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SPECTRAL INVARIANTS OF AFFINE HYPERSURFACES

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Abstract. Let M be a smooth compact manifold of dimension m with smooth, possibly empty, boundary ∂M. If g is a Riemannian metric on M and if ∇ is an affine connection, let $D = D(g, \nabla)$ be the trace of the normalized Hessian; if ∂M is empty, then we impose Dirichlet boundary conditions. The structures (g, ∇) arise naturally in the context of affine differential geometry and we give geometric conditions which ensure that D is formally self-adjoint in this setting. We study the asymptotics of the heat equation trace; we have that $a_m(D)$ is an affine invariant. We use the asymptotics of the heat equation to study the affine geometry of affine hypersurfaces.

§0 Introduction

Let M be a smooth compact manifold of dimension $m \geq 2$ with smooth, possibly empty, boundary ∂M. Let ∇ be a Ricci symmetric, torsion free connection on the tangent bundle of M. Let g be a Riemannian metric on M. Let D be the trace of the normalized Hessian defined by g and ∇; see §1.3 for details. If the boundary of M is non-empty, we impose Dirichlet boundary conditions; it is also possible to use suitable modified Neumann boundary conditions. Let $a_n(D)$ be the coefficients in the asymptotic expansion of the heat trace, see §1.4 for details. In [**2**], we showed that if two connections ∇ and $\tilde{\nabla}$ are projectively equivalent and if two metrics g and $\tilde{g}$ are conformally equivalent, then $a_m(D(g, \nabla)) = a_m(D(\tilde{g}, \tilde{\nabla}))$.

Here is a brief outline to the paper. In §1, we shall present a brief review of results from [**2**] and [**3**] which we shall need. In §2, we review affine differential geometry. We define the metric g and the two torsion free Ricci symmetric tensors

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${}^1\nabla$ and ${}^2\nabla$ which are associated to a relative normalization $\{x, X, y\}$ of a non-degenerate hypersurface. The operators $D(g,\nabla)$ for $\nabla = {}^1\nabla$ or for $\nabla = {}^2\nabla$ need not be self-adjoint. In §3, we study conditions on the hypersurface being studied to ensure that these operators are self-adjoint. In §4, we use the invariants of the heat equation to define invariants of affine differential geometry. In §5, we study the spectral geometry of the Gauss map.

§1 Heat equation asymptotics

1.1 Notational conventions. We adopt the following notational conventions. Let Greek indices ν and μ range from 1 through m and index local coordinate frames for the tangent and cotangent bundles of M; let Greek indices α and β range from 1 through $m-1$ and index local coordinate frames for the tangent and cotangent bundles of the boundary. Let Roman indices i and j range from 1 through m and index local orthonormal frames for the tangent and cotangent bundles of M; let Roman indices a and b range from 1 through $m-1$ and index local orthonormal frames for the tangent and cotangent bundles of ∂M. We shall assume ∂_m is perpendicular to the boundary; for the moment we do not assume that it is a unit normal vector field. We adopt the Einstein convention and sum over repeated indices. We shall assume the coordinates are chosen near the boundary so that $g_{\alpha m} \equiv 0$; this normalization is preserved by conformal rescaling. Let Γ_∇ and Γ_g be the Christoffel symbols of the connection ∇ and of the Levi-Civita connection determined by g;

$$\nabla_{\partial_\nu}\partial_\mu = \Gamma_{\nabla,\nu\mu}{}^\sigma\partial_\sigma \text{ and } {}^g\nabla_{\partial_\nu}\partial_\mu = \Gamma_{g,\nu\mu}{}^\sigma\partial_\sigma.$$

The difference Θ of these two connections is tensorial. Since the two connections are torsion free we have

$$\Theta_{\nu\mu}{}^\sigma := \Gamma_{\nabla,\nu\mu}{}^\sigma - \Gamma_{g,\nu\mu}{}^\sigma \text{ satisfies } \Theta_{\nu\mu}{}^\sigma = \Theta_{\mu\nu}{}^\sigma.$$

Let L be the second fundamental form along the boundary of the metric g;

$$L_{\alpha\beta} = ({}^g\nabla_{\partial_\alpha}\partial_\beta, \partial_m) = -\tfrac{1}{2}\partial_m g_{\alpha\beta}.$$

We impose Dirichlet boundary conditions on all operators henceforth;

$$\text{domain}(D) = \{f \in C^\infty(M) : f|_{\partial M} = 0\}.$$

1.2 Projective equivalence. Let TM, TM^* and $S^2M \subset T^*M \otimes T^*M$ be the tangent, cotangent and symmetric 2 cotensor bundles over M. We say that two metrics $\tilde{g}$ and g are *conformally equivalent* if there exists a smooth function ψ on M so that $\tilde{g} = e^{2\psi}g$. We say that two connections ∇ and $\tilde{\nabla}$ are *projectively equivalent* if there exists a smooth closed 1-form

$\theta = \theta(\tilde{\nabla}, \nabla)$ so that:

$$(\tilde{\nabla}_u - \nabla_u)v = \theta(u)v + \theta(v)u.$$

We note that two connections are projectively equivalent if and only if their unparametrized geodesics coincide. If ∇ is a torsion free connection on TM, let

$$R_\nabla(u,v) : w \to (\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]})w, \text{ and}$$
$$\rho_\nabla(u,w) := -\operatorname{tr}(v \to R_\nabla((u,v)w)$$

be the full curvature tensor and the Ricci tensor of the connection ∇. A connection is said to be *Ricci symmetric* if $\rho_\nabla(u,w) = \rho_\nabla(w,u)$ for all tangent vectors u and w; we restrict to torsion free Ricci symmetric connections henceforth.

1.3 The Hessian. The Hessian H_∇ is a second order operator from the space of smooth functions on M to the space of smooth symmetric 2 tensors on M which is defined by the equation:

$$(H_\nabla f)(u,v) := u(v(f)) - \nabla_u v(f).$$

If $\omega = \omega_{\nu\mu} dx^\nu \circ dx^\mu$ is a symmetric 2 tensor, let $\operatorname{tr}_g \omega := g^{\nu\mu}\omega_{\nu\mu}$ be the contraction of ω. We contract the Hessian and normalize by adding a suitable multiple of the Ricci tensor to define a second order operator $D = D(g,\nabla)$ of Laplace type on $C^\infty(M)$:

$$Df := -\operatorname{tr}_g\{H_\nabla(f) + (m-1)^{-1} f \rho_\nabla\}.$$

Although D need not be self-adjoint in general, it satisfies an important transformation rule. Let $\tilde{g} = e^{2\psi}g$ be a metric which is conformally equivalent to g and let $\tilde{\nabla}$ be a connection which is projectively equivalent to ∇. Choose a local primitive ϕ so $d\phi = \theta(\tilde{\nabla}, \nabla)$. We refer to [**2**, Lemma 2.1] for the proof of the following identity:

$$D(\tilde{g}, \tilde{\nabla}) = e^{-2\psi+\phi} D(g, \nabla) e^{-\phi}.$$

1.4 Heat equation. The fundamental solution $u(x;t) = e^{-tD}\phi(x)$ of the heat equation satisfies the equations:

$$(\partial_t + D)u(x;t) = 0, \;\; u(x;0) = \phi(x), \text{ and } u(y;t) = 0 \text{ for } y \in \partial M.$$

The operator e^{-tD} for $t > 0$ is trace class on $L^2(M)$. As $t \downarrow 0$, there is an asymptotic series of the form:

$$\operatorname{tr}_{L^2}(e^{-tD}) \sim \sum_{n\geq 0} a_n(D) t^{(n-m)/2}.$$

The coefficients $a_n(D)$ are locally computable invariants which will comprise the focal point of our discussion. Let $dx = dx(g)$ and $dy = dy(g)$ be the Riemannian measures on the interior of M and on the boundary of M. We refer to [**2**, 6] for the proof of the following result:

1.5 Theorem.

(1) *Let D be an operator of Laplace type. There exist local invariants $a_n(x,D)$ defined for $x\in M$ and $a_n^{bd}(y,D)$ defined for $y\in\partial M$ so that we have $a_n(D)=\int_M a_n(x,D)dx+\int_{\partial M}a_n^{bd}(y,D)dy$. If n is odd, then the interior invariants $a_n(x,D)$ vanish.*

(2) *Let $\tilde g$ and g be conformally equivalent metrics. Let $\tilde\nabla$ and ∇ be projectively equivalent torsion free Ricci symmetric connections. Then we have that $a_m(D(g,\nabla))=a_m(D(\tilde g,\tilde\nabla))$.*

To describe the local formulae for the invariants of Theorem 1.5, it is convenient at this point to express the operator D invariantly. We refer to [**2**], [**6**] for the proof of the following assertion.

1.6 Lemma. *Let $D=D(g,\nabla)$. Let $\Theta=\nabla-{}^g\nabla$.*

(1) *There exists a unique connection ∇_D on $C^\infty(M)$ and a unique function $E\in C^\infty(M)$ so that $D=-(\operatorname{tr}_g(\nabla_D^2)+E)$.*

(2) *The connection 1 form ω_D of ∇_D is given by $\omega_{D,\delta}:=-\frac12 g_{\nu\delta}g^{\mu\sigma}\Theta_{\nu\sigma}{}^{\nu}$.*

(3) *We have $E:=\frac{1}{(m-1)}g^{\nu\mu}\rho_{\nabla,\nu\mu}-g^{\nu\mu}(\partial_\mu\omega_{D,\nu}+\omega_{D,\nu}\omega_{D,\mu}-\omega_{D,\sigma}\Gamma_{g,\nu\mu}{}^{\sigma})$.*

Let e_m be the inward unit normal vector field on the boundary of M. Let R_{ijkl}, $\rho_{ij}:=R_{ikkj}$, and $\tau:=\rho_{ii}$ be the curvature tensor, the Ricci tensor, and the scalar curvature of the Levi-Civita connection. Let ${}^D\Omega$ be the curvature tensor of the connection ${}^D\nabla$. We refer to [**3**, Theorems 1.1 and 1.2] for the proof of the following Theorem:

1.7 Theorem. *Let M be a manifold with smooth boundary. Adopt the notation of Lemma 1.6.*

(1) $a_0(D)=(4\pi)^{-m/2}\int_M dx$.

(2) $a_1(D)=-\frac14(4\pi)^{-(m-1)/2}\int_{\partial M}dy$.

(3) $a_2(D)=\frac16(4\pi)^{-m/2}\{\int_M(6E+\tau)dx+\int_{\partial M}2L_{aa}dy\}$.

(4) $$a_3(D)=-\tfrac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}(96E+16\tau-8\rho_{mm}+7L_{aa}L_{bb}$$
$$-10L_{ab}L_{ab})dy.$$

(5) $$a_4(D)=\tfrac{1}{360}(4\pi)^{-m/2}\{\int_M(60\tau E+180E^2+30\Omega^2+5\tau^2-2\rho^2+2R^2)dx$$
$$+\int_{\partial M}(-180e_m(E)-30e_m(\tau)+120EL_{aa}+20\tau L_{aa}-4\rho_{mm}L_{bb}$$
$$-12R_{ambm}L_{ab}+4R_{abcb}L_{ac}+\tfrac{40}{21}L_{aa}L_{bb}L_{cc}-\tfrac{88}{7}L_{ab}L_{ab}L_{cc}$$
$$+\tfrac{320}{21}L_{ab}L_{bc}L_{ca})dy\}.$$

We note that information concerning the invariants a_5 is available; see [**4**] for details.

§2 Operators defined by Affine Differential Geometry

2.1 Affine differential geometry of nondegenerate hypersurfaces. We refer to [**1**], [**5**], [**8**], [**9**], [**11**], and [**13**] for further details concerning this material. Let $\mathcal{A}$ be a real affine space which is modeled on a vector space V of dimension $m+1$, and let V^* be the dual space. The tangent space and cotangent space at a point $a \in \mathcal{A}$ are modeled on the vector space V and the dual vector space V^*, i.e. $T_a\mathcal{A} = V$ and $T_a^*\mathcal{A} = V^*$. Let $\langle \cdot, \cdot \rangle : V^* \times V \to \mathbb{R}$ be the natural pairing between V^* and V. Let x be a smooth immersion of M into $\mathcal{A}$. Let

$$C(M)_P = \{X \in V^* : \langle X, dx(v) \rangle = 0, \forall v \in T_P M^m\}$$

be the conormal space at a point $P \in M$; let $C(M)$ be the conormal line bundle over M. We assume that $C(M)$ is trivial and choose a non-vanishing conormal field X on M. We say that the hypersurface $x(M)$ is *regular* if and only if there exists a conormal field X such that $\operatorname{rank}(X, dX) = m+1$; if this condition is satisfied for one non-vanishing conormal field, it is satisfied for every non-vanishing conormal field so this notion is affinely invariant. We assume $x(M)$ is regular henceforth; this implies that X is an immersion from M to V^* such that the position field X is transversal to $X(M)$. Define $y = y(X) : M \to V$ by the conditions $\langle X, y \rangle = 1$ and $\langle dX, y \rangle = 0$. The triple (x, X, y) is called a *hypersurface with relative normalization.* Note that y need not be an immersion.

The relative structure equations given below contain the fundamental geometric quantities of relative hypersurface theory; two connections ${}^1\nabla$ and ${}^2\nabla$, the relative shape (Weingarten) operator B, and two symmetric forms g and $\hat{B}$. Let ${}^{\mathcal{A}}\nabla$ be the flat affine connection on $\mathcal{A}$. We have:

$$dy(v) = -dx(B(v)), \qquad \text{(Weingarten equation)}$$
$${}^{\mathcal{A}}\nabla_w dx(v) = dx({}^1\nabla_w v) + g(v, w)y, \qquad \text{(Gauss equation)}$$
$${}^{\mathcal{A}}\nabla_w dX(v) = dX({}^2\nabla_w v) - \hat{B}(v, w)X. \qquad \text{(Gauss equation)}$$

We shall assume that the metric g is positive definite henceforth; this means that the immersed hypersurface $x(M)$ is locally strongly convex. We will also assume M to be compact; if the boundary of M is empty and if M is simply connected, then M is a hyperovaloid. The relative shape operator B is self-adjoint with respect to g and is related to the Weingarten form $\hat{B}$ by the identity:

$$\hat{B}(v, w) = g(B(v), w) = g(v, B(w)).$$

We define a (1,2) difference tensor A, a totally symmetric relative cubic form $\hat{A}$, and the Tchebychev form $\hat{T}$ as follows:

$$A := \tfrac{1}{2}({}^1\nabla - {}^2\nabla), \ \hat{A}(v, w, z) := g(A(v, w), z), \text{ and } \hat{T}(z) := \tfrac{1}{m}\operatorname{tr}_g(A(z, \cdot)).$$

Let ';' denote multiple covariant differentiation with respect to the Levi-Civita connection ${}^g\nabla$. The Tchebychev tensor $\hat{T}$ has the symmetry property $\hat{T}_{i;j} = \hat{T}_{j;i}$;

see [**12**] for further details. Both the induced connection ${}^1\nabla$ and the conormal connection ${}^2\nabla$ are torsion free Ricci symmetric connections on TM. They are conjugate relative to g, i.e.

$$\tfrac{1}{2}({}^1\nabla + {}^2\nabla) = {}^g\nabla.$$

We call the triple $\{{}^1\nabla, g, {}^2\nabla\}$ a conjugate triple. Let A be the difference tensor defined above. We then have ${}^1\nabla = {}^g\nabla + A$ and ${}^2\nabla = {}^g\nabla - A$. Let $H := m^{-1}B_{ii}$ be the normalized mean curvature and recall the notation for the Ricci tensor from section §1.2. We have:

$$\rho_{{}^1\nabla} = mHg - B,\ \ \rho_{{}^2\nabla} = (m-1)B,\ \text{and}$$
$$\operatorname{tr}_g(\rho_{{}^1\nabla}) = \operatorname{tr}_g(\rho_{{}^2\nabla}) = m(m-1)H.$$

2.2 Definition.

(1) The *relative support function* ϱ with respect to $x_0 \in V$ is given by $\varrho = -\langle X, x - x_0\rangle$. If $b \in V$, define a generalized spherical function $F := \langle X, b\rangle$.

(2) A relative normalization is said to be *equiaffine* if the Tchebychev form $\hat{T}$ vanishes. A nondegenerate hypersurface with equiaffine normalization is called a *Blaschke hypersurface.* We denote the support function of this geometry by $\varrho(e)$.

(3) Consider a non degenerate hypersurface $x : M \longrightarrow V$ such that its position vector is transversal. Then $y(c) := -x$ is called the *centroaffine normal.* Following Nomizu we call such a hypersurface together with its centroaffine normalization $\{X(c), y(c)\}$ a *centroaffine hypersurface.* The associated geometry of $\{x, X(c), y(c)\}$ is invariant under the group $GL(n+1,\mathbb{R})$. Then

$$\hat{B} := \hat{B}(c) = g(c) = g,\ \ mH(c) = m,\ \text{and}\ \hat{T}(c) = \frac{m+2}{2m} d \lg|\varrho(e)|.$$

Recall that x is a *proper affine sphere* with center at $O \in V$ if and only if $\hat{T}(c) \equiv 0$. For a locally strongly convex hypersurface we choose the orientation such that $\varrho(e) > 0$.

2.3 Operators of Laplace type defined by relative normalizations. The connections ${}^1\nabla$ and ${}^2\nabla$ determined by a relative normalization (x, X, y) are torsion free Ricci symmetric connections. We assume the associated metric g is positive definite and use the construction described in §1.3 to define operators of Laplace type ${}^1D$ and ${}^2D$. These operators and their spectra are not affine invariants of the embedding x since they depend on the relative normalization chosen. However, Theorem 1.5 shows that the coefficients a_m where $m := \dim(M)$ are affine invariants. To study these invariants, we recall some notations and results from [**2**].

2.4 Lemma. *Let $\epsilon_1 = 1$ and let $\epsilon_2 = -1$. We adopt the notation of Lemma 1.6.*

(1) *We have $\Omega({}^rD) = 0$, $\Theta({}^rD) := \nabla - {}^g\nabla = \epsilon_r A$, $\omega({}^rD) = -\frac{1}{2}\epsilon_r m\hat{T}$, and $E({}^rD) = mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i}$; here $r = 1,2$ and $i = 1,\dots,m$.*

(2) *Let ${}^gD$ be determined by the Levi-Civita connection associated to the metric g. Then $\Omega({}^gD) = 0$, $\Theta({}^gD) = 0$, $\omega({}^gD) = 0$, and $E({}^gD) = \frac{1}{m-1}\tau_g$.*

We use Lemma 2.4 and Theorem 1.7 to determine the heat equation asymptotics in this setting:

2.5 Theorem. *Let M be a manifold with smooth boundary. Let $\{x, X, y\}$ be a relative normalization of a regular embedding x. Assume the associated quadratic form g is positive definite so M is strictly convex.*

(1) $a_0({}^rD) = (4\pi)^{-m/2}\mathrm{vol}(M)$.

(2) $a_1({}^rD) = -\frac{1}{4}(\pi)^{-(m-1)/2}\mathrm{vol}(\partial M)$.

(3) $a_2({}^rD) = (4\pi)^{-m/2}\int_M\{\frac{1}{6}\tau_g + mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i}\}dx$
$+\frac{1}{6}(4\pi)^{-m/2}\int_{\partial M} 2L_{aa}dy$.

(4) $a_3({}^rD) = -\frac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}\{96(mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i})$
$+16\tau_g + 8R_{g,amam} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab}\}dy$.

(5) $a_4({}^rD) = (4\pi)^{-m/2}\frac{1}{360}\int_M\{60\tau_g(mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i})$
$+180(mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i})^2$
$+60(mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i})_{;jj}$
$+12\tau_{g;kk} + 5\tau_g^2 - 2|\rho_g|_g^2 + 2|R_g|_g^2\}dx$
$+\frac{1}{360}(4\pi)^{-m/2}\int_{\partial M}\{-120(mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i})_{;m}$
$-18\tau_{g;m} + 120(mH - \frac{1}{4}m^2|\hat{T}|_g^2 + \frac{1}{2}\epsilon_r m\hat{T}_{i;i})L_{aa}$
$+20\tau_g L_{aa} + 4R_{g,amam}L_{bb} - 12R_{g,ambm}L_{ab} + 4R_{g,abcb}L_{ac} + 24L_{aa:bb}$
$+\frac{40}{21}L_{aa}L_{bb}L_{cc} - \frac{88}{7}L_{ab}L_{ab}L_{cc} + \frac{320}{21}L_{ab}L_{bc}L_{ca}dy\}$.

§3 Affine geometries where the operators ${}^1D$ and ${}^2D$ are self-adjoint

We begin our study with the following result:

3.1 Theorem. *Let M be a manifold with smooth boundary. Let $\{x, X, y\}$ be a relative normalization of a regular embedding x. Assume the associated quadratic form g is positive definite so M is locally strictly convex.*

(1) *The operator ${}^1D + {}^2D$ is self adjoint.*

(2) *Let x be a hyperovaloid and let $\{x, X, y\}$ be a relative normalization. The following assertions are equivalent:*

   2-a) *The operator ${}^1D$ is self-adjoint.*

   2-b) *The operator ${}^2D$ is self-adjoint.*

   2-c) *We have ${}^1D = {}^2D$.*

   2-d) *The Tchebychev tensor T vanishes identically.*

(3) *The Tchebychev tensor T vanishes identically if and only if the relative normalization $\{x, X, y\}$ is equiaffine.*

(4) *We have the identity: $\int_M(f \cdot {}^1D\tilde{f} - \tilde{f} \cdot {}^1Df) = \int_M(\tilde{f} \cdot {}^2Df - f \cdot {}^2D\tilde{f})$.*

Proof. Let Δ_g be the scalar Laplacian defined by the metric g. Since ${}^1\nabla = {}^g\nabla + A$, since ${}^2\nabla = {}^g\nabla - A$, and since $T = \operatorname{tr}_g A$, we see that

$$ {}^1Df = \Delta_g f - mT^r f_{;r} + mHf \text{ and } {}^2Df = \Delta_g f + mT^r f_{;r} + mHf. $$

Thus ${}^1D + {}^2D = 2\Delta_g + 2mHf$. Since $2mH$ is a term of order *zero* and $2\Delta_g$ is self-adjoint, the first assertion follows. Let $\mathcal{V}f := T^r f_{;r}$; this operator is self-adjoint if and only if the tensor T vanishes. Assertion (2) now follows. We refer to Proposition 4.13 in [**8**] for the proof of assertion (3). Assertion (4) follows since the operator ${}^1D + {}^2D$ is self-adjoint. □

We use similar techniques to prove the next result

3.2 Theorem.

(1) *Let $\{x, X, y\}$ be the Euclidean normalization. Then the following assertions are equivalent:*

1-a) *The Gauss-Kronecker curvature $K = K_n$ is constant.*

1-b) *We have ${}^1D = {}^2D$.*

1-c) *The operator ${}^1D$ or the operator ${}^2D$ is self-adjoint.*

(2) *Let x be a compact centroaffine hypersurface with nonempty boundary. The following assertions are equivalent:*

2-a) *We have ${}^1D = {}^2D$.*

2-b) *We have that ${}^1D$ or ${}^2D$ are self-adjoint.*

2-c) *We have that x is a proper affine sphere.*

(3) *Let x be a compact centroaffine hypersurface without boundary. Then the following assertions are equivalent:*

3-a) *We have ${}^1D = {}^2D$.*

3-b) *We have that ${}^1D$ or ${}^2D$ is self-adjoint.*

3-c) *We have that x is a hyperellipsoid.*

Proof. For a hypersurface with Euclidean normalization non-degeneracy means that

$$ K = K_n \neq 0 \text{ and } T = -\tfrac{1}{2n} d \lg |K|; $$

see [**13**], (6.1.2.1) for details. Thus if K is constant and non-zero, we have T vanishes identically, ${}^1D = {}^2D$, and these operators are self-adjoint. On the other hand, if $K \neq 0$ and if one of the other conditions is satisfied, then necessarily K is constant. This proves the first assertion.

If the hypothesis of (2) are satisfied, one can see that $T(c) \equiv 0$ so that therefore x is a proper affine sphere.

If x is compact without boundary a proof like that in case (2) together with the well known result of Blaschke and Deicke [**8**, p. 121] imply the third assertion. □

§4 Spectral invariants of affine geometry

In this section we use the heat equation asymptotics of the operators ${}^rD$ $(r = 1, 2)$ and ${}^gD$ on hypersurfaces M with non-empty boundary immersed in an affine space $\mathcal{A}$ to study their geometry.

Recall that if x is a compact, locally strongly convex Blaschke hypersurface, then ${}^1D = {}^2D$. We use Theorem 2.5 to establish the following Lemma:

4.1 Lemma. *Let x be a compact, locally strongly convex Blaschke hypersurface with boundary. Then we have*

(1) $a_0(D) = (4\pi)^{-m/2} vol(M)$.

(2) $a_1(D) = -\frac{1}{4}(4\pi)^{-(m-1)/2} vol(\partial M)$.

(3) $a_2(D) = \frac{1}{6}(4\pi)^{-m/2}\{\int_M(\tau_g + 6mH)dx + 2\int_{\partial M} L_{aa}dy\}$.

(4) $a_3(D) = -\frac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}\{96mH + 16\tau_g + 8R_{g,amam} + 7L_{aa}L_{bb} -10L_{ab}L_{ab}\}dy$.

(5) $a_4(D) = (360)^{-1}(4\pi)^{-m/2}\{\int_M(60mH\tau_g + 180m^2H^2 + 60mH_{;jj} + 12\tau_{g;kk} +5\tau_g^2 - 2|\rho_g|_g^2 + 2|R_g|_g^2)dx + \int_{\partial M}(-120mH_{;m} - 18\tau_{g;m} + 120mHL_{aa} +20\tau_gL_{aa} + 4R_{g,amam}L_{bb} - 12R_{g,ambm}L_{ab} + 4R_{g,abcb}L_{ac} +\frac{40}{21}L_{aa}L_{bb}L_{cc} - \frac{88}{7}L_{ab}L_{ab}L_{cc} + \frac{320}{21}L_{ab}L_{bc}L_{ca})dy\}$.

We can use this Lemma to draw the following conclusion:

4.2 Theorem. *Let x be a compact, locally strongly convex Blaschke hypersurface with boundary. Then:*

(1) *Let (x, X, y) be a relative normalization. Let $c(m) := 384(4\pi)^{(m-1)/2}$.*
 - 1-a) *If $m < 5$, then $c(m)\{a_3(D) - a_3({}^gD)\} \leq \int_{\partial M} 96mJ$.*
 - 1-b) *If $m = 5$, then $c(m)\{a_3(D) - a_3({}^gD)\} = \int_{\partial M} 96mJ$.*
 - 1-c) *If $m > 5$, then $c(m)\{a_3(D) - a_3({}^gD)\} \geq \int_{\partial M} 96mJ$.*

(2) *Equality in assertions* (1-a) *and* (1-c) *holds if and only if the normalization is equiaffine.*

(3) *If the normalization is equiaffine, then $a_3(D) - a_3({}^gD) \geq 0$. Equality holds if and only if $\int_{\partial M} J = 0$.*

Proof. For the operator ${}^gD$, we have $\Theta = \omega = \Omega = 0$ and $E_D = \frac{1}{m-1}\tau_g$. Thus we may use the formulas given previously to see that

$$a_3(D) = -c(m)\int_{\partial M}\{\tfrac{16(5+m)}{m-1}\tau_g + 8R_{g,amam} + 7L_{aa}L_{bb} - 10L_{ab}L_{ab}\}$$
$$c(m)(a_3(D) - a_3({}^gD)) = \int_{\partial M}\{\tfrac{96}{m-1}\tau_g - 96(mH - \tfrac{1}{4}m^2|\hat{T}|^2)\}$$

We use the Theorema egregium in relative geometry (see [13], 4.12.2.2) to see that

$$\kappa = J + H - \tfrac{m}{m-1}|\hat{T}|^2 \text{ i.e. } \tau_g = m(m-1)(J + H) - m^2|\hat{T}|^2.$$

We combine these two displays to see that

$$c(m)(a_3(D) - a_3({}^gD)) = 96m \int_{\partial M} \{J + \tfrac{m(m-5)}{4(m-1)}|\hat{T}|^2\}.$$

As the metric g is positive definite, we have that $J \geq 0$ and $|\hat{T}|^2 \geq 0$; the first assertion holds. Recall that $\hat{T} = 0$ characterizes an equiaffine normalization. The second assertion now follows as $J \geq 0$. □

4.3 Corollary. *Let x be a compact, locally strongly convex Blaschke hypersurface with boundary. Assume that $m \geq 3$, that the affine mean curvature H is constant on M, and that $a_3(D) = a_3({}^gD)$. We may then conclude that $x(M)$ lies on a quadric.*

Proof. The previous result shows that $\int_{\partial M} J = 0$. The result now follows from Theorem 3.1.6.5 in [**8**]. □

4.4 Remark. One can prove an analogous result assuming $H_1 := H \neq 0$ and the quotient $\frac{H_2}{H_1}$ is a non-zero constant on M where H_r is the r-th $(r = 1, \ldots, m)$ normed elementary symmetric function of the affine principal curvatures (apply 3.1.6.8 in [**8**]).

4.5 Volumes of convex bodies. Let M be a compact Blaschke hyperovaloid without boundary. If $f, f^{\#}$ are smooth functions on M, then Theorem 3.1 implies the following integral formula for the operator $D = {}^1D = {}^2D$:

$$(1) \quad \int D(ff^{\#}) = \int \{fDf^{\#} + \langle \text{grad} f, \text{grad} f^{\#}\rangle\} = \int \{f^{\#}Df + \langle \text{grad} f, \text{grad} f^{\#}\rangle\}.$$

Let ϱ and F be as defined in §2.2. We then have $D\varrho = {}^2D\varrho = m$ and $DF = {}^2DF = 0$ on M. We refer to [**13**, §4.13] for details. Recall that, for a hyperovaloid and for any choice of basepoint $x_0 \in V$, the volume of the convex body $\mathcal{K}$ enclosed by M is given by

$$\text{vol}(\mathcal{K}) = \tfrac{1}{m+1} \int_M \varrho.$$

4.6 Theorem. *Let $x : M \longrightarrow \mathcal{A}$ be a Blaschke hyperovaloid. Then*

(1) *We have $m \int H\varrho^2 = m(m+1)\text{vol}(\mathcal{K}) + \int \|\text{grad}\varrho\|^2$.*
(2) *We have $\int H\varrho^2 \geq (m+1)\text{vol}(\mathcal{K})$. Furthermore, equality holds if and only if x is a hyperellipsoid.*

Proof. Let $1 \in C^\infty(M)$ denote the constant function. Then $D1 = mH$. Note D is self-adjoint and the boundary of M is empty. We use equation (1) and the observations made in §4.5 to see that

$$\begin{aligned} m \int_M \varrho^2 H &= \int_M \varrho^2 D1 = \int_M D(\varrho^2) = \int_M (\varrho D\varrho + \|\text{grad}\varrho\|^2) \\ &= \int_M (m\varrho + \|\text{grad}\varrho\|^2) = m(m+1)\text{vol}(\mathcal{K}) + \int_M \|\text{grad}\varrho\|^2. \end{aligned}$$

Assume that equality holds in the second assertion. Then $\varrho = \text{const}$. We apply Lemma 7.2.4 of [**13**] to see that x is an affine sphere. We use a Theorem of Blaschke and Deicke (see Theorem 2.4.7 [**7**]) to see x is a hyperellipsoid. □

Recall the affine isoperimetric inequality for Blaschke hyperovaloids ([**8**], p. 237). Denote by σ_{m+1} the volume of the unit ball in Euclidean $(m+1)$-space. Let $\text{Area} := \int 1$ be the affine area of the hyperovaloid. We then have the inequality

$$(\text{Area})^{m+2} \leq \{(m+1)\sigma_{m+1}\}^2 \cdot \{(m+1)\text{vol}(\mathcal{K})\}^m.$$

Equality holds exactly for hyperellipsoids. We can now establish the result:

4.7 Corollary. *Let x be a Blaschke hyperovaloid.*

1. *We have $(\text{Area})^{m+2} \leq \{(m+1)\sigma_{m+1}\}^2 \cdot \{\int H\varrho^2\}^m$; equality holds if and only if x is a hyperellipsoid.*
2. *We have $\{\int H\varrho\}^{m+2} \leq \{(m+1)\sigma_{m+1}\}^2 \cdot \{\int H\varrho^2\}^m$; equality holds if and only if x is a hyperellipsoid.*
3. *Assume that the affine Weingarten operator has maximal rank. This implies that the m-th curvature function, the affine Gauss-Kronecker curvature, is nonzero. Then $\int \frac{H_{m-1}}{H_m} \leq \int H\varrho^2$; equality holds exactly for hyperellipsoids.*

Proof. The first assertion is obvious from the previous discussion; the second assertion follows from the affine Minkowski formula $\int 1 = \int H\varrho$ ([**8**, p. 165]) and the third from a related formula $\int \frac{H_{m-1}}{H_m} = \int \varrho$ (l.c., p.169). For the discussion of equality compare the proof of Theorem 4.6. □

We now turn our attention to centroaffine normalizations. The following formula are immediate from our previous calculations.

4.8 Lemma. *Let x have centroaffine normalization. We have*

1. $a_0({}^1D) = (4\pi)^{-m/2}\mathit{vol}(M)$.
2. $a_1({}^1D) = -\frac{1}{4}(4\pi)^{-(m-1)/2}\mathit{vol}(\partial M)$.
3. $a_2({}^1D) = (4\pi)^{-m/2}\{\int_M(\frac{1}{6}\tau_g - \frac{1}{4}m^2|\hat{T}|^2 + \frac{1}{2}m\hat{T}_{i;i})dx + 2\int_{\partial M} L_{aa}dy\}$ $+a_0({}^1D)$.
4. $a_3({}^1D) = -\frac{1}{384}(4\pi)^{-(m-1)/2}\int_{\partial M}\{-24m^2|\hat{T}|^2 + 16\tau_g + 8R_{g,amam} + 7L_{aa}L_{bb}$ $-10L_{ab}L_{ab}\}dy + a_1({}^1D)$.

§5 The geometry of affine Gauss maps

In this section we consider a Blaschke hypersurface $\{x, X, y\}$ and its two affine Gauss maps $X : M \longrightarrow V^*$ and $y : M \longrightarrow V$, see [**13**, §4.6]. Then X is an immersion with transversal position vector (also denoted by X), while y is an immersion if and only if the equiaffine Weingarten operator B satisfies $\text{rank}(B) =$

m. In the latter case, both Gauss maps define centroaffine hypersurfaces in the sense of §2.2 above, i.e. $\check{X} := -X$ and $\check{y} := -y$ are their centroaffine normals, respectively.

If $\operatorname{rank}(B) = m$ the Gauss structure equations take the form

$$\bar{\nabla}_v dX(w) = dX({}^2\nabla_v w) + h(\check{X})(v,w)\check{X}$$
$$\bar{\nabla}_v dy(w) = dy({}^1\check{\nabla}_v w) + h(\check{y})(v,w)\check{y}.$$

The centroaffine metrics of both hypersurfaces X and y coincide. This means that $h(\check{X}) = \hat{B} = h(\check{y})$ and we have

5.1 Lemma.

(1) *We have that* $\{{}^1\check{\nabla}, \hat{B}, {}^2\nabla\}$ *is a conjugate triple. The connection* ${}^1\check{\nabla}$ *is torsion-free and Ricci symmetric and satisfies the relation given in* [**10**, section 5]*:* ${}^1\check{\nabla}_u v = B^{-1}({}^1\nabla_u(Bv))$.

(2) *We have that* ${}^1\check{\nabla}$ *is the induced connection of the Gauss map* y*, and that* ${}^2\nabla$ *is the associated conormal connection.*

(3) *We have that* ${}^2\nabla$ *is the induced connection of the conormal Gauss map* X*, and that* ${}^1\check{\nabla}$ *is the associated conormal connection.*

5.2 Lemma. *Let* $\{x, X, y\}$ *be a Blaschke hypersurface with* $\operatorname{rank}(B) = m$. *Then we have the following statements are equivalent:*

(1) *The equiaffine Gauss-Kronecker curvature* $H_m = H_m(e) = \det(B)$ *is a nonzero constant.*

(2) *The map* y *defines a proper affine sphere.*

(3) *The map* X *defines a proper affine sphere.*

(4) *The Tchebychev field* $\check{T} = \check{T}(y)$ *vanishes.*

Proof. Lemma 5.2 follows from the relation $2m\check{T} = d\lg|H_m(e)|$. This follows from Lemma 5.1 and the result $T(e) \equiv 0$; see [**7**, p. 182] for further details. □

For a Blaschke hypersurface with $\operatorname{rank}(B) = m$ and associated conjugate triple $\{{}^1\check{\nabla}, \hat{B}, {}^2\nabla\}$ we have associated operators ${}^1\check{D}$, ${}^2D$ and ${}^{\hat{B}}D$ according to the definitions in section 1.3 above. The following result is proved analogously with previously established results:

5.3 Theorem. *Let* x *be a Blaschke hyperovaloid with* $\operatorname{rank}(B) = m$. *Then*

(1) *We have that the operators* ${}^1\check{D}$ *and* ${}^2\check{D} = {}^2D$ *satisfy the global conjugacy relation:* $\int_M (f \cdot {}^1\check{D}\tilde{f} - \tilde{f} \cdot {}^1\check{D}f) = \int_M (\tilde{f} \cdot {}^2\check{D}f - f \cdot {}^2\check{D}\tilde{f})$.

(2) *The following assertions are equivalent:*

  2-a) *The operator* ${}^1\check{D}$ *is self-adjoint.*

  2-b) *The operator* ${}^2D = {}^2\check{D}$ *is self-adjoint.*

  2-c) *The immersion* x *defines a hyperellipsoid.*

Proof. Since the centroaffine normalization is a relative normalization we apply Theorem 3.1 to establish the first assertion. We have that the operator ${}^r\check{D}$ is self-adjoint if and only if the map y defines a hyperellipsoid. We use Lemma 5.2 to see that this implies $H_m(e) = \text{const}$. We use [**8**, Theorem 3.1.26] to conclude x is a hyperellipsoid and show 2-a) or 2-b) implies 2-c); the converse is immediate. □

5.4 Theorem. *Let $x : M \longrightarrow \mathcal{A}$ be a locally strongly convex Blaschke hypersurface with boundary. Then the following assertions are equivalent.*

(1) *We have* $H_m(e) = \text{const} \neq 0$.
(2) *We have* ${}^1\check{D} = {}^2\check{D}$.
(3) *We have the operator* ${}^1\check{D}$ *is self-adjoint.*
(4) *We have the operator* ${}^2\check{D}$ *is self-adjoint.*

Proof. Use the relationship $2m\check{T} = d\lg|H_m(e)|$ given above; the proof then is analogous to the proof of Theorem 3.2 (1). □

References

[1] W. Blaschke, *Vorlesungen über Differentialgeometrie; II. Affine Differentialgeometrie*, Springer-Verlag, Berlin, 1923.

[2] N. Bokan, P.B. Gilkey and U. Simon, *Applications of spectral geometry to affine and projective geometry*, Beiträge zur Algebra und Geometrie **35** (1994), 283–314.

[3] T. Branson and P.B. Gilkey, *The asymptotics of the Laplacian on a manifold with boundary*, Comm. in PDE **15** (1990), 245–272.

[4] T. Branson, P. Gilkey, and D. Vassilevich, *The asymptotics of the Laplacian on a manifold with boundary II*, Suppl. fasc. 2, Boll. Unione Mat. Ital. **11-B** (1997), 39–67.

[5] E. Calabi, *Hypersurfaces with maximal affinely invariant area*, Amer. J. Math. **104** (1982), 91–126.

[6] P. Gilkey, *Invariance Theory, the Heat Equation, and the Atiyah-Singer Index theorem (2^{nd} edition)*, ISBN 0-8493-7874-4, CRC Press, Boca Raton, Florida, 1994.

[7] A.M. Li, U. Simon and B. Chen, *A two step Monge-Ampère procedure for solving a fourth order PDE for affine hypersurfaces with constant curvature*, J. Reine Ang. Math. **487** (1997), 179–200.

[8] A.M. Li, U. Simon and G. Zhao, *Global affine differential geometry of hypersurfaces*, De Gruyter, Berlin and New York, 1993.

[9] K. Nomizu and T. Sasaki, *Affine differential geometry*, Cambridge Univ. Press, 1993.

[10] K. Nomizu and U. Simon, *Notes on conjugate connections*, Geom.& Top. of Submanifolds **IV** (eds.: F. Dillen et al.) World Scientific Singapore (1992), 152–172.

[11] P.A. and A.P. Schirokow, *Affine Differentialgeometrie*, Teubner Verlag, Leipzig, 1962.

[12] U. Simon, *Connections and conformal structure in affine differential geometry*, Differential geometry and its Applications, Proc. of the Conf. August 24–30, (1986), Brno, Czechoslovakia, 315–327.

[13] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, *Introduction to the affine differential geometry of hypersurfaces*, Lecture Notes Science University Tokyo, 1991, ISBN 3798315299.

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