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A GENERALIZATION OF THE NOTION OF REPRODUCTIVITY

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ABSTRACT. In Section 1 we state the known theory of reproductive equations in general. The key result is that every equation, having at least one solution, is equivalent to some reproductive equation. In Section 2 we extend the notion of reproductive equations to the class of equations $Eq(x)$ which are solved by means of some given equation, denoted by $Eq_1(\varrho)$. In that case we also prove that such an $Eq(x)$, having at least one solution, is equivalent to some reproductive equation.

1. Reproductivity of an equation $x = f(x)$

Let A be a set (or a class) and $\mathrm{Eq}(x)$ an equation in $x \in A$. Eq can be understood as a unary relation over A. Let S be the set (class) of all solutions of $\mathrm{Eq}(x)$. Then we have:

$$(\forall x \in A)(\mathrm{Eq}(x) \Leftrightarrow x \in S) \tag{1}$$

Let us assume that P is another set (class) and $\phi : P \to S$ a surjection. Using such P and a function ϕ formula (1) can be reformulated thus

$$(\forall x \in A)(\mathrm{Eq}(x) \Leftrightarrow (\exists p \in P)\, x = \phi(p)) \tag{2}$$

In this case the formula $x = \phi(p)$, where p is any element of P, gives all the solutions of $\mathrm{Eq}(x)$, and accordingly we have the following definition:

DEFINITION 1. If (2) holds, then the formula

$$x = \phi(p), \quad p \text{ is any element of } P$$

is called a *formula of the general solution of the equation* $\mathrm{Eq}(x)$.

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Let us concentrate on the $\Rightarrow$-part of the formula (2), i.e., $(\forall x \in A)(\mathrm{Eq}(x) \Rightarrow (\exists p \in P)x = \phi(p))$ which is equivalent to $(\forall x \in A)(\exists p \in P)(\mathrm{Eq}(x) \Rightarrow x = \phi(p))$, since p does not appear in the first part of the implication, i.e., in $\mathrm{Eq}(x)$. Using the axiom of choice, we can introduce a new function $\psi : A \to P$ such that:

$$(\forall x \in A)(\mathrm{Eq}(x) \Rightarrow x = \phi(\psi(x))) \tag{3}$$

Let $f : A \to A$ be defined by $f(x) = \phi(\psi(x))$. Then (3) transforms into:

$$(\forall x \in A)(\mathrm{Eq}(x) \Rightarrow x = f(x)) \tag{4}$$

It is easy to see that $(\forall x \in A)(\mathrm{Eq}(x) \Leftrightarrow (\exists p \in P)\, x = f(p))$ is also true. So, the formula:

$$x = f(p), \quad p \in A \tag{5}$$

is the formula of the general solution of $\mathrm{Eq}(x)$. But, since (4) is true, we say that (5) is *a formula of the general reproductive solution*. The notion of a reproductive solution appeared in 1919 in Löwenheim paper on Boolean equations [**1**]. In Prešić [**1**] we gave the following definition of a *reproductive equation*.

Definition 2. Equation $x = f(x)$ in $x \in A$ is reproductive iff the equality $(\forall x \in A)\, f(f(x)) = f(x)$ holds.

It is easy to see that:

Lemma 1. *If $x = f(x)$ is a reproductive equation, then all of its solutions are given by the formula* (5).

Example 1. Consider the system of equations

$$x = x \cup y, \quad y = x \cap y \tag{$*1$}$$

in $x, y \in \mathcal{B}$, where $\mathcal{B}$ is a Boolean algebra. Let $f : \mathcal{B}^2 \to \mathcal{B}^2$ be defined by:

$$\text{If } X = (x, y), \text{ then } f(X) = (x \cup y, x \cap y)$$

Then $(*1)$ transforms into $X = f(X)$, $(X \in \mathcal{B}^2)$, which is easily seen to be reproductive.

Example 2. Consider the functional equation

$$\phi(x, y) = \phi(y, x) \tag{$*2$}$$

in $\phi : R^2 \to R$. Let Func be the set of all functions $\phi : R^2 \to R$ and $f : \mathrm{Func} \to \mathrm{Func}$ be defined by:

$$\text{If } X \text{ is } \phi \text{ then } f(X) \text{ is the function } \begin{pmatrix} (x,y) \\ \frac{\phi(x,y)+\phi(y,x)}{2} \end{pmatrix}$$

Equation $(*2)$ becomes $X = f(X)$, which is easily seen to be reproductive. Indeed, let $X = \phi$; then $f(X) = \psi$, where $\psi(x,y) = (\phi(x,y) + \phi(y,x))/2$. Moreover,

$$\begin{aligned} f(f(X)) = f(\psi) &= \begin{pmatrix} (x,y) \\ \frac{\psi(x,y)+\psi(y,x)}{2} \end{pmatrix} = \begin{pmatrix} (x,y) \\ \frac{\phi(x,y)+\phi(y,x)+\phi(x,y)+\phi(y,x)}{4} \end{pmatrix} \\ &= \begin{pmatrix} (x,y) \\ \frac{\phi(x,y)+\phi(y,x)}{2} \end{pmatrix} = \begin{pmatrix} (x,y) \\ \psi(x,y) \end{pmatrix} = \psi, \end{aligned}$$

so $f^2 = f$.

The next important fact (see Prešić [**1**]) is the following theorem:

THEOREM 1. *If* $\mathrm{Eq}(x)$ *has at least one solution in* $x \in A$, *then it has an equivalent reproductive equation.*

Proof. Let $S \subseteq A$ be the set (class) of all the solutions for $\mathrm{Eq}(x)$. It is sufficient to define $f : A \to A$ thus:

If $X \in S$, then $f(X) = X$; if $X \in A \setminus S$, then $f(X)$ is any element in S

It is easy to see that the equality $X = f(X)$ is reproductive and equivalent to $\mathrm{Eq}(x)$.

We would like to point out that Theorem 1 was often a leading idea used in solving many classes of equations (functional, Boolean, on finite sets, etc.). We mention one such result concerning the so called *linear homogeneous functional equations on groups* (see Prešić [**2**], [**3**]):

$$a_1(x)\phi(\theta_1(x)) + \cdots + a_n(x)\phi(\theta_n(x)) = 0$$

where $x \in S$ (S given set); $\theta_i : S \to S$ forms a group of order n, and a_i are given mappings from S into a given field F. The function $\phi : S \to F$ is unknown.

For this equation, an equivalent reproductive equation is effectively described (see also Kuczma [**1**, p. 268]).

2. Reproductivity of an equation $\mathrm{Eq}(x)$ given by a formula of the form $(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = f(x,\varrho))$

In mathematics there are many cases when a given equation $\mathrm{Eq}(x)$ is solved *up to some other equation,* say $\mathrm{Eq}_1(\varrho)$. In other words, $\mathrm{Eq}(x)$ is solved by means of the equation $\mathrm{Eq}_1(\varrho)$.

EXAMPLE 3. Equation $(x-1)^2 = 7$ in x real can be solved by means of the following equation $\mathrm{Eq}_1(\varrho) : \varrho^2 = 7$. Namely, we have the equivalence

$$(x-1)^2 = 7 \Leftrightarrow (\exists \varrho \in R)(\mathrm{Eq}_1(\varrho),\ x = 1 + \varrho)$$

Similarly, equation $(x-y)^2 = 7$ in $x, y \in R$ can be solved using the same equation $\mathrm{Eq}_1(\varrho)$. We have:

$$(x-y)^2 = 7 \Leftrightarrow (\exists p \in R)(\exists \varrho \in R)(\mathrm{Eq}_1(\varrho),\ y = p,\ x = p + \varrho)$$

There is a 'parameter' p in the previous formula and a quantifier $(\exists p \in R)$, since equation $x - y = \varrho$ does not have a unique solution in x, y.

Bearing in mind this simple example we shall in general define the meaning of these sentences:

x is an $\mathrm{Eq}_1(\varrho)$-solution of the equation $\mathrm{Eq}(x)$.

Equation $\mathrm{Eq}(x)$ is solved by means of the equation $\mathrm{Eq}_1(\varrho)$.

Assume that B is some other set (class) with a unary relation Eq_1. Further, let P be another set (class).

DEFINITION 3. (i) Let $x \in A$ be determined by an equality of the form $x = \mu(\varrho)$, where $\mu : B \longrightarrow A$ is a certain function. We say that x is an $\mathrm{Eq}_1(\varrho)$-solution of the equation $\mathrm{Eq}(x)$ iff the following implication is true

$$(x = \mu(\varrho), \mathrm{Eq}_1(\varrho)) \Rightarrow \mathrm{Eq}(x)$$

(ii) Equation $\mathrm{Eq}(x)$ in $x \in A$ is solved by means of the equation $\mathrm{Eq}_1(\varrho)$ iff the following equivalence is true[1]

$$\mathrm{Eq}(x) \Leftrightarrow (\exists p \in P)(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = \phi(p, \varrho)) \tag{6}$$

where $\phi : P \times B \to A$ is a given function.

We can see how natural is the previous definition if we consider the following fact:

If (6) is true, then all the $\mathrm{Eq}_1(\varrho)$*-solutions of* $\mathrm{Eq}(x)$ *are determined by* $x = \phi(p, \varrho)$*, where* $p \in P$ *is an arbitrary element and* $\varrho \in B$ *is any solution of* $\mathrm{Eq}_1(\varrho)$.

According to this we introduce the following definition.

DEFINITION 4. If the equivalence (6) holds, then the formula $x = \phi(p, \varrho)$, where $p \in P$ is an arbitrary element and $\varrho \in B$ is any solution of the equation $\mathrm{Eq}_1(\varrho)$, is called a formula of the general solution of the equation $\mathrm{Eq}(x)$ by means of the equation $\mathrm{Eq}_1(\varrho)$.

Notice that if $\mathrm{Eq}(x)$ has a unique solution, then P is superfluous and (6) becomes

$$\mathrm{Eq}(x) \Leftrightarrow (\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = \phi(\varrho))$$

where $\phi : B \to A$ is some function.

Concerning Definition 3 more examples follow. First, we notice that Example 3 can be extended to the case in Galois theory when one considers the question whether a given algebraic equation is solvable by radicals.

[1] We can put $(\forall x \in A)$ before the formula (6)

Example 4. In the class of all groupoids take equation $\mathrm{gr}(X)$ with the meaning: X *is a group*. We do not know a general solution to $\mathrm{gr}(X)$. But various *representation theorems* solve this equation up to some simpler equation. One such theorem, the so called Cayley representation theorem, can be stated thus:

$$\mathrm{gr}(X) \Leftrightarrow (\exists \mathcal{G})(\mathcal{G} \text{ is a permutation group, } X \cong \mathcal{G})$$

assuming that in some sense $\cong$ (isomorphism) is an equality of algebras.

Similarly, the same is true for various representation theorems in many fields of mathematics.

Example 5. Consider the Pexider functional equation (for more on Pexider equation see Krapež and Taylor [**1**])

$$f(x+y) = g(x) + h(y) \tag{7}$$

where $f, g, h : R \to R$ are unknown functions. As is well known this equation reduces to the Cauchy equation $\phi(x+y) = \phi(x) + \phi(y)$. Indeed, from (7) we get:

$$g(x) = f(x) - h(0), \quad h(x) = f(x) - g(0) \tag{$*1$}$$

Equation (7) then becomes $f(x+y) = f(x) - h(0) + f(y) - g(0)$, i.e.

$$f(x+y) - g(0) - h(0) = (f(x) - g(0) - h(0)) + (f(y) - g(0) - h(0)) \tag{$*2$}$$

Defining function ϕ by $\phi(x) = f(x) - g(0) - h(0)$ equation $(*2)$ transforms into the Cauchy functional equation. If we introduce two constants by $C_1 = g(0)$, $C_2 = h(0)$ then we have the following assertion.

$(*3)$ *Any solution (f, g, h) of (7) satisfies the condition:*

$$f(x) = \phi(x) + C_1 + C_2, \; g(x) = \phi(x) + C_1, \; h(x) = \phi(x) + C_2$$

where C_1, C_2 are some constants and ϕ is a solution to the Cauchy functional equation.

This can be written down as

$$\begin{aligned} (*4) \quad & (\forall x, y \in R) f(x+y) = g(x) + h(y) \Rightarrow \\ & (\exists C_1, C_2)(\exists \phi)(\forall x, y \in R)[\phi(x+y) = \phi(x) + \phi(y), \; f(x) = \phi(x) + C_1 + C_2, \\ & \qquad g(x) = \phi(x) + C_1, \; h(x) = \phi(x) + C_2] \end{aligned}$$

As is usual when we prove some implication like $(*4)$, then we check whether the functions in the consequence part of the formula do make a solution. In this example this means whether the following equivalence is true:

$$\begin{aligned} (8) \quad & f(x+y) = g(x) + h(y) \Leftrightarrow \\ & (\exists C_1, C_2)(\exists \phi)[\phi(x+y) = \phi(x) + \phi(y), \; f(x) = \phi(x) + C_1 + C_2, \\ & \qquad g(x) = \phi(x) + C_1, \; h(x) = \phi(x) + C_2] \end{aligned}$$

It can be easily checked that equivalence (8) is indeed true.

Next, we prove that this equivalence is of the form (6). First, equation $f(x+y) = g(x) + h(y)$ is $\mathrm{Eq}(X)$ in $X \in A$ where A is the set of all triples (f, g, h) of real functions $R \to R$. Further, let B be a set of all real functions and $\mathrm{Eq}_1(\varrho)$ be the Cauchy equation[2] $(\forall x, y \in R)\phi(x+y) = \phi(x) + \phi(y)$. Let P be R^2 and

[2] ϕ stands instead of ϱ

$\Phi : P \times B \to A$ be defined in the following way:

If p, denoted by (C_1, C_2), is any element of P and $\phi \in B$, then $\Phi(p,\phi)$ is the triple (f,g,h) where $f(x) = \phi(x) + C_1 + C_2,\ g(x) = \phi(x) + C_1,\ h(x) = \phi(x) + C_2$

Using this notation (8) becomes

$$\mathrm{Eq}(X) \Leftrightarrow (\exists p)(\exists \phi)(\mathrm{Eq}_1(\phi),\ X = \Phi(p,\phi))$$

which is of the form (6). Therefore (f,g,h) as given above is a solution of the Pexider equation *by means of the equation* $\mathrm{Eq}_1(\varrho)$, *i.e., the Cauchy equation.*

Let us return to the general case i.e., to the equation $\mathrm{Eq}(x)$ for which is supposed that (6) is true, i.e.:

$$(\forall x \in A)(\mathrm{Eq}(x) \Leftrightarrow (\exists p \in P)(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = \phi(p,\varrho))) \tag{9}$$

The $\Rightarrow$-part of (9) is equivalent to

$$(\forall x \in A)(\exists p \in P)(\mathrm{Eq}(x) \Rightarrow (\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = \phi(p,\varrho)))$$

Using the axiom of choice we can define a mapping $\psi : A \to P$ such that

$$(\forall x \in A)(\mathrm{Eq}(x) \Rightarrow (\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = \phi(\psi(x),\varrho))) \tag{$*5$}$$

A new function $f : A \times B \to A$ is defined by: $f(x,\varrho) = \phi(\psi(x),\varrho)$. Then $(*5)$ transforms into:

$$(\forall x \in A)(\mathrm{Eq}(x) \Rightarrow (\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = f(x,\varrho))) \tag{$*6$}$$

On the other hand, concerning $\Leftarrow$-part of (9) we have the following implication chain

$$\begin{aligned}&(\forall x \in A)((\exists p \in P)(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = \phi(p,\varrho)) \Rightarrow \mathrm{Eq}(x))\\ &\qquad\qquad \Rightarrow (\forall x \in A)(\forall p \in P)((\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = \phi(p,\varrho)) \Rightarrow \mathrm{Eq}(x))\end{aligned}$$

The variable p is not free in $\mathrm{Eq}(x)$ (using a logical law for the quantifier $(\exists p \in P)$).

$$\Rightarrow (\forall x \in A)((\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = \phi(\psi(x),\varrho)) \Rightarrow \mathrm{Eq}(x)).$$

"Replacing p by $\psi(x)$" (using a logical law for the quantifier $(\forall p \in P)$).

Thus we have the following implication

$$(\forall x \in A)((\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = f(x,\varrho)) \Rightarrow \mathrm{Eq}(x)).$$

Combining this implication and $(*6)$ we obtain the following equivalence

$$(\forall x \in A)(\mathrm{Eq}(x) \Leftrightarrow (\exists p \in P)(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = f(p,\varrho)))$$

which states that the formula:

$$x = f(p, \varrho),\ p \in A,\ \varrho \in B\ \{\textit{while } \varrho \textit{ is a solution of } \mathrm{Eq}_1(\varrho)\}$$

gives all the solutions of $\mathrm{Eq}(x)$ by means of the solutions of the equation $\mathrm{Eq}_1(\varrho)$. Furthermore, since $(*6)$ is true, we can say that this formula is reproductive.

Now we concentrate on the formula

$$(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = f(x, \varrho)) \tag{10}$$

the right-hand part of the implication $(*6)$, which is understood as an equation in $x \in A$. In the sequel, whenever we give an x-solution, for instance x_0, we shall also specify a value ϱ_0, as a corresponding 'witness' for the quantifier $(\exists \varrho \in B)$. That means that the pair $< x_0, \varrho_0 >$ shall satisfy the conditions $\mathrm{Eq}_1(\varrho_0), x_0 = f(x_0, \varrho_0)$. Now we define when formula (10) is reproductive:

Definition 5. The formula $(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = f(x, \varrho))$ is reproductive iff the following condition

$$(\forall x \in A)(\forall \varrho \in B)(\mathrm{Eq}_1(\varrho) \Rightarrow f(x, \varrho) = f(f(x, \varrho), \varrho)))$$

holds.

The reason for this definition is that like Lemma 1 we have the following fact:

Lemma 2. *If the formula* (10) *is reproductive, then all of its solutions by means of the equation* $\mathrm{Eq}_1(\varrho)$ *are given by the following formula*

$$x = f(p, \varrho) \tag{11}$$

$p \in A$, $\varrho \in B$ *are any elements provided* $\mathrm{Eq}_1(\varrho)$

Indeed, if $p_0 \in A$, and $\varrho_0 \in B$ with property $\mathrm{Eq}_1(\varrho_0)$ are any elements, then x defined by $x = f(p_0, \varrho_0)$ satisfies the formula $(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = f(x, \varrho))$, since we can take ϱ_0 as a witness for quantifier $(\exists \varrho \in B)$ and the equality $x = f(x, \varrho)$ reduces to the true equality $f(x_0, \varrho_0) = f(f(x_0, \varrho_0), \varrho_0)$. Conversely, if x_1 is a solution of $(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = f(x, \varrho))$, then a certain ϱ_1 is a witness for $(\exists \varrho \in B)$, consequently the equality $x_1 = f(x_1, \varrho_1)$ holds. This equality is an instance of (11) when $p = x_1, \varrho = \varrho_1$.

It is important that we have a theorem similar to Theorem 1:

Theorem 2. *Let* $\mathrm{Eq}(x)$ *be an equation in* $x \in A$ *whish is solvable by means of*[3] $\mathrm{Eq}_1(\varrho)$*, and which has at least one* $\mathrm{Eq}_1(\varrho)$*-solution. Then there is a formula*

$$(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x = f(x, \varrho))) \quad (f : A \times B \to A \textit{ is some function})$$

[3]i.e., the equivalence of type (6) is valid

which is equivalent to $\mathrm{Eq}(x)$ *and reproductive in the sense of Definition 5.*

Proof. Denote by S the set of all the $\mathrm{Eq}_1(\varrho)$-solutions of $\mathrm{Eq}(x)$. We define the function f as follows:
If $x \in S$, *then for all* $\varrho \in B$ *such that* $\mathrm{Eq}_1(\varrho)$, *we define* $f(x,\varrho) = x$; *for other* ϱ'*s* $f(x,\varrho)$ *is arbitrary. If* $x \notin S$, *then for all* $\varrho \in B$ *such that* $\mathrm{Eq}_1(\varrho)$, *we define* $f(x,\varrho)$ *as some* $x' \in S$; *for the remaining* ϱ'*s* $f(x,\varrho)$ *is arbitrary.*

First, we prove the equivalence:

$$\mathrm{Eq}(x) \Leftrightarrow (\exists \varrho \in B)(\mathrm{Eq}_1(\varrho),\ x = f(x,\varrho))$$

The $\Rightarrow$-part. Let $\mathrm{Eq}(x_0)$, i.e. $x_0 \in S$ and ϱ_0 some element in B such that $\mathrm{Eq}_1(\varrho_0)$. Further, by the definition of f, the value of $f(x_0,\varrho_0)$ is x_0. So: $\mathrm{Eq}_1(\varrho_0), x_0 = f(x_0,\varrho_0)$, which yields $(\exists \varrho \in B)(\mathrm{Eq}_1(\varrho), x_0 = f(x_0,\varrho))$.

The $\Leftarrow$-part. Let for some ϱ_0 and x_0 be $\mathrm{Eq}_1(\varrho_0), x_0 = f(x_0,\varrho_0)$. According to the definition of f we conclude that $x_0 \in S$, i.e. $\mathrm{Eq}(x_0)$.

Second, we prove that f satisfies

$$(*7) \qquad f(f(x,\varrho),\varrho) = f(x,\varrho) \quad \text{whenever} \quad \mathrm{Eq}_1(\varrho)$$

for all $x \in A, \varrho \in B$. Indeed, let $\mathrm{Eq}_1(\varrho)$. Then:

1° If $x \in S$, then $f(x,\varrho) = x$ and $(*7)$ follows.
2° If $x \notin S$, then $f(x,\varrho) = x'$, where $x' \in S$ so:
(i) $f(f(x,\varrho),\varrho) = f(x',\varrho) = x'$ (since $x' \in S$); (ii) $f(x,\varrho) = x'$
From (i) and (ii) $(*7)$ follows.

Now, we can use Definition 5 and Theorem 2 to guide us in solving a given equation $\mathrm{Eq}(x)$ by means of another given equation $\mathrm{Eq}_1(\varrho)$. Briefly, we follow the following plan. We attempt to find a formula For with these properties:

For is a logical consequence of $\mathrm{Eq}(x)$
For is equivalent to $\mathrm{Eq}(x)$
For is reproductive in the sense of Definition 5.

However, comparing this idea with the idea from Example 5 used in solving Pexider equation, it might seem that it is rather artificial. Returning to Pexider equation we disprove this view.

The main step was the conclusion $(*3)$, stated as the implication $(*4)$. But was it necessary to introduce constants C_1, C_2, which are just denotations for $g(0)$, $h(0)$ respectively? We might say that this introduction was merely for psychological reasons—therefore unnecessary. Without C_1, C_2 formula $(*4)$ would read:

$$\begin{aligned}
&(\forall x,y \in R) f(x+y) = g(x) + h(y)\\
&\qquad \Rightarrow (\exists \phi)(\forall x,y \in R)[\phi(x+y) = \phi(x) + \phi(y), f(x) = \phi(x) + g(0) + h(0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad g(x) = \phi(x) + g(0),\ h(x) = \phi(x) + h(0)]
\end{aligned}$$

Denote this implication by $P \Rightarrow Q$ temporarily. The converse of this implication is also true and moreover the part Q is easily seen to be reproductive. Therefore, all the solutions (f,g,h) are given by:

$$f(x) = \phi(x) + G(0) + H(0),\ g(x) = \phi(x) + G(0),\ h(x) = \phi(x) + H(0)$$

where ϕ is any solution of the Cauchy functional equation and G, H arbitrary functions. Therefore, whether we prefer C_1, C_2 to $G(0)$, $H(0)$ is just a matter of taste.

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