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# DOMINATING PROPERTIES OF STAR COMPLEMENTS

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ABSTRACT. Let $G$ be a finite graph with an eigenvalue $\mu$ of multiplicity $m$. A set $X$ of $m$ vertices in $G$ is called a *star set* for $\mu$ in $G$ if $\mu$ is not an eigenvalue of the *star complement* $G - X$. Various dominating properties of the vertices in $G-X$ are established and discussed in the context of memoryless communication networks.

## 1. Introduction

Let $G$ be a finite simple graph with vertex set $V = \{1, 2, \dots, n\}$ and $(0,1)$-adjacency matrix $A$. Let $\mu$ be an eigenvalue of $G$, and let $P$ be the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$, the corresponding eigenspace of $A$. Then $\mathcal{E}(\mu)$ is spanned by $P\mathbf{e}_1, P\mathbf{e}_2, \dots, P\mathbf{e}_n$, where $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. The subset $X$ of $V$ is called a *star set* for $\mu$ in $G$ if the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$, equivalently [**5**, Theorem 7.2.9] if $|X| = \dim \mathcal{E}(\mu)$ and $\mu$ is not an eigenvalue of $G - X$. In this situation the graph $G - X$ is called a *star complement* for $\mu$; it is the subgraph of $G$ induced by $\overline{X}$, where $\overline{X} = V \smallsetminus X$, and this note is concerned with dominating properties of such a set $\overline{X}$. It is shown in [**5**, Chapter 7] that if $\mu \neq 0$ then $\overline{X}$ is a dominating set in $G$; moreover if $\mu \notin \{-1, 0\}$ then $\overline{X}$ is a location-dominating set, meaning that the $\overline{X}$-neighbourhoods in vertices in $X$ are distinct and non-empty. Here we generalize these results and note the consequences for memoryless communication networks of the type discussed in [**1**]. We suppose throughout that $G$ has no isolated vertices and (for $v \in V$) we write $W_k(v)$ for the set of vertices reachable from $v$ by a walk of length $k$. We write $G_k$ for the graph with vertex set $V$ in which distinct vertices $u, v$ are adjacent if and only if there exists a $u$-$v$ walk of length $k$ in $G$. We say that the subset $D$ of $V$ is a *$k$-dominating set* if $W_k(v) \cap D \neq \emptyset$ for all $v \in \overline{D}$, equivalently if $D$ is a dominating set in $G_k$; and we define $\beta_k(G)$ to be the least size of a $k$-dominating

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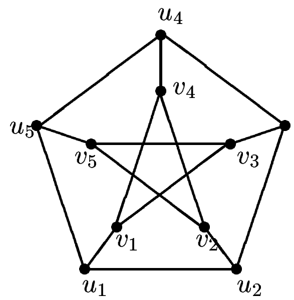


FIGURE 1. The graph of Example 1.

set in $G$. Note that $\beta_{k+2}(G) \leq \beta_k(G)$ for all $k \in \mathbb{N}$ because $W_k(v) \subseteq W_{k+2}(v)$ for all $v \in V$. Now a smallest dominating set in $G_k$ has size at most $\frac{1}{2}n$ provided $G_k$ has no isolated vertices (cf. [**6**, Section 2]). Accordingly $\beta_k(G) \leq \frac{1}{2}n$ if $k$ is odd, and the same bound holds for even $k$ provided $G$ has no component which is a star. In order to describe sharper bounds implicit in [**1**], suppose that $G$ represents a communication network where (i) initially each vertex in $D$ holds the same bit of information, and (ii) at each increment of time any bit of information at a vertex $v$ is transmitted to each neighbour of $v$ (but not retained at $v$). If $G$ is primitive (i.e. connected and non-bipartite) then (cf. [**1**]) we may define $f^*(G,r)$ $(r \geq 1)$ as the shortest time in which the information can be disseminated from a set $D$ of $r$ vertices to all vertices of $G$. If $k = f^*(G,r)$ then $G$ has a $k$-dominating set of size $r$, and so $\beta_k(G) \leq r$. By [**1**, Theorem 6.7] we have $f^*(G,r) < 1 + 2(n-r)/(r+1)$. It follows that $r(k+1) < 2n - k + 1$ and hence that $\beta_k(G) \leq \lfloor (2n-k)/(k+1) \rfloor$, where $k = f^*(G,r) \leq n$.

We shall see that if $X$ is a star set (in any graph $G$ without isolated vertices) then $\overline{X}$ is a $k$-dominating set if $k$ is odd or if $k$ is even and $G$ has no component which is a star. We point out two consequences of this result. First, such a $k$-dominating set $\overline{X}$ can be found in polynomial time with respect to $n$ [**4**], whereas the problem of finding $\beta_k(G)$ is NP-complete [**3**, p. 190]. Secondly, the foregoing bounds for $\beta_k(G)$ can be improved when $G$ has an eigenvalue of relatively high multiplicity. An improvement to roughly $\sqrt{2n}$ is the best that can be achieved in this way; for it is known that if $\mathcal{E}(\mu)$ has codimension $t > 1$ then $n \leq \frac{1}{2}(t^2+5t-4)$ and that an upper bound of order $\frac{1}{2}t^2$ is asymptotically best possible [**7**, Section 2]. The corresponding lower bound of order $\sqrt{2n}$ for $|\overline{X}|$ is asymptotically best possible and attained in the strongly regular graph $L(K_t)$, with $n = \frac{1}{2}t(t-1)$ and $|\overline{X}| = \text{codim}\ \mathcal{E}(-2) = t$.

We shall see that most strongly regular graphs have an eigenvalue (of multiplicity $> \frac{1}{2}n$) for which $\overline{X}$ has a stronger dominating property. To define this property, suppose that $D$ is a $k$-dominating set in $G$ with vertices $1, 2, \ldots, s$, and write $A^k = \left(a_{ij}^{(k)}\right)$. We say that $D$ is a *k-location dominating set* in $G$ if for any pair $u, v$ of vertices in $\overline{D}$ we have

$$\left(a_{u1}^{(k)}, a_{u2}^{(k)}, \dots, a_{us}^{(k)}\right) \neq \left(a_{v1}^{(k)}, a_{v2}^{(k)}, \dots, a_{vs}^{(k)}\right).$$

Note that (i) if $k = 1$, $D$ is just a location dominating set, (ii) for $k > 1$ the notion of a $k$-location dominating set in $G$ differs from that of a location-dominating set in $G_k$. We can now extend an idea of Slater [**8**] to the model of a communication network. Suppose that at time $t = 0$ one vertex of $\overline{D}$ holds one bit of information. If $D$ is a $k$-location dominating set it is possible by measuring signal strengths at the vertices in $D$ at time $t = k$ to identify the vertex in $\overline{D}$ at which the information originated. In particular, if $D$ is a $k$-location dominating set for all $k \in \mathbb{N}$ then the capability to distinguish vertices in $\overline{D}$ persists.

We conclude this section with an example.

EXAMPLE 1.1. In the Petersen graph of Fig. 1, let $D_1 = \{u_2, u_5, v_1\}$, $D_2 = \{u_5, v_1, v_3\}$. Then $D_1$ is a minimal dominating set, but not a 2-dominating set because $W_2(u_1) \cap D_1 = \emptyset$. On the other hand $D_2$ is a 2-dominating set which is not a 1-dominating set because $W_1(v_2) \cap D_2 = \emptyset$. Also, $\{u_5\}$ is both a 3-dominating set and a 4-dominating set. Finally, it is a consequence of Proposition 2.5 that $\{u_1, u_2, u_3, u_4, u_5\}$ is a $k$-location dominating set for all $k \in \mathbb{N}$.

## 2. Star sets and their complements

PROPOSITION 2.1. *Let $G$ be a graph without isolated vertices, let $X$ be a star set in $G$, and let $k \in \mathbb{N}$.*
*(i) If $k$ is odd then $\overline{X}$ is a $k$-dominating set.*
*(ii) If $k$ is even and no component of $G$ is a star then $\overline{X}$ is a $k$-dominating set.*

PROOF. (i) Since $G$ has no isolated vertices. it follows from [**5**, Theorem 7.3.1] that $\overline{X}$ is a 1-dominating set, and hence a $(2h+1)$-dominating set for all $h \in \mathbb{N}$.

(ii) Let $X$ be a star set for the eigenvalue $\mu$, and let $P$ be the orthogonal projection of $\mathbb{R}^n$ onto $\mathcal{E}(\mu)$. It suffices to show that $\overline{X}$ is a 2-dominating set. Since $\mu^k P\mathbf{e}_u = A^k P\mathbf{e}_u$, we have

$$\mu^2 P\mathbf{e}_u - \sum\{a_{uj}^{(2)} P\mathbf{e}_j : j \in W_2(u)\} = \mathbf{0} \tag{2.1}$$

for each vertex $u$. If now $u \in X$ and $W_2(u) \cap \overline{X} = \emptyset$ then (1) is a relation on the linearly independent vectors $P\mathbf{e}_h$ $(h \in X)$. Accordingly either (a) $\mu = 0$ and $W_2(u) = \emptyset$, or (b) $\mu \neq 0$ and $W_k(u) = \{u\}$. Now (a) cannot hold because $W_2(u) \supseteq \{u\}$; and (b) cannot hold for if $W_2(u) = \{u\}$ then $G$ has as a component a star with centre $u$. It follows that $W_2(u) \cap \overline{X} \neq \emptyset$ for all $u \in X$. □

PROPOSITION 2.2. *Let $X$ be a star set for the eigenvalue $\mu$ of $G$. If $\mu^k \notin \mathbb{Z}$, then $\overline{X}$ is a $k$-location dominating set.*

PROOF. For $u \in X$ we have

$$\mu^k P\mathbf{e}_u - \sum\{a_{uj}^{(k)} P\mathbf{e}_j : j \in W_k(u)\} = \mathbf{0}. \tag{2.2}$$

Since $\mu^k \notin \mathbb{Z}$ and the vectors $P\mathbf{e}_j$ $(j \in X)$ are linearly independent, we have $W_k(u) \nsubseteq X$. It follows that $X$ is a $k$-location dominating set. Now suppose, by

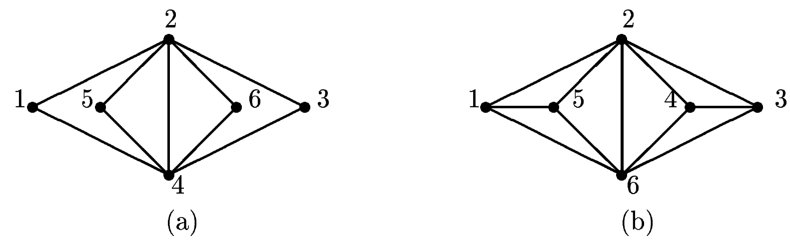


FIGURE 2. The graphs of Examples 2 and 3.

way of contradiction, that for some distinct $u, v \in X$ we have $a_{uj}^{(k)} = a_{vj}^{(k)}$ for all $j \in \overline{X}$. If we subtract from equation (2) the corresponding equation with $v$ in place of $u$ we obtain

$$\mu^k P\mathbf{e}_u - \sum_{j \in X} a_{uj}^{(k)} P\mathbf{e}_j = \mu^k P\mathbf{e}_v - \sum_{j \in X} a_{vj}^{(k)} P\mathbf{e}_j.$$

On equating coefficients of $P\mathbf{e}_u$ we find that

$$\mu^k = a_{uu}^{(k)} - a_{vu}^{(k)}. \tag{2.3}$$

In particular, $\mu^k$ is an integer, contrary to assumption. □

We may use equation (3) to establish further sufficient conditions for $\overline{X}$ to be a $k$-location dominating set for particular values of $k$.

PROPOSITION 2.3. *Let $G$ be a graph in which no component is trivial or a star, and let $X$ be a star set for $\mu$ in $G$.*
*(i) If $G$ is bipartite, $k$ is odd and $\mu > 0$ then $\overline{X}$ is a $k$-location dominating set.*
*(ii) If $\mu^2 \neq d - a_{uv}^{(2)}$ whenever $u, v$ are vertices in $X$ of degree $d$ then $\overline{X}$ is a 2-location dominating set.*

PROOF. By Proposition 2.1, the hypotheses ensure that in each case $\overline{X}$ is a $k$-dominating set for all $k \in \mathbb{N}$. In case (i), if $\overline{X}$ is not a $k$-location dominating set then by equation (3), $\mu^k = -a_{vu}^{(k)}$ for some $u, v \in X$, a contradiction. In case (ii), if $\overline{X}$ is not a 2-location dominating set then by equation (3), $\mu^2 - \deg(u) = -a_{vu}^{(k)} = -a_{uv}^{(k)} = \mu^2 - \deg(v)$ for some $u, v \in X$, a contradiction. □

The two following examples demonstrate that the restriction on $\mu$ in Proposition 2.3(ii) is essential.

EXAMPLE 2.1. For the graph in Fig. 2(a), take $\mu = 0$ and $X = \{1, 3, 6\}$. Then $\overline{X}$ is not a 2-location dominating set because $a_{1j}^{(2)} = a_{6j}^{(2)}$ $(j = 2, 4, 5)$. Also, $\overline{X}$ is not a 1-location dominating set because vertices 1 and 6 have the same $\overline{X}$-neighbourhoods.

EXAMPLE 2.2. For the graph in Fig. 2(b), take $\mu = -1$ and $X = \{1, 2, 3\}$. Here $\overline{X}$ is a 1-location dominating set, but not a 2-location dominating set because $a_{1j}^{(2)} = a_{3j}^{(2)}$ $(j = 4, 5, 6)$. We have deg(1)=deg(3)=3 and $\mu^2 = 3 - a_{13}^{(2)}$.

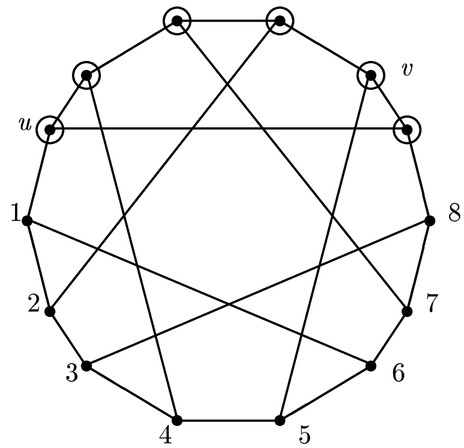


FIGURE 3. The graph of Example 4.

PROPOSITION 2.4. *Let $G$ be a regular graph of degree $r \geq 1$, and let $X$ be a star set for $\mu$ in $G$.*
*(i) If $G$ has no 4-cycles and $\mu^2 \notin \{r-1, r\}$ then $\overline{X}$ is a 2-location dominating set.*
*(ii) If $G$ has girth $> 6$ and if $\mu^3 \notin \{1-2r, -1, 0\}$ then $\overline{X}$ is a 3-location dominating set.*

PROOF. (i) Deny. Then by equation (3), $\mu^2 - r = -a_{vu}^{(2)}$ for some $u, v \in X$. This is a contradiction because $a_{vu}^{(2)} \in \{0, 1\}$.
(ii) Deny. Then by equation (3), $\mu^2 = -a_{vu}^{(3)}$ for some $u, v \in X$. This is a contradiction because

$$a_{vu}^{(3)} = \begin{cases} 2r-1 & \text{if } d(u,v) = 1, \\ 0 & \text{if } d(u,v) = 2, \\ 1 & \text{if } d(u,v) = 3, \\ 0 & \text{if } d(u,v) > 3. \end{cases}$$

□

EXAMPLE 2.3. The Heawood graph illustrated in Fig. 3 demonstrates that the condition $\mu^2 \neq r-1$ of Proposition 2.4(i) is essential. Here $r = 3$ and the eigenvalues are $\pm 3$, of multiplicity 1, and $\pm\sqrt{2}$, of multiplicity 6. The circled vertices constitute a star set $X$ for $\pm\sqrt{2}$, but $\overline{X}$ is not a 2-location dominating set because $a_{uj}^{(2)} = a_{vj}^{(2)}$ $(1 \leq j \leq 8)$.

In the application to a communication network described in Section 1, a $k$-location dominating set is usually required to be relatively small. The next result shows that with minor exceptions this requirement is fulfilled by the complements of star sets in strongly regular graphs, since such graphs have an eigenvalue of relatively large multiplicity. Explicitly, if $G$ is a strongly regular graph with parameters $(n, r, e, f)$ such that $(n-1)(f-e) \neq 2r$ then $G$ has an integer eigenvalue of multiplicity greater than $\frac{1}{2}n$ (see [**2**, Chapter 2]).

PROPOSITION 2.5. *Let $G$ be a strongly regular graph with parameters $(n, r, e, f)$ such that $(n-1)(f-e) \neq 2r$ and $e \neq f \neq r$. If $X$ is a star set for an eigenvalue $\neq r$ then $\overline{X}$ is a $k$-location dominating set for each $k \in \mathbb{N}$.*

PROOF. By Proposition 2.1, $\overline{X}$ is a $k$-dominating set for each $k \in \mathbb{N}$. Our hypotheses here ensure that the distinct eigenvalues of $G$ are $r$, $\rho_1$, $\rho_2$, where $\rho_1$, $\rho_2$ are non-zero integers such that $\rho_1 + \rho_2 \neq 0$. Also, by definition, $G$ is not complete and so $-1$ is not an eigenvalue. In particular, $\overline{X}$ is a location-dominating set, and so we may assume that $k > 1$. We know (by distance-regularity) that if $d(u, v) = d(i, j)$ then $a_{uv}^{(k)} = a_{ij}^{(k)}$. To prove that the converse holds as well, we make use of the fact that $(A - \rho_1 I)(A - \rho_2 I)$ is a multiple of the all-1 matrix $J$ [**2**, Chapter 2]. Since $AJ = rJ$, we know that for each $k \in \mathbb{N}$, $A^k$ is a linear combination of $A$, $I$ and $J$, say $A^k = \alpha_k A + \beta_k I + \gamma_k J$. Since $A$, $I$ and $J$ are linearly independent, we have

$$\begin{pmatrix} \alpha_{k+1} \\ \beta_{k+1} \end{pmatrix} = M \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix}, \text{ where } M = \begin{pmatrix} \alpha_2 & 1 \\ \beta_2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Since $M$ has eigenvalues $\rho_1$, $\rho_2$, we have

$$\begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} = \frac{1}{\rho_1 - \rho_2} \left\{ \rho_1^k \begin{pmatrix} 1 \\ -\rho_2 \end{pmatrix} + \rho_2^k \begin{pmatrix} -1 \\ \rho_1 \end{pmatrix} \right\}.$$

In particular, for $k > 1$ we have $\beta_k = -\rho_1 \rho_2 \alpha_{k-1} \neq 0$. Moreover $\alpha_k \neq \beta_k$ for otherwise $\rho_1^k(1 + \rho_2) = \rho_2^k(1 + \rho_1)$, which yields the contradiction $\rho_1 = \rho_2$. Since $A^k$ has $(u, v)$-entry

$$a_{uv}^{(k)} = \begin{cases} \beta_k + \gamma_k & \text{if } d(u,v) = 0 \\ \alpha_k + \gamma_k & \text{if } d(u,v) = 1 \\ \gamma_k & \text{if } d(u,v) = 2 \end{cases},$$

we deduce that $a_{uv}^{(k)} = a_{ij}^{(k)}$ if and only if $d(u, v) = d(i, j)$.

Now suppose that $u, v$ are vertices in $X$ such that $a_{uj}^{(k)} = a_{vj}^{(k)}$ for all $j \in \overline{X}$. Then $d(u, j) = d(v, j)$ for all $j \in \overline{X}$. But since $\overline{X}$ is a location-dominating set, there exists $j \in \overline{X}$ such that $j$ is adjacent to precisely one of $u, v$, a contradiction. It follows that $\overline{X}$ is a $k$-location dominating set for each $k \in \mathbb{N}$. □

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