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# CLASSICAL INTEGRABLE MECHANICAL SYSTEMS AND THEIR INTEGRABLE PERTURBATIONS

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**Abstract.** Perturbation technic for constructing new integrable systems which are close to the most celebrated integrable systems in classical mechanics are developed. Analytical conditions for the periodicity of billiard trajectories with the ellipsoid are given, generalizing the Cayley condition for the Poncelet Theorem.

## 1. Introduction

Several mathematical formalisms are used to describe the motion of mechanical systems. The Lagrangian and the Hamiltonian approaches are among the most important ones. In the theory of the Hamiltonian systems, the completely integrable cases have a very important position. Their significance is stressed by the fact that integrable systems occur so rarely both in reality, and in theory. According to their regular behavior, as it is stated by Liouville-Arnold's theorem (see Theorem 1, and also [**1,2**]), completely integrable systems serve in perturbation theory, as the first step in investigation of close nonintegrable problems.

This paper is devoted to the perturbation technics of constructing new integrable systems as perturbations of the most celebrated integrable systems in classical mechanics. It presents a part of the scientific activity in the Mathematical Institute SANU in the Seminar on Mathematical Methods in Mechanics from 1995 to 1997. based on the enthusiasm of Nikola Burić, Borislav Gajić, Božidar Jovanović, Milena Radnović,...

## 2. Basic notions of Hamiltonian mechanics

Hamiltonian formalism can be introduced briefly in a frame of the Poisson geometry. Suppose we have a $2n$-dimensional manifold $N$ (in the natural systems $N = T^*M$) and a Poisson bracket defined on $C^\infty(N)$:

$$\{,\} : C^\infty(N) \times C^\infty(N) \to C^\infty(N)$$

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with the well known properties of linearity, skew-symmetricity, and which satisfies the Laibnitz and Jacobi rules. The system is described by a Hamiltonian function $H \in C^\infty(N)$ (in the natural systems $H = T + U$ is a sum of kinetic and potential energy). The equations of motion are

$$\dot{x} = \{x, H\} \tag{1}$$

The definition of the first integral will be given in an analytical, geometrical and algebraical language. Analytically, the first integral is a function $F : N \to R$ with the property:

$$\frac{d}{dt}(F(x(t))) = 0.$$

In a geometric manner $F$ is such that the vector field which induces the motion is tangent to the surfaces $F^{-1}(c)$, for all $c \in R$. Finally, the algebraic definition is: $F$ is the first integral of the equations (1) if and only if

$$\{F, H\} = 0.$$

Specially, the Hamiltonian $H$ is a first integral.

*Definition¿* A Hamiltonian system $(N^{2n}, H)$ is completely integrable if and only if it has $n$ functionally independent first integrals $F_1, \dots, , F_n$ which are in involution:

$$\{F_i, F_j\} = 0.$$

Let $M_f = \{x : F_i(x) = f_i, i = 1, \dots, m\}$. The Liouville-Arnold's theorem (see [**1**,**2**]) is the following:

THEOREM 1. *If $(N^{2n}, H)$ is a completely integrable system then:*

*a) $M_f$ is invariant under the Hamiltonian flow;*

*b) every component of $M_f$ is diffeomorphic to $T^k \times R^{n-k}$;*

*c) there are coordinates $\varphi_1, \dots, \varphi_k$ mod $2\pi$, $y_1, \dots, y_{n-k}$ on $T^k \times R^{n-k}$ in which the Hamilton's equations on $M_f$ take the form*

$$\dot{\varphi}_m = \omega_{m_i}, \dot{y}_s = c_{s_i} \quad (\omega, c = \text{const})$$

## 3. The billiard system within an ellipse

A billiard system describes a particle moving freely within some enclosure, with the billiard low of elastical reflection at the boundary. That means that the impact and reflection angles are equal.

The billiard system within an ellipse in $R^2$

$$\frac{x^2}{a} + \frac{y^2}{b} = 1$$

is completely integrable. As it is well known, this system with two degrees of freedom has, beside the energy integral, an additional integral

$$F_0 = \frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} - \frac{(\dot{x}y - x\dot{y})^2}{ab}.$$

We want to find a potential $V = V(x,y)$ such that the billiard system with a perturbated Hamiltonian has the integral

$$\tilde{F_0} = F_0 + f_0, \tag{2}$$

where $f_0$ is a function depending on the coordinates only:

$$f_0 = f_0(x,y,z).$$

The function $\tilde{F_0}$ remains unchanged under reflections since the perturbation is chosen to depend only on the coordinates.

The equations of motion are

$$\ddot{x} = -V_x, \ddot{y} = -V_y. \tag{3}$$

From the condition that $\tilde{F_0}$ from (2) is an integral of equations (3) we have:

$$\begin{aligned}\dot{F} = 0 \Leftrightarrow &- 2\frac{\dot{x}}{a}V_x - 2\frac{\dot{y}}{b}V_y - \frac{2y\dot{x}}{ab}(xV_y - yV_x) + \\ &+ \frac{2x\dot{y}}{ab}(xV_y - yV_x) + f_x\dot{x} + f_y\dot{y} = 0\end{aligned}$$

Comparing the terms with $\dot{x}$ and $\dot{y}$ we have:

$$\begin{aligned}f_x &= \frac{2}{a}V_x + \frac{2y}{ab}(xV_y - yV_x) \\ f_y &= \frac{2}{b}V_y - \frac{2x}{ab}(xV_y - yV_x).\end{aligned}$$

The compatibility condition $f_{xy} = f_{yx}$ gives us the equation for $V$:

$$\lambda V_{xy} + 3(yV_x - xV_y) + (y^2 - x^2)V_{xy} + xy(V_{xx} - V_{yy}) = 0 \tag{4}$$

where $\lambda = a - b$. We look for solutions in the form of Laurent polynomials:

$$V(x,y,\lambda) = \sum a_{m,n}(\lambda)x^m y^n. \tag{5}$$

Substituting (5) in (4) we get

$$\lambda mn a_{m,n} = (n+m)(m a_{n-2,m} - n a_{n,m-2}). \tag{6}$$

We give a complete description of solutions in this class. For details see [**7**]. Similar results for the cases of constant positive and negative curvature are given in [**8**].

*Example* 1. The simplest new potential is

$$V_2(x,y,\lambda)=\frac{\lambda-x^2}{\lambda y^4}.$$

*Example* 2. The example of the polynomial potential is

$$W_2(x,y,\lambda)=\beta x^2+\alpha y^2+\frac{2}{\lambda}(\alpha-\beta)x^2y^2+\frac{\alpha-\beta}{\lambda}(x^4+y^4).$$

The extension of this method for the systems with more than two degrees of freedom was given in [**9**]. The billiard system within an ellipsoid in $R^3$

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1,\quad a<b<c$$

is completely integrable. Starting from the following integrals

$$\begin{aligned}F_1&=\dot{x}^2+\frac{(x\dot{y}-y\dot{x})^2}{a-b}+\frac{x\dot{z}-z\dot{x})^2}{a-c}\\ F_2&=\dot{y}^2+\frac{(x\dot{y}-y\dot{x})^2}{b-a}+\frac{(y\dot{z}-z\dot{y})^2}{b-c}\\ F_3&=\dot{z}^2+\frac{(x\dot{z}-z\dot{x})^2}{c-a}+\frac{(y\dot{z}-z\dot{y})^2}{c-b}\end{aligned}\tag{7}$$

we are looking for a potential $V=V(x,y,z)$ such that the perturbated system has integrals of the form

$$\tilde{F}_i=F_i+f_i\tag{8}$$

where $f_i=f_i(x,y,z)$ depend only on coordinates.

The system we consider has more than two degrees of freedom. So, it is not obvious that new perturbated integrals are still in involution. But we have the following

Lemma 1. *If $F_i$ are given by* (7) *and $\tilde{F}_i$ from* (8) *are the integrals of the perturbated system, then $\tilde{F}_i$ and $\tilde{F}_j$ are in involution.*

*Proof.* Using the linearity of the Poisson bracket, it follows

$$\{F_1+f_1,F_2+f_2\}=\{F_1,F_2\}+\{F_1,f_2\}-\{F_2,f_1\}+\{f_1,f_2\}.$$

We have $\{F_1,F_2\}=0$ and $\{f_1,f_2\}=0$, by the assumptions. So, we need to prove

$$\{F_1,f_2\}=\{F_2,f_1\}.$$

As $f_2$ depends on the coordinates only, it follows that

$$\{F_1, f_2\} = -\langle (F_{1\dot{x}}, F_{1\dot{y}}, F_{1\dot{z}}), (f_x^2, f_y^2, f_z^2)\rangle.$$

Using the explicit formulae for $F_{i\dot{x}}, F_{i\dot{y}}, F_{i\dot{z}}$ from (7) and

$$\begin{aligned}
f_x^2 &= \frac{2y}{b-a}(yV_x - xV_y)\\
f_y^2 &= 2V_y + \frac{2x}{b-a}(xV_y - yV_x) + \frac{2z}{b-c}(zV_y - yV_z)\\
f_z^2 &= \frac{2y}{b-c}(yV_z - zV_y)\\
f_x' &= 2V_x + \frac{2y}{a-b}(yV_x - xV_y) + \frac{2z}{a-c}(zV_x - xV_z)\\
f_y' &= \frac{2x}{a-b}(xV_y - yV_x)\\
f_z' &= \frac{2x}{a-c}(xV_z - zV_x)
\end{aligned}$$

after some computation we get the proof of the Lemma.

## 4. Potential perturbations of the Jacobi problem

One of the celebrated classical integrable system is the system of a particle moving under inertia on the ellipsoid

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 1.$$

It has an additional integral of motion, found by Ioahimstal

$$I = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)\left(\frac{\dot{x}^2}{a} + \frac{\dot{y}^2}{b} + \frac{\dot{z}^2}{c}\right).$$

Kozlov's idea was to analyse whether the equations of motion under the influence of a force with potential $V$:

$$\ddot{x} = \lambda\frac{x}{a} - V_x, \ddot{y} = \lambda\frac{y}{b} - V_y, \ddot{z} = \lambda\frac{z}{c} - V_z$$

have an integral $F$ of the form

$$F = I + f,$$

where $f$ is a function depending on the coordinates. From the condition $\dot{F} = 0$ we can get the system:

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)V_{xy}\frac{a-b}{ab} - 3\frac{y}{b^2}\frac{V_x}{a} + 3\frac{x}{a^2}\frac{V_y}{b} + \left(\frac{x^2}{a^3} - \frac{y^2}{b^3}\right)V_{xy} + \\
+\frac{xy}{ab}\left(\frac{V_{yy}}{a} - \frac{V_{xx}}{b}\right) + \frac{zx}{ca^2}V_{zy} - \frac{zy}{cb^2}V_{zx} = 0$$

$$
\begin{aligned}
(9)\qquad & \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)V_{yz}\frac{b-c}{bc}-3\frac{z}{c^2}\frac{V_y}{b}+3\frac{y}{b^2}\frac{V_z}{c}+\left(\frac{y^2}{b^3}-\frac{z^2}{c^3}\right)V_{yz}+\\
&\qquad +\frac{yz}{bc}\left(\frac{V_{zz}}{b}-\frac{V_{yy}}{c}\right)+\frac{xy}{ab^2}V_{xz}-\frac{xz}{ac^2}V_{xy}=0
\end{aligned}
$$

$$
\begin{aligned}
& \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)V_{zx}\frac{c-a}{ac}-3\frac{x}{a^2}\frac{V_z}{c}+3\frac{z}{c^2}\frac{V_x}{a}+\left(\frac{z^2}{c^3}-\frac{x^2}{a^3}\right)V_{zx}+\\
&\qquad +\frac{xz}{ac}\left(\frac{V_{xx}}{c}-\frac{V_{zz}}{a}\right)+\frac{zy}{bc^2}V_{xy}-\frac{yx}{ba^2}V_{yz}=0
\end{aligned}
$$

Solutions of (9) in the form of Laurent polynomials

$$
V(x,y,z)=\sum_{r\geq n,m,p\geq s} a_{m,n.p}x^n y^m z^p
$$

are given in the next theorem.

We will assume $a \neq b \neq c$.

Theorem 2. [**6**] *The general formula for the elements of the basic Laurent polynomial solutions with* $a_{m_0,0,-m_0-2}=\alpha$ *is:*

$$
\begin{aligned}
& a_{m_0+2k,2s,-m_0-2-2(k+s)} = \\
& = (-1)^s\binom{s+k-1}{k}\cdot\frac{c^{k+s}(c-a)^s(c-b)^k}{b^k a^s(b-a)^{k+s}}\cdot\frac{\prod_{i=1}^{k+s}(m_0+2i)}{\prod_{i=1}^{s}2i\prod_{j=1}^{k}(m_0+2j)}
\end{aligned}
$$

*where* $k \leq s$, *and* $m_0+2(k+s)<0$.

*Proof.* By putting the form of $V$ into (9) we get a system of difference equations.

The proof follows from the next five lemmas.

Lemma 2. *The system with variables* $a_{m-2,n,p}, a_{m,n-2,p}$ *and* $a_{m,n,p-2}$ *is singular for arbitrary* $m,n,p,a,b,c$.

Let us call the *level* of an element $a_{m,n,p}$ the sum $m+n+p$, and the *degree* of an element $a_{m,n,p}$ of a fixed level the sum $m+n$.

Lemma 3. *Among the nonzero elements of a fixed level if* $a_{m,n,p}$ *has minimal degree then* $m=0$ *or* $n=0$.

Lemma 4. *If* $m_0,p$ *are such that* $m_0\neq 0$ *and* $a_{m_0,0,p}$ *is a nonzero element of minimal degree and of level* $k=m_0+p$ *for arbitrary* $a,b,c$, *then* $k=-2$.

If $m_0 \neq 0$ then according to lemma 4 the nonzero elements are of level $-2$.

LEMMA 5. (a) *The coefficients with the first index equal $m_0$ are:*

$$a_{m_0,2k,-m_0-2-2k} = (-1)^k \frac{c^k(c-a)^k \prod_{i=1}^k (m_0+2i)}{b^k(b-a)^k \prod_{i=1}^k 2i} a_{m_0,0,-m_0-2}$$

(b) *The index $m_0$ is iven negative integer: $m_0 = -2k$, $k \in N$.*

We introduce the following function:

$$K(m_0, 2n) = 1 \quad K(m_0+2, 2n) = n \quad n \geq 2$$

and by induction

$$K(m_0+2k, 2n) = \sum_{s=k-1}^{n} K(m_0 + 2(k-1), 2s).$$

Some elementary combinatorics gives us

LEMMA 6. *The function $K$ defined above has the following expression:*

$$K(m_0+2k, 2n) = \binom{n+k-1}{k}. \qquad \square$$

*Example* 3. One of the obtained potentials obtained above is

$$V_4(x,y,z) = x^{-4}\left(z^2 + \frac{c(c-a)}{b(b-a)} y^2\right).$$

By using the Levi-Civita criterion one can prove

**Theorem 3.** [**16**] *The systems are integrable by the method of separation of variables in the elliptic coordinate system.*

## 5. Periodical trajectories of the billiard system

Analitical conditions of Cayley's type for periodical trajectories have been derived by Radnović and the author ([**17**], [**18**]).

Suppose we have two conics given in the plane, and a polygon inscribed in one and circumscribed about the other conic. Poncelet's theorem [**20**] states that then exists infinitely many such polygons. Cayley established an analytical condition for determining whether there existed a polygon inscribed in one and circumscribed about the other conic [**20**]. Griffiths and Harris derived Cayley's theorem using the analytical condition on a point to be of finite order in an elliptic curve group structure [**21**].

One can consider the billiard system inside an ellipsoid in space of any dimension. The trajectory of that mechanical system in $R^d$ is tangent to $d-1$ quadrics confocal with the outer hyper-surface [**22**]. Generalization of Poncelet's theorem is valid in $d$-dimensional case also: there exists a closed trajectory with $d-1$ given confocal caustics if and only if infinitely many such trajectories exist, and all of them have the same period [**22**].

We found a generalization of Cayley's theorem, i.e. the analytical condition on a billiard trajectory within an ellipsoid in $R^d$ to be periodic, by modifying Griffiths' and Harris' ideas and applying them to isospectral curves of Veselov and Moser. (The case $d=3$ was treated in [**17**] and $d>3$ in [**18**].) One of the consequences of the result we obtained can be restated in a simple manner – there is no periodic billiard trajectories within an ellipsoid in $d$-dimensional space with non-degenerate caustics, with period less than $d$. If a trajectory is closed after $k$ bounces, $k<d$, then it is placed in one of the coordinate $k$-planes and can be examined as in the $k$-dimensional case. Let us note that in the planar case, the condition above obtained is not identical to the classical Cayley formula, although they are similar by their form.

Veselov and Moser in [**23**] considered the ellipsoidal billiard as a system with discrete time. They used the connection of the billiard within the ellipsoid $(Ax,x)=1$, in $R^d$ with the discrete Heisenberg $XYZ$ model.

The isospectral curve, the basic tool of algebro-geometric integration, related to the Heisenberg system is found in [**9**]. It can be shown that this curve is of the form

$$\Gamma:\ \nu^2=\prod_{i=1}^{d-1}(\mu-\alpha_i)\prod_{j=1}^{d}(\mu-J_j^2),$$

where $J=\mathrm{diag}(J_1,\dots,J_d)$, $J=A^{-\frac{1}{2}}$ and $\alpha_1,\dots,\alpha_{d-1}$ are such that every segment of the given billiard trajectory is tangent to the same $d-1$ quadrics.

The sequence $q_k$ can be uniquely recovered from the divisor sequence $D_{k+1}=D_k+P_\infty-P_0$ on the Jacobian variety on the curve $\Gamma$, where $P_\infty$ corresponds to the value $\mu=\infty$ and $P_0$ to $\mu=0,\lambda=(q_k,J^{-1}q_{k+1})^{-1}$.

The connection between periodical billiard trajectories and periodic sequences of divisors is described in the following lemma.

LEMMA 7. [**17**] *If the billiard is periodic with the period $n$, then the divisor sequence $D_k$ joined to the corresponding Heisenberg $XYZ$ system is also periodic, with period $2n$.*

By this lemma, by investigating when $2n(P_\infty-P_0)=0$ holds on the Jacobian variety of the spectral curve $\Gamma$ we obtain the following

THEOREM 4. [**18**] *The condition on billiard trajectory inside the ellipsoid $(J^{-2}x,x)=1$ in $R^d$, with non-degenerate caustics (5), to be periodic with period $n\geq d$ is*

$$\mathrm{rank}\begin{bmatrix} B_{n+1} & B_n & \dots & B_{d+1}\\ B_{n+2} & B_{n+1} & \dots & B_{d+2}\\ \dots & \dots & \dots & \dots\\ B_{2n-1} & B_{2n-2} & \dots & B_{n+d-1}\end{bmatrix} < n-d+1,$$

*where* $\sqrt{(x-\alpha_1)\dots(x-\alpha_{d-1})(x-J_1^2)\dots(x-J_d^2)} = B_0+B_1x+B_2x^2+\dots$. *There are no such trajectories with period less than* $d$.

This is a sort of a generalization of Cayley's condition. It is very similar, but even in the case $d = 2$, not exactly the same as Cayley's formula. The theorem 4 does not give a complete description of periodical billiard trajectories. For example, it is obvious that there always exists a trajectory with period 2 – with both bounces on the same coordinate axis.

THEOREM 5. [**18**] *If the billiard trajectory inside the ellipsoid* $(J^{-2}x, x) = 1$ *is periodic with period* $n < d$*, then all the bounces are placed in one of the coordinate* $n$*-dimensional planes.*

All periodic trajectories of the billiard system either satisfy the condition from theorem 4 and have at least $d$ bounces, or they are placed in a coordinate $k$-plane. The analytic condition in the latter case can be obtained by appropriate application of theorem 4.

## 6. Perturbations of the Chaplygin problem

The method of integrable potential perturbations can successfully be applied to nonholonomic systems. The Suslov, Chaplygin and Veselov-Veselova problems (see [**2**–**5**]) have been considered by Gajić, Jovanović and the author.

Nonholonomic systems are not Hamiltonian. In general, a nonholonomic system does not have an invariant measure in an $m$-dimensional phase space, and one need $m-1$ functionally independent integrals for the complete integrability.

We are interested in integrable potential perturbations of the Chaplygin problem of a balanced, dynamically asymmetric ball ($I_1 \neq I_2 \neq I_3$) rolling on a rough surface. The nonholonomic constraint is given by the condition that the velocity of the point of contact is equal to zero. The equations of the motion in a potential field with potential $V(\gamma)$ are [**1,13**]:

$$\dot{k} + \omega \times k = \gamma \times \frac{\partial V}{\partial \gamma}, \qquad \dot{\gamma} = \gamma \times \omega, \tag{10}$$

where $k = I\omega + ma^2\gamma \times (\omega \times \gamma)$ is the angular momentum of the ball relative to the point of contact, $a$ is the radius, $m$ is the mass and $I$ is the inertia tensor of the ball relative to its center. The equations (10) have the invariant measure with the density

$$M = \frac{1}{\sqrt{(ma^2)^{-1} - \langle \gamma, (I + ma^2E)^{-1}\gamma\rangle}},$$

where $E$ is the identity matrix. They always possess the following three integrals

$$F_1 = \frac{1}{2}\langle k, \omega\rangle + V(\gamma), \qquad F_2 = \langle k, \gamma\rangle, \qquad F_3 = \langle \gamma, \gamma\rangle (= 1).$$

Chaplygin considered the motion without the potential force. He found the fourth integral $F_4 = \langle k, k\rangle$, and solved the problem by quadratures [**3**].

We are looking for the fourth integral, in the form

$$\tilde{F}_4 = \frac{1}{2}\langle k, k\rangle + F(\gamma).$$

The condition $\dot{\tilde{F}}_4 = 0$ is equivalent to the following system:

$$\begin{aligned}
K_1\left(\frac{\partial V}{\partial\gamma_3}\gamma_2 - \frac{\partial V}{\partial\gamma_2}\gamma_3\right) &= \frac{\partial F}{\partial\gamma_3}\gamma_2 - \frac{\partial F}{\partial\gamma_2}\gamma_3\\
K_2\left(\frac{\partial V}{\partial\gamma_1}\gamma_3 - \frac{\partial V}{\partial\gamma_3}\gamma_1\right) &= \frac{\partial F}{\partial\gamma_1}\gamma_3 - \frac{\partial F}{\partial\gamma_3}\gamma_1\\
K_3\left(\frac{\partial V}{\partial\gamma_2}\gamma_1 - \frac{\partial V}{\partial\gamma_1}\gamma_2\right) &= \frac{\partial F}{\partial\gamma_2}\gamma_1 - \frac{\partial F}{\partial\gamma_1}\gamma_2,
\end{aligned}\tag{11}$$

where $K_i = I_i + ma^2$, $i = 1, 2, 3$. The first two equations in (11) are same as the equations in the Suslov problem. We shall derive polynomial solutions. We have that any polynomial solution should be of even degree in $\gamma_3$. Using the symmetry in $\gamma_1$, $\gamma_2$ and $\gamma_3$ of (11), we get that it should be of even degree in $\gamma_1$, $\gamma_2$ also.

Theorem 6. *The equations* (10) *of Chaplygin problem, for $I_1 \neq I_2 \neq I_3$, are integrable for potentials $V = \sum_L a_L V_{2L}(\gamma|A_L, B_L, C_L)$, where:*

$$V_{2L} = \sum_{\substack{m+n+k=L\\ m,\ n,\ k\geq 0}} \left(\binom{n+k}{k}c_{n+k} - \binom{m+k}{k}d_{m+k}\right)\gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}.\tag{12}$$

*The corresponding fourth integral is $\tilde{F}_4 = \frac{1}{2}\langle k, k\rangle + \sum_L a_L F_{2L}(\gamma|A_L, B_L, C_L)$, where:*

$$F_{2L} = \sum_{\substack{m+n+k=L\\ m,\ n,\ k\geq 0}} \left(K_2\binom{n+k}{k}c_{n+k} - K_1\binom{m+k}{k}d_{m+k}\right)\gamma_1^{2m}\gamma_2^{2n}\gamma_3^{2k}.$$

*c-s and d-s depend on $A_L, B_L, C_L$ by:*

$$\begin{aligned}
c_{L-i} &= \binom{L}{i}(1-i)(I_1 - I_3)A + \binom{L-1}{i-1}(I_1 - I_3)B\\
d_i &= \binom{L}{i}(I_3 - I_2)iA + \binom{L-1}{i-1}(I_2 - I_3)B + \binom{L}{i}C.
\end{aligned}$$

*Examples.* i) For $N = 2$ the solution is the Klebsh potential

$$V_2(\gamma) = a_1\gamma_1^2 + a_2\gamma_2^2 + a_3\gamma_3^2, \qquad a_1(I_2 - I_3) + a_2(I_3 - I_1) + a_3(I_1 - I_2) = 0.$$

ii) For $N = 4$ the new integrable potential is $V_4(\gamma) = \sum\limits_{i+j+k=2} a_{2i2j2k}\gamma_1^{2i}\gamma_2^{2j}\gamma_3^{2k}$,

$$\begin{aligned}
a_{400} &= c_0 - d_2 = (2I_2 - I_3 - I_1)A + (I_1 - I_2)B - C,\\
a_{040} &= c_2 - d_0 = (I_1 - I_3)A - C,\\
a_{004} &= c_2 - d_2 = (I_1 + 2I_2 - 3I_3)A + (I_3 - I_2)B - C,\\
a_{220} &= c_1 - d_1 = 2(I_2 - I_3)A + (I_1 - I_2)B - 2C,\\
a_{202} &= c_1 - 2d_2 = 4(I_2 - I_3)A + (I_1 - 2I_2 + I_3)B - 2C,\\
a_{022} &= 2c_2 - d_1 = 2(I_1 + I_2 - 2I_3)A + (I_3 - I_2)B - 2C.
\end{aligned}$$

Interesting generalizations of the Suslov's problem were suggested recently by Jovanović [**12**].

## 7. L-A pair for the Kowalevskaya gyrostat in a magnetic field

Starting from a Reyman and Semenov-Tian-Shansky L-A pair for the Kowalevskaya top, we got an L-A pair for the Kowalevskaya gyrostat in a magnetic field [**19**]. The system is given by the Hamiltonian

$$H = \frac{1}{2}\left(M_1^2 + M_2^2 + 2M_3^2 + 2\gamma M_3\right) - p_i - \delta_2.$$

The algebra is generated by $M_i, p_i, \delta_i$ and relations

$$\{M_i, M_j\} = \epsilon_{ijk}M_k, \{M_i, p_j\} = \epsilon_{ijk}p_k, \{M_i, \delta_j\} = \epsilon_{ijk}\delta_k$$

Theorem 7. *The Hamilton's system is equivalent to*

$$\dot{L}(\lambda) = -\left[L(\lambda), A(\lambda)\right],$$

*where*

$$L(\lambda) = i\begin{bmatrix} -\gamma & \frac{p_- - i\delta_-}{\lambda} & M_- & \frac{-p_3 + i\delta_3}{\lambda} \\ \frac{p_+ - i\delta_+}{\lambda} & \gamma & \frac{p_3 + i\delta_3}{\lambda} & -M_+ \\ M_+ & \frac{-p_3 + i\delta_3}{\lambda} & -T & \frac{-p_+ + i\delta_+}{\lambda} + 2\lambda \\ \frac{p_3 + i\delta_3}{\lambda} & -M_- & \frac{p_- + i\delta_-}{\lambda} & T \end{bmatrix}$$

*and*

$$A(\lambda) = \frac{i}{2}\begin{bmatrix} T & 0 & M_- & 0 \\ 0 & -T & 0 & -M_+ \\ M_+ & 0 & -T & -2\lambda \\ 0 & -M_- & 2\lambda & T \end{bmatrix}$$

*where* $T = 2M_3 + \gamma$.

The resulting matrices have all the symmetries necessary for the procedure of algebro-geometric integration.

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