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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SOME DIFFERENCE EQUATIONS

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Abstract. We consider the oscillation and asymptotic behaviour of nonoscillatory solutions of a class of nonlinear difference equations.

1. Introduction

We consider a nonlinear difference equation

$$\Delta(r_n \Delta(u_n + p_n u_{n-k})) = q_n f(u_{n-l}), \qquad n = 0, 1, 2, \dots \tag{1}$$

where Δ denotes the forward difference operator, i.e., $\Delta v_n = v_{n+1} - v_n$ for any sequence (v_n) of real number, k and l are nonnegative integers, (p_n) and (q_n) are sequences of real numbers with $q_n \geq 0$ eventually, (r_n) is a sequence of positive numbers and

$$\sum_{n=0}^{\infty} \frac{1}{r_n} = \infty. \tag{2}$$

The function f is real valued function satisfying $uf(u) > 0$ for $u \neq 0$.

By a solution of (1) we mean a sequence (u_n) which is defined for $n \geq -\max\{k, l\}$ and satisfies (1) for all large n. A nontrivial solution (u_n) of (1) is said to be oscillatory if for every positive integer n_0 there exists $n \geq n_0$ such that $u_n u_{n+1} \leq 0$. Otherwise it is called nonoscillatory.

Recently, there has been considerable interest in the study of oscillation and asymptotic behaviour of solutions of difference equations; see for example [**2**], [**3**], [**5**–**15**] and the references cited therein. For the general theory of difference equations one can refer to [**1**] and [**4**].

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Our purpose is to study the oscillatory and asymptotic behaviour of nonoscillatory solutions of equation (1). The obtained results extend those contained in [**14**].

2. Main results

Here we give some oscillatory and asymptotic properties of solution of (1).

We will need the following assumptions:

$$f(u) \text{ is bounded away from zero if } u \text{ is bounded away from zero,} \tag{3}$$

$$\sum_{n=0}^{\infty} q_n = \infty. \tag{4}$$

The following lemma describes some asymptotic properties of the sequence (z_n) defined as follows:

$$z_n = u_n + p_n u_{n-k}, \tag{5}$$

where (u_n) is a nonoscillatory solution of (1).

LEMMA. *Assume that* (3) *and* (4) *hold and there exists a constant* P_1 *such that* $P_1 \le p_n \le 0$.

(*a*) *If* (u_n) *is an eventually positive solution of* (1), *then the sequences* (z_n) *and* $(r_n \Delta z_n)$ *are eventually monotonic and either*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = \infty \tag{6}$$

or

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = 0, \qquad \Delta z_n < 0 \text{ and } z_n > 0. \tag{7}$$

(*b*) *If* (u_n) *is an eventually negative solution of* (1), *then the sequences* (z_n) *and* $(r_n \Delta z_n)$ *are eventually monotonic and either*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = -\infty \tag{8}$$

or

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} r_n \Delta z_n = 0, \qquad \Delta z_n > 0 \text{ and } z_n < 0. \tag{9}$$

Proof. Let (u_n) be an eventually positive solution of (1), say $u_{n-k} > 0$ and $u_{n-l} > 0$ for $n \ge n_0$. From (1) we have

$$\Delta(r_n \Delta z_n) = q_n f(u_{n-l}) \ge 0 \qquad \text{for } n \ge n_0 \tag{10}$$

that is $(r_n \Delta z_n)$ is nondecreasing, which implies that (Δz_n) is eventually of constant sign and in consequence (z_n) is eventually monotonic.

First suppose there exists $n_1 \geq n_0$ such that $\Delta z_{n_1} \geq 0$, then since $q_n \equiv 0$ eventually, there exists $n_2 \geq n_1$ such that $r_n \Delta z_n \geq r_{n_2} \Delta z_{n_2} = c > 0$ for $n \geq n_2$. Summing the above inequality, by (2) we have

$$z_n \geq z_{n_2} + c \sum_{i=n_2}^{n-1} \frac{1}{r_i} \to \infty \qquad n \to \infty,$$

hence $z_n \to \infty$ as $n \to \infty$.

Since $u_n \geq z_n$, so $u_n \to \infty$ as $n \to \infty$. Then summing (10) we get

$$r_n \Delta z_n = r_{n_2} \Delta z_{n_2} + \sum_{i=n_2}^{n-1} q_i f(u_{i-l})$$

which in view of (3) and (4), implies that $r_n \Delta z_n \to \infty$ as $n \to \infty$, and thus (6) holds.

Now, if $\Delta z_n < 0$ for $n \geq n_0$, then $r_n \Delta z_n \to L \leq 0$ as $n \to \infty$. Summing (10) from n to m and letting $m \to \infty$ gives

$$\sum_{i=n}^{\infty} q_i f(u_{i-l}) = L - r_n \Delta z_n < \infty.$$

The last inequality together with (3) and (4) implies

$$\lim_{n \to \infty} \inf u_n = 0. \tag{11}$$

Suppose that $L < 0$. Then we have $r_n \Delta z_n \leq L$ for $n \geq n_0$. Also, we can choose $n_3 \geq n_0$ such that $z_{n_3} < 0$. Summing the above inequality we get

$$z_n \leq z_{n_3} + L \sum_{i=n_3}^{n-1} \frac{1}{r_i} < L \sum_{i=n_3}^{n-1} \frac{1}{r_i} \quad \text{for } n > n_3$$

and, by assumption, we obtain

$$P_1 u_{n-k} \leq p_n u_{n-k} < z_n < L \sum_{i=n_3}^{n-1} \frac{1}{r_i}, \qquad n > n_3$$

so

$$u_{n-k} > \frac{L}{P_1} \sum_{i=n_3}^{n-1} \frac{1}{r_i} \to \infty \qquad n \to \infty,$$

which contradicts (11). Thus $\lim_{n \to \infty} r_n \Delta z_n = 0$. Next we show that $z_n > 0$ for $n \geq n_0$. If not, then there exists $n_4 \geq n_0$ such that $z_{n_4} \leq 0$, then since $\Delta z_n < 0$ for $n \geq n_0$ $z_n < z_{n_5} < 0$ for $n \geq n_5 \geq n_4$ that is

$$u_n < z_{n_5} - p_n u_{n-k} \quad \text{for } n \geq n_5 \tag{12}$$

By (11), there is an increasing sequence of positive integers (n_i) such that $u_{n_i-k} \to 0$ as $i \to \infty$. This together with the assumption about (p_n) and (12) implies that there exists i_0 such that $u_{n_{i_0}} < z_{n_5}/2 < 0$, contradicting $u_n > 0$ eventually.

Since (z_n) is decreasing, $z_n \to L_1 \geq 0$. If $L_1 > 0$, then $u_n \geq z_n \geq L_1$, contradicting (11) Thus (7) holds and (a) is proved.

The proof of (b) is similar to that of (a) and hence will be omitted.

THEOREM 1. *Suppose that* (3) *and* (4) *holds. If there exists a constant* P_2 *such that* $P_2 \leq p_n \leq -1$, *then every nonoscillatory solution* (u_n) *of* (1) *satisfies* $|u_n| \to \infty$ *as* $n \to \infty$.

Proof. If (u_n) is an eventually positive solution of (1) such that (u_n) does not tend to ∞ as $n \to \infty$, then (6) cannot hold since $z_n \leq u_n$ eventually. Thus, by Lemma (a) (7) holds. Moreover, from the proof of (7) we have (11) holding. But

$$0 < z_n = u_n + p_n u_{n-k} \leq u_n - u_{n-k},$$

so $u_n > u_{n-k}$ which contradicts (11). This completes the proof for $u_n > 0$. The proof is similar when (u_n) is eventually negative.

From Theorem 1 we immediately obtain

COROLLARY 1. *Under the assumptions of Theorem* 1 *all bounded solutions of* (1) *are oscillatory.*

THEOREM 2. *Suppose that there exists a constant* P_3 *such that* $-1 < P_3 \leq p_n \leq 0$ *and that* f *is a nondecreasing continuous function such that*

$$\int_0^{\pm a} \frac{du}{f(u)} < \infty, \qquad a > 0. \tag{13}$$

If

$$\sum_{n=n_0}^{\infty} \frac{1}{r_{n-l}} \sum_{i=n-l}^{n} q_i = \infty, \tag{14}$$

then every nonoscillatory solution (u_n) *of* (1) *satisfies either* $|u_n| \to \infty$ *or* $u_n \to 0$ *as* $n \to \infty$.

Proof. Assume that (u_n) is an eventually positive solution of (1) which does not satisfy our assertion. Then for (z_n) defined in (5) we see from (1), that $\Delta(r_n \Delta z_n) \geq 0$ eventually that is $(r_n \Delta z_n)$ is nondecreasing and $(z_n)$is eventually monotonic. Now if (z_n) is eventually nonpositive, then the assumption concerning (p_n) implies $u_n \leq -p_n u_{n-k} \leq -P_3 u_{n-k}$ so $u_{n+k} \leq -P_3 u_n$ for all n sufficiently large, say for $n \geq n_0$. It then follows by induction that for all $n \geq n_0$ we have $u_{n+ik} \leq (-P_3)^i u_n$ for every positive integer i. Since $0 < -P_3 < 1$, the last inequality implies that $u_n \to 0$ as $n \to \infty$ which contradicts our assumption. Also, if there exists $n_1 \geq n_0$ such that $\Delta z_{n_1} \geq 0$, then there is $n_2 \geq n_1$ such that

$r_n\Delta z_n \geq r_{n_2}\Delta z_{n_2} > 0$ for $n \geq n_2$ which, by (2), implies that $z_n \to \infty$ as $n \to \infty$. Since $u_n \geq z_n$ we have $u_n \to \infty$ as $n \to \infty$, again a contradiction to our assumptions on (u_n).

Therefore we have $z_n > 0$ and $\Delta z_n < 0$ for $n \geq n_0$. Since $0 < z_n \leq u_n$ and f is nondecreasing from (1) we get

$$\Delta(r_n\Delta z_n) \geq q_n f(z_{n-l}) \qquad \text{for } n \geq n_1 = n_0 + l$$

Summing the above inequality we obtain

$$r_{n+1}\Delta z_{n+1} - r_{n-l}\Delta z_{n-l} \geq \sum_{i=n-l}^{n} q_i f(z_{i-l})$$

and so

$$\sum_{i=n-l}^{n} q_i f(z_{i-l}) \leq -r_{n-l}\Delta z_{n-l} \qquad n \geq n_1.$$

In view of monotonicity of (z_n) and f we see that

$$\frac{f(z_{n-l})}{r_{n-l}} \sum_{i=n-l}^{n} q_i \leq -\Delta z_{n-l},$$

and further

$$\frac{1}{r_{n-l}} \sum_{i=n-l}^{n} q_i \leq \frac{-\Delta z_{n-l}}{f(z_{n-l})} \leq \int_{z_{n+1-l}}^{z_{n-l}} \frac{du}{f(u)}, \qquad n \geq n_1.$$

Summing the last inequality from n_1 to n by (13) we get

$$\sum_{j=n_1}^{n} \frac{1}{r_{j-l}} \sum_{i=n-l}^{n} q_i \leq \int_{z_{n+1-l}}^{z_{n_1-l}} \frac{du}{f(u)} < \int_{0}^{z_{n_1-l}} \frac{du}{f(u)} < \infty,$$

which contradicts (14). The proof is similar when (u_n) is eventually negative.

COROLLARY 2. *Under the assumptions of Theorem 2 any bounded solution of* (1) *is either oscillatory or tends to zero as* $n \to \infty$.

THEOREM 3. *Assume that there exist constants* P_3 *and* P_4 *such that either*

$$-1 < P_3 \leq p_n \leq 0 \tag{15}$$

or

$$0 \leq p_n \leq P_4 < 1. \tag{16}$$

Then every unbounded solution (u_n) *of* (1) *is either oscillatory or satisfies* $|u_n| \to \infty$ *as* $n \to \infty$.

Proof. Let (u_n) be an unbounded solution of (1) which is eventually positive, say $u_{n-k} > 0$ and $u_{n-l} > 0$ for $n \geq n_0$. Then as before we have $\Delta(r_n \Delta z_n) \geq 0$ for $n \geq n_0$, so $(r_n \Delta z_n)$ is nondecreasing and hence (z_n) is monotonic.

First assume that (15) holds. Then it follows that $z_n > 0$ for $n \geq n_1 \geq n_0$. Otherwise, there exists $n_2 \geq n_1$ such that $u_n + p_n u_{n-k} = z_n \leq 0$ for $n \geq n_2$ and (15) implies that $u_n \leq -P_3 u_{n-k} \leq u_{n-k}$. This implies that (u_n) is bounded, a contradiction.

Further we claim that (Δz_n) is eventually positive. Otherwise, (z_n) is decreasing and hence is bounded from above, say $0 < z_n \leq M$ for some constant M. Therefore $u_n = z_n - p_n u_{n-k} \leq M - P_3 u_{n-k}$. Since (u_n) is unbounded there is an increasing sequence of positive integers (n_i) such that $u_{n_i} \to \infty$ as $i \to \infty$ and $u_{n_i} = \max_{n_1 \leq n \leq n_i} u_n$. Then we have

$$u_{n_i} \leq M - P_3 u_{n_i - k} \leq M - P_3 u_{n_i},$$

so $(1 + P_3) u_{n_i} \leq M$ for all i which is impossible in view of (15)

Finally, observe, as in the proof of Lemma, that $(r_n \Delta z_n)$ nondecreasing and (Δz_n) eventually positive implies that $z_n \to \infty$ as $n \to \infty$ and hence $u_n \to \infty$ as $n \to \infty$ since $u_n \geq z_n$.

Now assume that (16) holds. Then it is clear that $z_n > 0$ for $n \geq n_0$. Also we see that (Δz_n) is eventually positive. In fact, if not, then (z_n) is decreasing and so is bounded from above and since $z_n \geq u_n$ (u_n) is bounded, a contradiction.

As previously we conclude that $z_n \to \infty$ as $n \to \infty$. Since $z_n \leq u_n + P_4 z_{n-k} \leq u_n + P_4 z_n$ we have $(1 - P_4) z_n \leq u_n$ which in view of (16), implies $u_n \to \infty$ as $n \to \infty$.

A similar argument treats the case of eventually negative solution.

THEOREM 4. *Suppose that there exist constants* P_5 *and* P_6 *such that* $P_5 \leq p_n \leq P_6 < -1$ *and* f *is a nondecreasing continuous function such that*

$$\int_{\varepsilon}^{\infty} \frac{du}{f(u)} < \infty, \qquad \int_{-\varepsilon}^{-\infty} \frac{du}{f(u)} < \infty, \qquad \varepsilon > 0. \tag{17}$$

If

$$\sum_{n=n_0}^{\infty} \frac{1}{r_{n-l}} \sum_{i=n-k+1}^{\infty} q_i = \infty \qquad \text{when } l \geq k, \tag{18}$$

or

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{i=n}^{\infty} q_i = \infty \qquad \text{when } l < k \tag{19}$$

then all bounded solutions of (1) *are oscillatory.*

Proof. Assume that there exists a bounded nonoscillatory solutions (u_n) of (1) and let $u_n > 0$ eventually, say $u_{n-k-l} > 0$ for $n \geq n_0$. Then as before for the sequence (z_n) defined in (5) it follows that $(r_n \Delta z_n)$ is a nondecreasing sequence and in consequence (z_n) is eventually monotonic. We show first that (z_n) is eventually negative. If there exists $n_1 \geq n_0$ such that $z_{n_1} > 0$, then by the assumptions we get $u_{n_1} = z_{n_1} - p_{n_1} u_{n_1-k} > -P_6 u_{n_1-k}$. Then it follows by induction that $u_{n_1+ik} > (-P_6)^i u_{n_1}$, which implies $u_{n_i+ik} \to \infty$ as $i \to \infty$ contradicting the boundedness of (u_n). Therefore (z_n) is eventually negative, say for $n \geq n_0$. Now we observe that $\Delta z_n < 0$ for $n \geq n_0$. If not, then a similar argument as in the proof of Lemma leads to the fact that $z_n \to \infty$ contradicting $z_n < 0$ for $n \geq n_0$. By assumption, we have $P_5 u_{n-k} \leq p_n u_{n-k} < z_n < 0$, which implies that $0 < z_{n+k}/P_5 < u_n$ for $n \geq n_0$.

In view of monotonicity of f from (1) we see that

$$\Delta(r_n \Delta z_n) \geq q_n f\Big(\frac{z_{n+k-l}}{P_5}\Big) \qquad \text{for } n \geq n_1 = n_0 + l \tag{20}$$

Summing (20) from $n-k$ to $m > n-k$ we obtain

$$r_{m+1}\Delta z_{m+1} - r_{n-k}\Delta z_{n-k} \geq \sum_{i=n-k}^{m} q_i f\Big(\frac{z_{i+k-l}}{P_5}\Big).$$

After letting $m \to \infty$, we have

$$-r_{n-k}\Delta z_{n-k} \geq \sum_{i=n-k}^{\infty} q_i f\Big(\frac{z_{i+k-l}}{P_5}\Big) \geq \sum_{i=n-k+1}^{\infty} q_i f\Big(\frac{z_{i+k-l}}{P_5}\Big),$$

from which we get

$$-r_{n-k}\Delta z_{n-k} \geq f\Big(\frac{z_{n+1-l}}{P_5}\Big) \sum_{i=n-k+1}^{\infty} q_i. \tag{21}$$

Since $(r_n \Delta z_n)$ is nondecreasing, for $l \geq k$ we have $r_{n-l}\Delta z_{n-l} \leq r_{n-k}\Delta z_{n-k}$ and further from (21) we obtain

$$\frac{1}{r_{n-l}} \sum_{i=n-k+1}^{\infty} q_i \leq -\frac{\Delta z_{n-l}}{f\Big(\frac{z_{n+1-l}}{P_5}\Big)} \quad \text{for } n \geq n_1. \tag{22}$$

In view of monotonicity of (z_n) and f for $z_{n-l}/P_5 \leq u \leq z_{n+1-l}/P_5$ we have

$$\frac{1}{f(u)} \geq \frac{1}{f\Big(\frac{z_{n+1-l}}{P_5}\Big)}$$

and so

$$\int_{z_{n-l}/P_5}^{z_{n+1-l}/P_5} \frac{du}{f(u)} \geq \frac{1}{P_5} \frac{\Delta z_{n-l}}{f\left(\frac{z_{n+1-l}}{P_5}\right)} \qquad \text{for } n \geq n_1. \tag{23}$$

Now using (23) in (22) and summing both sides from n_1 to n we get

$$\sum_{j=n_1}^{n} \frac{1}{r_{j-l}} \sum_{i=j-k+1}^{\infty} q_i \leq -P_5 \int_{z_{n_1-l}/P_5}^{z_{n+1-l}/P_5} \frac{du}{f(u)}, \qquad n \geq n_1$$

which in view of (17) contradicts the condition (18).

If $l < k$, then summing (20) from n to $m > n$ and letting $m \to \infty$ we obtain

$$-r_n \Delta z_n \geq \sum_{i=n}^{\infty} q_i f\left(\frac{z_{i+k-l}}{P_5}\right) \geq f\left(\frac{z_{n+k-l}}{P_5}\right) \sum_{i=n}^{\infty} q_i. \tag{24}$$

Since $n + k - l \geq n + 1$, it follows that

$$f\left(\frac{z_{n+1}}{P_5}\right) \leq f\left(\frac{z_{n+k-l}}{P_5}\right).$$

Therefore from (24) we get

$$\frac{1}{r_n} \sum_{i=n}^{\infty} q_i \leq -\frac{\Delta z_n}{f\left(\frac{z_{n+1}}{P_5}\right)} \qquad \text{for } n \geq n_1$$

and the rest of the proof follows analogously to that as above.

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