence we conclude that the curve α belongs to a 2-parameter family of curves. One set of the solutions of the above system of equations is $b_1 = c_1 = e_2 == e_3 = 0$, with b_2, b_3, d_1, e_1, f_2 related by

$$(1) \qquad \tfrac{1}{2}(2b_2^2 + 2b_3^2) + \tfrac{9}{2}(-e_1^2 - d_1^2) + 49f_2^2 = \pm 1,$$
$$(3) \qquad 9(d_1^2 - e_1^2) + 28b_2f_2 = 0,$$
$$(7) \qquad 14b_3f_2 + 9d_1e_1 = 0,$$

so we find that α has the form

$$\alpha(s) = (d_1\cos(3s) + e_1\sin(3s), b_2\cos(s) - b_3\sin(s) + f_2\cos(7s),$$
$$b_3\cos(s) + b_2\sin(s) + f_2\sin(7s)).$$

Thus this curve lies on the quadric

$$9x^2 + 7(y^2 + z^2) = \tfrac{9}{2}(d_1^2 + e_1^2) + 7(b_2^2 + b_3^2 + f_2^2).$$

Consequently, some of the curves with frequency ratio $p : t : q = 1 : 3 : 7$, lie on a quadric in E_1^3. In the sequel, assuming that $g(f, f) = g(h, h) = 0$, we obtain a contradiction.

(iii.1.2) $t - p \neq 2p$. Then we find that the curve α has the form

$$\alpha(s) = (d_1\cos(ts) + e_1\sin(ts), b_2\cos(ps) - b_3\sin(ps) + f_2\cos(qs),$$
$$b_3\cos(ps) + b_2\sin(ps) + f_2\sin(qs)),$$

where b_2, b_3, d_1, e_1, f_2 satisfy the relations

$$(1) \qquad t^2(d_1^2 - e_1^2) + 4pqb_2f_2 = 0,$$
$$(2) \qquad t^2d_1e_1 + 2pqb_3f_2 = 0,$$
$$(3) \qquad p^2(b_2^2 + b_3^2) + \tfrac{t^2}{2}(-e_1^2 - d_1^2) + q^2f_2^2 = \pm 1.$$

It follows that α lies on the quadric

$$t^2x^2 + pq(y^2 + z^2) = \tfrac{t^2}{2}(d_1^2 + e_1^2) + pq(b_2^2 + b_3^2 + f_2^2).$$

(iii.2) $2t = p + q$. It follows that $q - t = t - p$. We shall distinguish the subcases: (iii.2.1) $q - t = t - p = 2p$; (iii.2.2) $q - t = t - p \neq 2p$.

(iii.2.1) $q - t = t - p = 2p$. It follows that $p : t : q = 1 : 3 : 5$. If $g(f, f) = g(h, h) > 0$, we obtain that the curve α has the form:

$$\alpha(s) = (b_1\cos(s) + c_1\sin(s) + d_1\cos(3s) + e_1\sin(3s),$$
$$b_2\cos(s) + c_2\sin(s) + e_3\cos(3s) + e_2\sin(3s) + f_2\cos(5s),$$
$$b_3\cos(s) + c_3\sin(s) - e_2\cos(3s) + e_3\sin(3s) + f_2\sin(5s)),$$

where b_1, b_2, b_3, c_1, c_2, c_3, d_1, e_1, e_2, e_3, f_2 satisfy the relations

$$\begin{aligned}
&(1) \quad -b_1^2 + b_2^2 + b_3^2 - c_1^2 + c_2^2 + c_3^2 + 9(2(e_2^2 + e_3^2) - e_1^2 - d_1^2) + 50f_2^2 = \pm 2,\\
&(2) \quad -c_1^2 + c_2^2 + c_3^2 + b_1^2 - b_2^2 - b_3^2\\
&\qquad\qquad +6(-b_1 d_1 + b_2 e_3 - b_3 e_2 - c_1 e_1 + c_2 e_2 + c_3 e_3) + 60 e_3 f_2 = 0,\\
&(3) \quad 9(d_1^2 - e_1^2) + 10 f_2(c_3 - b_2) = 0,\\
&(4) \quad b_1 c_1 - b_2 c_2 - b_3 c_3\\
&\qquad\qquad -3(-c_1 d_1 + c_2 e_3 - c_3 e_2 + b_1 e_1 - b_2 e_2 - b_3 e_3) + 30 e_2 f_2 = 0,\\
&(5) \quad 3(-c_1 e_1 + c_2 e_2 + c_3 e_3 + b_1 d_1 - b_2 e_3 + b_3 e_2) + 5 f_2(b_2 + c_3) = 0,\\
&(6) \quad 3(-b_1 e_1 + b_2 e_2 + b_3 e_3 - c_1 d_1 + c_2 e_3 - c_3 e_2) + 5 f_2(c_2 - b_3) = 0,\\
&(7) \quad 5 f_2(b_3 + c_2) - 9 d_1 e_1 = 0.
\end{aligned}$$

One set of solutions of the above system of equations is $b_1 = c_1 = e_2 = e_3 = 0$ with $b_2 = -c_3$, $b_3 = c_2$, d_1, e_1, f_2 related by

$$\begin{aligned}
&(1) \quad 2b_2^2 + 2b_3^2 - 9(d_1^2 + e_1^2) + 50 f_2^2 = \pm 2,\\
&(3) \quad 9(d_1^2 - e_1^2) - 20 b_2 f_2 = 0,\\
&(7) \quad 10 b_3 f_2 - 9 d_1 e_1 = 0,
\end{aligned}$$

where d_1 and e_1 are not both 0 and $f_2 \neq 0$. So we get that α has the form

$$\alpha(s) = (d_1 \cos(3s) + e_1 \sin(3s), b_2 \cos(s) + b_3 \sin(s) + f_2 \cos(5s),\\
b_3 \cos(s) - b_2 \sin(s) + f_2 \sin(5s)),$$

thus it lies on the quadric

$$9x^2 - 5(y^2 + z^2) = \tfrac{9}{2}(d_1^2 + e_1^2) - 5(b_2^2 + b_3^2 + f_2^2).$$

Hence, α belongs to a 4-parameter family of curves with frequency ratio $p : t : q = 1 : 3 : 5$. Next assuming that $g(f, f) = g(h, h) = 0$, it can be proved that α belongs to a 5-parameter family of curves with frequency ratio $p : t : q = 1 : 3 : 5$.

(iii.2.2) $q - t = t - p \neq 2p$. Now we shall distinguish the subcases:

(iii.2.2.1) $q - p = 2p$; (iii.2.2.2) $q - p \neq 2p$.

(iii.2.2.1) $q - p = 2p$. It follows that $p : t : q = 1 : 2 : 3$. Then we get that α belongs to a 2-parameter or to a 3-parameter family of curves with frequency ratio $p : t : q = 1 : 2 : 3$. It is easy to prove that some of these curves lie on quadrics.

(iii.2.2.2) $q - p \neq 2p$. Then we get that α has the form

$$\alpha(s) = (d_1 \cos(ts) + e_1 \sin(ts), b_2 \cos(ps) + b_3 \sin(ps) + f_2 \cos(qs),\\
b_3 \cos(ps) - b_2 \sin(ps) + f_2 \sin(qs)),$$

where b_2, b_3, d_1, e_1, f_2 satisfy the relations

$$p^2(b_2^2+b_3^2)+\tfrac{t^2}{2}(-e_1^2-d_1^2)+q^2f_2^2=\pm1, \tag{1}$$
$$t^2(d_1^2-e_1^2)-4pqb_2f_2=0, \tag{2}$$
$$-2pqb_3f_2+t^2d_1e_1=0. \tag{3}$$

Thus we obtain that α lies on the quadric

$$t^2x^2-pq(y^2+z^2)=\tfrac{t^2}{2}(e_1^2+d_1^2)-pq(b_2^2+b_3^2+f_2^2).$$

(iii.3) $2p=q-t$. It follows that $q-p=p+t$. Now we shall distinguish the subcases: (iii.3.1) $2p=q-t=t-p$; (iii.3.2) $2p=q-t\neq t-p$.

(iii.3.1) $2p=q-t=t-p$. This subcase is equivalent to the subcase (iii.2.1), which was already considered.

(iii.3.2) $2p=q-t\neq t-p$. In this subcase, we obtain that α has the form

$$\begin{aligned}\alpha(s)=(b_1\cos(ps)+c_1\sin(ps),e_3\cos(ts)+e_2\sin(ts)+f_2\cos(qs),\\ -e_2\cos(ts)+e_3\sin(ts)+f_2\sin(qs)),\end{aligned}$$

where b_1, c_1, e_2, e_3, f_2 satisfy the relations

$$\tfrac{p^2}{2}(-b_1^2-c_1^2)+t^2(e_2^2+e_3^2)+q^2f_2^2=\pm1, \tag{1}$$
$$p^2(b_1^2-c_1^2)+4tqe_3f_2=0, \tag{2}$$
$$p^2b_1c_1-2tqe_2f_2=0. \tag{3}$$

Hence, α lies on the quadric

$$p^2x^2+tq(y^2+z^2)=\tfrac{p^2}{2}(b_1^2+c_1^2)+tq(e_2^2+e_3^2+f_2^2).$$

Case (iv). The corresponding proof follows the cases of Lemma 3.2 and the same methods as in Theorem 3.1. Hence we shall distinguish the subcases: (iv.1) $2t=q-p$; (iv.2) $2t=p+q$; (iv.3) $2p=q-t$.

(iv.1) $2t=q-p$. It follows that $q-t=t+p$. In this subcase, we shall consider the subcases: (iv.1.1) $t-p=2p$; (iv.1.2) $t-p\neq 2p$.

(iv.1.1) $2p=t-p$. It follows that $p:t:q=1:3:7$. Assuming that $g(f,f)=-g(h,h)>0$, we find that α has the form:

$$\begin{aligned}\alpha(s)=(&b_1\cosh(s)+b_2\sinh(s)+e_2\cosh(3s)+e_1\sinh(3s)+f_2\sinh(7s),\\ &b_2\cosh(s)+b_1\sinh(s)+e_1\cosh(3s)+e_2\sinh(3s)+f_2\cosh(7s),\\ &b_3\cosh(s)+c_3\sinh(s)+d_3\cosh(3s)+e_3\sinh(3s)),\end{aligned}$$

where b_1, b_2, b_3, c_3, e_1, e_2, e_3, d_3, f_2 satisfy the relations

$$
\begin{aligned}
&(1) && \tfrac{1}{2}(2(b_1^2-b_2^2)+c_3^2-b_3^2)+\tfrac{9}{2}(2(e_2^2-e_1^2)+e_3^2-d_3^2)-49f_2^2=\pm 1,\\
&(2) && b_3^2+c_3^2+6(c_3e_2-b_3d_3+2(b_1-b_2)e_1)=0,\\
&(3) && 9(d_3^2+e_3^2)-28b_2f_2=0,\\
&(4) && b_3c_3+3(c_3d_3-b_3e_3+2(b_1e_1-b_2e_2))=0,\\
&(5) && b_3d_3+c_3e_3-14e_1f_2=0,\\
&(6) && c_3d_3+b_3e_3+14e_2f_2=0,\\
&(7) && 9e_3d_3+14b_1f_2=0.
\end{aligned}
$$

Therefore, α belongs to a 2-parameter family of curves. One set of solutions of the above system of equations is $e_1=e_2=b_3=c_3=0$, with b_1, b_2, e_3, d_3, f_2 related by

$$
\begin{aligned}
&(1) && b_1^2-b_2^2+\tfrac{9}{2}(e_3^2-d_3^2)-49f_2^2=\pm 1,\\
&(3) && 9(d_3^2+e_3^2)-28b_2f_2=0,\\
&(7) && 9e_3d_3+14b_1f_2=0.
\end{aligned}
$$

So we get that α has the form

$$
\begin{aligned}
\alpha(s)=(&b_1\cosh(s)+b_2\sinh(s)+f_2\sinh(7s),\\
&b_2\cosh(s)+b_1\sinh(s)+f_2\cosh(7s),\ d_3\cosh(3s)+e_3\sinh(3s)),
\end{aligned}
$$

where d_3 and e_3 are not both 0, $f_2\neq 0$. Hence, α lies on the quadric

$$7(x^2-y^2)+9z^2=7(b_1^2-b_2^2-f_2^2)+\tfrac{9}{2}(d_3^2-e_3^2).$$

Consequently, some of the curves with frequency ratio $p:t:q=1:3:7$ lie on quadrics. Further, assuming that $g(f,f)=-g(h,h)<0$ or $g(f,f)=g(h,h)=0$, it can be proved that α belongs respectively to a 2-parameter or a 4-parameter family of curves with frequency ratio $p:t:q=1:3:7$.

(iv.1.2) $t-p\neq 2p$. Then we find that α has the form

$$
\begin{aligned}
\alpha(s)=(&b_1\cosh(ps)+b_2\sinh(ps)+f_2\sinh(qs),\\
&b_2\cosh(ps)+b_1\sinh(ps)+f_2\cosh(qs),\ d_3\cosh(ts)+e_3\sinh(ts)),
\end{aligned}
$$

where b_1, b_2, d_3, e_3, f_2 satisfy the relations

$$
\begin{aligned}
&(1) && p^2(b_1^2-b_2^2)+\tfrac{t^2}{2}(e_3^2-d_3^2)-q^2f_2^2=\pm 1,\\
&(2) && t^2(e_3^2+d_3^2)-4pqb_2f_2=0,\\
&(3) && 2pqb_1f_2+t^2e_3d_3=0,
\end{aligned}
$$

d_3 and e_3 are not both 0, $f_2 \neq 0$. Hence we find that α lies on the quadric

$$pq(x^2 - y^2) + t^2 z^2 = pq(b_1^2 - b_2^2 - f_2^2) + \tfrac{t^2}{2}(d_3^2 - e_3^2).$$

(iv.2) $2t = p + q$. It follows that $q - t = t - p$. In this subcase, we shall distinguish the subcases: (iv.2.1) $q-t = t-p = 2p$; (iv.2.2) $q-t = t-p \neq 2p$.

(iv.2.1) $q - t = t - p = 2p$. It follows that $p : t : q = 1 : 3 : 5$. In this subcase, assuming that $g(f, f) = -g(h, h) > 0$, we find that α has the form:

$$\begin{aligned}\alpha(s) = (&b_1 \cosh(s) + c_1 \sinh(s) + e_2 \cosh(3s) + e_1 \sinh(3s) + f_2 \sinh(5s),\\ &b_2 \cosh(s) + c_2 \sinh(s) + e_1 \cosh(3s) + e_2 \sinh(3s) + f_2 \cosh(5s),\\ &b_3 \cosh(s) + c_3 \sinh(s) + d_3 \cosh(3s) + e_3 \sinh(3s)),\end{aligned}$$

where b_1, b_2, b_3, c_1, c_2, c_3, d_3, e_1, e_2, e_3, f_2 satisfy the relations

$$\begin{aligned}
&(1) && \tfrac{1}{2}(-c_1^2 + c_2^2 + c_3^2 + b_1^2 - b_2^2 - b_3^2) + \tfrac{9}{2}(2(e_2^2 - e_1^2) + e_3^2 - d_3^2) - 25f_2^2 = \pm 1,\\
&(2) && \tfrac{1}{2}(-c_1^2 + c_2^2 + c_3^2 - b_1^2 + b_2^2 + b_3^2)\\
& && \qquad +3(-c_1e_1 + c_2e_2 + c_3e_3 + b_1e_2 - b_2e_1 - b_3d_3) - 30f_2e_1 = 0,\\
&(3) && 9(e_3^2 + d_3^2) + 10f_2(b_2 - c_1) = 0,\\
&(4) && -b_1c_1 + b_2c_2 + b_3c_3\\
& && \qquad +3(-c_1e_2 + c_2e_1 + c_3d_3 + b_1e_1 - b_2e_2 - b_3e_3) + 30e_2f_2 = 0,\\
&(5) && 3(-b_1e_2 + b_2e_1 + b_3d_3 - c_1e_1 + c_2e_2 + c_3e_3) - 5f_2(c_1 + b_2) = 0,\\
&(6) && 3(-c_1e_2 + c_2e_1 + c_3d_3 - b_1e_1 + b_2e_2 + b_3e_3) + 5f_2(c_2 + b_1) = 0,\\
&(7) && 9e_3d_3 + 5f_2(c_2 - b_1) = 0.
\end{aligned}$$

Therefore, α belongs to a 4-parameter family of curves. One set of solutions of the above system of equations is $b_3 = c_3 = e_1 = e_2 = 0$, with b_1, b_2, c_1, c_2, e_3, d_3, f_2 related by

$$\begin{aligned}
&(1) && c_1 = -b_2,\\
&(2) && c_2 = -b_1,\\
&(3) && 9e_3d_3 - 10b_1f_2 = 0,\\
&(4) && 9(e_3^2 + d_3^2) + 20b_2f_2 = 0,\\
&(5) && b_1^2 - b_2^2 + \tfrac{9}{2}(e_3^2 - d_3^2) - 25f_2^2 = \pm 1,
\end{aligned}$$

d_3 and e_3 not both 0 and $f_2 \neq 0$. Hence we get that α has the form

$$\begin{aligned}\alpha(s) = (&b_1 \cosh(s) - b_2 \sinh(s) + f_2 \sinh(5s),\\ &b_2 \cosh(s) - b_1 \sinh(s) + f_2 \cosh(5s),\ d_3 \cosh(3s) + e_3 \sinh(3s)),\end{aligned}$$

thus it lies on the quadric

$$-5(x^2 - y^2) + 9z^2 = -5(b_1^2 - b_2^2 - f_2^2) + \tfrac{9}{2}(d_3^2 - e_3^2).$$

Further, assuming that $g(f,f) = g(h,h) = 0$ or $g(f,f) = -g(h,h) < 0$, it can be proved that α belongs respectively to a 5-parameter or a 4-parameter family of curves with frequency ratio $p : t : q = 1 : 3 : 5$.

(iv.2.2) $q - t = t - p \neq 2p$. Next we distinguish the subcases: (iv.2.2.1) $q - p = 2p$; (iv.2.2.2) $q - p \neq 2p$.

(iv.2.2.1) $q - p = 2p$. It follows that $p : t : q = 1 : 2 : 3$. In this subcase, we find that α belongs to one of two a 2-parameter families of curves or to a 3-parameter family of curves with frequency ratio $1 : 2 : 3$. It is easy to prove that some of them lie on quadrics. (iv.2.2.2) $q - p \neq 2p$. In this subcase, we find that α has the form:

$$\alpha(s) = (b_1 \cosh(ps) - b_2 \sinh(ps) + f_2 \sinh(qs),\\ b_2 \cosh(ps) - b_1 \sinh(ps) + f_2 \cosh(qs),\ d_3 \cosh(ts) + e_3 \sinh(ts)),$$

where b_1, b_2, d_3, e_3, f_2 satisfy the relations

$$(1) \qquad p^2(b_1^2 - b_2^2) + \tfrac{t^2}{2}(e_3^2 - d_3^2) - q^2 f_2^2 = \pm 1,$$

$$(2) \qquad t^2(e_3^2 + d_3^2) + 4pqb_2 f_2 = 0,$$

$$(3) \qquad t^2 d_3 e_3 - 2pqb_1 f_2 = 0,$$

d_3 and e_3 are not both zero, b_1 and b_2 are not both zero and $f_2 \neq 0$. Therefore, α lies on the quadric

$$-pq(x^2 - y^2) + t^2 z^2 = -pq(b_1^2 - b_2^2 - f_2^2) + \tfrac{t^2}{2}(d_3^2 - e_3^2).$$

(iv.3) $2p = q - t$. It follows that $q - p = p + t$. Now, we shall distinguish the subcases: (iv.3.1) $2p = q - t = t - p$; (iv.3.2) $2p = q - t \neq t - p$.

(iv.3.1) $2p = q - t = t - p$. This subcase is equivalent to the subcase (iv.2.1), which was already considered.

(iv.3.2) $2p = q - t \neq t - p$. Assuming that $g(f,f) = -g(h,h) > 0$, we find that α has the form

$$\alpha(s) = (e_2 \cosh(ts) + e_1 \sinh(ts) + f_2 \sinh(qs),\\ e_1 \cosh(ts) + e_2 \sinh(ts) + f_2 \cosh(qs),\ b_3 \cosh(ps) + c_3 \sinh(ps)),$$

where b_3, c_3, e_1, e_2, f_2 satisfy the relations

$$(1) \qquad \tfrac{p^2}{2}(c_3^2 - b_3^2) + t^2(e_2^2 - e_1^2) - q^2 f_2^2 = \pm 1,$$

$$(2) \qquad p^2(b_3^2 + c_3^2) - 4tqe_1 f_2 = 0,$$

$$(3) \qquad p^2 b_3 c_3 + 2tqe_2 f_2 = 0,$$

b_3 and c_3 are not both zero, e_1 and e_2 are not both zero and $f_2 \neq 0$. Hence, α lies on the quadric

$$tq(x^2 - y^2) + p^2 z^2 = tq(e_2^2 - e_1^2 - f_2^2) + \tfrac{p^2}{2}(b_3^2 - c_3^2).$$

This completes the proof of the Theorem 3.2. $\square$

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