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PERTURBED INTERPOLATION PROBLEMS

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Abstract. A sort of perturbations of interpolation problems is used, which enables to transfer the known results on interior interpolation problems of Pick–Nevanlinna type to boundary value interpolation problems. It is shown that the nonnegativity and the so called absence-of-residues property of the Pick kernel are necessary and sufficient condition for solvability of a matricial directional boundary value interpolation problem of Pick–Nevanlinna type.

1. Introduction. The classical Pick–Nevanlinna interpolation problem [**1**], [**2**] involves finding a condition ensuring the existence of a complex function ψ, analytic and with a nonnegative imaginary part in the open upper half-plane $\mathbf{C}^+$ of the complex plane, taking prescribed values at points of a given finite set $E \subset \mathbf{C}^+$. The problem has been solved independently by Pick [**1**] and Nevanlinna [**2**]. Pick established the condition in the form of nonnegativity of a certain matrix (the so called Pick matrix), while Nevanlinna expressed his condition as a sequence of inequalities.

The Pick–Nevanlinna and similar interpolation problems have been studied very much. A variety of generalizations of the mentioned problem have been considered. Nevanlinna already considered multiple interpolation on an infinite set E [**2**]. Next generalization allowed the demanded function to be meromorphic [**3**]. In a latter paper [**4**] Nevanlinna "moved" the set E to the boundary of the domain $\mathbf{C}^+$. The so arisen problem is called the Pick–Nevanlinna boundary value interpolation problem. It turned out to be natural in such a problem to prescribe also some estimate of the radial derivative of the unknown function at each point in E. Loewner [**5**] solved the boundary problem with interpolation data given on a full interval of the real axis. Rosenblum and Rovnyak [**6**] studied a different type of boundary interpolation problems, in which E is a Borel subset of the boundary.

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Fedčina [**7**] introduced the next considerable generalization of the problem, where the unknown function is a matrix function and one prescribes not its full values but rather only values in given directions at points in E (directional or tangential matricial interpolation problem). Such problems were thereafter very intensively studied, in view of manifold applications to problems of technics and physics.

Among other numerous generalizations of the problem, we mention here only the problems on multilply connected domains and the interpolation problems for functions of several independent variables. As well, a great number of various approaches and methods were applied.

The interior interpolation problems (i.e., the problems in which $E \subset \mathbf{C}^+$) are better studied than the boundary problems (where $E \subset \mathbf{R}$). In this paper we consider a sort of perturbations of interpolation problems which enables us to transfer known results on the interior problems to the boundary ones. In this way we will establish our result which asserts that nonnegativity of an analogue to the Pick matrix (called Pick kernel), together with the so called absence-of-residues property, is necessary and sufficient for solvability of a directional matricial boundary value interpolation problem. Ball and Helton [**8**] proved a similar result, in a more general setting, by a different method.

2. Interpolation. Let us describe the problem which we will consider.

Let H be a finite dimensional Hilbert space and let $L(H)$ denote the set of all linear mappings from H into H. Let E be a finite set of points on the real axis $\mathbf{R}$, let for each $x \in E$ a subspace H_x of H be given, and two linear mappings: $\varphi : H_x \to H$ and $\varphi^1(x) : H_x \to H_x$, such that $P_x\varphi(x)P_x$ and $\varphi^1(x)$ are self-adjoint, where P_x denotes the orthogonal projection in H onto H_x.

The problem is to find an operator function $\psi : \mathbf{C}^+ \to L(H)$ having the properties: (i) ψ is analytic on $\mathbf{C}^+$; (ii) $\operatorname{Im}\psi(z) \geq 0$, $z \in \mathbf{C}^+$; (iii) for each $x \in E$, there exists $\lim\limits_{y\downarrow 0} \psi(x+iy) =: \psi(x)$, $\operatorname{Im}\psi(x) = 0$ and $\psi(x)/H_x = \varphi(x)$; (iv) for each $x \in E$, there exists $\lim\limits_{y\downarrow 0} \dfrac{1}{y} \operatorname{Im}\psi(x+iy) =: \psi'(x)$ and $\langle \psi'(x)a, a\rangle_H \leq \langle \varphi^1(x)a, a\rangle_H$, $a \in H_x$.

(The above radial limits $\psi(x)$ and $\psi'(x)$ exist at least in the sense of weak operator convergence. Since H is finite dimensional, it follows that these radial limits exist also in the sense of uniform operator convergence.)

Denote the above boundary value interpolation problem by $I(\varphi, \varphi^1)$.

In connection with the problem $I(\varphi, \varphi^1)$ we introduce an operator valued kernel $\mathcal{P}$ which will be called *Pick kernel.* Let

$$\mathcal{P}(\varphi, \varphi^1; x, t) = \mathcal{P}(x,t) := \frac{1}{t-x} P_t[\varphi(t)^* - \varphi(x)]P_x$$

for $t \neq x$, $x, t \in E$, and

$$\mathcal{P}(\varphi, \varphi^1; x, x) = \mathcal{P}(x,x) := P_x\varphi^1 P_x, \quad x \in E.$$

If $\mathcal{P} \geq 0$, i.e., if $\mathcal{P}$ is positive semidefinite, then there exists a unique functional Hilbert space $h(\varphi, \varphi^1) = h$, whose reproducing kernel is $\mathcal{P}$ [**9**]. (The proof given in [**9**] concerns the scalar case, but it works also in the general.) The elements of h are functions defined on E and valued in H. In fact, they are linear combinations of functions of the form $\mathcal{P}(x, \cdot)a$, $x \in E$, $a \in H_x$.

The kernel $\mathcal{P}$ will be said to have the *absence-of-residues property* if the following holds: whenever

$$\sum_{x \in D} \mathcal{P}(x, \cdot) a_x = 0 \tag{1}$$

for some set $D \subset E$ and some nonzero vectors a_x, $a_x \in H_x$, $x \in D$, and

$$a_v := \sum_{x \in D} \frac{1}{v - x} a_x \in H \tag{2}$$

for some $v \in E \setminus D$, then is

$$\varphi(v) a_v = \sum_{x \in D} \frac{1}{v - x} \varphi(x) a_x. \tag{3}$$

If an operator function $\psi : \mathbf{C}^+ \to B(H)$ satisfies (i) and (ii) (without assuming that H is finite dimensional), then there exists a unique functional Hilbert space $H(\psi)$ with the reproducing kernel

$$K(w, z) := \frac{1}{z - \bar{w}} [\psi(z) - \psi(w)^*], \quad z, w \in \mathbf{C}^+.$$

($B(H)$ stands for the set of all bounded linear operators on H.) In the scalar case: $\dim H = 1$, this fact is established in [**10**, Th. 5]) and in the general case it can be established by a procedure quite analogous to that in [**10**].

For convenience of the reader, we shall present this procedure here. The main tool will be the following representation of ψ, called the Herglotz representation: there exists a unique Hermitian nondecreasing, left continuous operator function $\mu : [-\infty, +\infty] \to B(H)$, such that $\mu(-\infty) = 0$ and

$$\psi(z) = \int_{-\infty}^{+\infty} \frac{1 + tz}{t - z} \, d\mu(t) + \operatorname{Re} \psi(i), \quad z \in \mathbf{C}^+, \tag{4}$$

where the integral exists in the sense of the uniform operator norm. Such a representation for the scalar case can be found, for example, in [**11**, pp. 40 and 67]. The general case follows easily from the scalar one. See also [**12**].

From (4) it follows that

$$K(w, z) = \int_{-\infty}^{+\infty} \frac{1 + t^2}{(t - z)(t - \bar{w})} \, d\mu(t), \quad z, w \in \mathbf{C}^+.$$

According to this, whenever $w_1, w_2, \dots, w_n$ are points in $\mathbf{C}^+$ and $a_1, a_2, \dots, a_n$ some vectors in H, then we have

$$\sum_{j,k=1}^{n} \langle K(w_j, w_k)a_j, a_k\rangle_H = \sum_{j,k=1}^{n} \int_{-\infty}^{+\infty} \frac{1+t^2}{(t-w_k)(t-\bar{w}_j)} \langle d\mu(t)a_j, a_k\rangle_H$$
$$= \int_{-\infty}^{+\infty} (1+t^2) \Big\langle d\mu(t) \sum_{j=1}^{n} \frac{1}{t-\bar{w}_j} a_j, \sum_{k=1}^{n} \frac{1}{t-\bar{w}_k} a_k \Big\rangle_H \geq 0.$$

This shows that $K \geq 0$, which implies, by [**9**], that there exists a unique functional Hilbert space whose reproducing kernel is K.

THEOREM 1. *A necessary and sufficient condition for the problem $I(\varphi, \varphi^1)$ to have a solution is that the kernel $\mathcal{P}$ is nonnegative and has the absence-of-residues property.*

Note that in the case $H_x \equiv H$, $x \in E$, the nonnegativity of $\mathcal{P}$ alone is necessary and sufficient for solvability of the problem $I(\varphi, \varphi^1)$, as in the scalar case. Namely, in that case the nonnegativity of $\mathcal{P}$ implies the absence-of-residues property. This can be proved in the same way as in the necessity part of the below proof, for the kernel K/E^2. (See the derivation of (6) from (5).)

Proof. Necessity. Assume that ψ is a solution to the problem. Then ψ satisfies (i) and (ii) and there exists the space $H(\psi)$. According to [**13**, Remark 1], (iii) and (iv) imply that the functions in z

$$K(x,z)a := \frac{1}{z-x}[\psi(z) - \psi(x)]a, \quad x \in E, \ a \in H,$$

belong to $H(\psi)$ and that $\langle K(x,\cdot)a, K(t,\cdot)b\rangle_{H(\psi)} = \langle K(x,t)a, b\rangle_H$, $x, t \in E$, $a, b \in H$, with $K(x,x) := \psi'(x)$, $x \in E$. Then we have $\mathcal{P}(x,t) = P_t K(x,t) P_x$ for $x \neq t$, $x, t \in E$, and $\mathcal{P}(x,x) \geq P_x K(x,x) P_x$, $x \in E$, which shows that $\mathcal{P} \geq \mathcal{K}$, where $\mathcal{K}(x,t) := P_t K(x,t) P_x$, $x, t \in E$. Thus, the nonnegativity of $\mathcal{P}$ follows from the nonnegativity of (the extended) K.

In order to show that $\mathcal{P}$ has the absence-of-residues property, let (1) and (2) hold. Since $\mathcal{P} \geq \mathcal{K}$ (as we have seen above), it follows that

$$\sum_{x \in D} \mathcal{K}(x,\cdot)a_x = 0,$$

which implies

$$\sum_{x \in D} K(x,\cdot)a_x = 0.$$

In particular, we have

$$\sum_{x \in D} \langle K(x,\cdot)a_x, K(v,\cdot)a\rangle_{H(\psi)} = 0 \tag{5}$$

for any $a \in H$. From (5), we obtain further

$$\sum_{x \in D} \frac{1}{v-x}[\langle \psi(v)a_x, a\rangle_H - \langle \psi(x)a_x, a\rangle_H] = 0,$$

and, according to (iii),

$$\langle \varphi(v)a_v, a\rangle_H - \Big\langle \sum_{x \in D} \frac{1}{v-x}\varphi(x)a_x, a\Big\rangle_H = 0. \tag{6}$$

Since (6) holds for any vector a in H, it follows that (3) holds. The absence-of-residues property of $\mathcal{P}$ is established.

Sufficiency. First case: $H_x \equiv H$, $x \in E$; $\mathcal{P} > 0$. For any $y > 0$ set $\varphi(x+iy) := \varphi(x) + i\varphi^1(x)$, $x \in E$, and

$$\mathcal{P}_y(x,t) := \frac{1}{t-x+2iy}[\varphi(t+iy) - \varphi(x+iy)^*], \quad x, t \in E.$$

It is clear that

$$\lim_{y \downarrow 0} \mathcal{P}_y = \mathcal{P}.$$

Since $\mathcal{P} > 0$, i.e., $\mathcal{P}$ is positive definite, we can make $\mathcal{P}_y$ positive by choosing a sufficiently small y. (Namely, we may replace the positivity of $\mathcal{P}_y$, resp. $\mathcal{P}$, by the positivity of the Gram matrix G_y, resp. G, of the set of vectors $\mathcal{P}_y(x,\cdot)e_j$, $x \in E$, $1 \le j \le \dim H$, resp. $\mathcal{P}(x,\cdot)e_j$, $x \in E$, $1 \le j \le \dim H$, using an orthonormal basis (e_j) in H, and then make the Sylvester determinants of G_y positive, as those of G are positive.) For any $y > 0$ we pose the problem of finding an operator function $\psi : \mathbf{C}^+ \to L(H)$ having the properties (i), (ii) and

$$\psi(x+iy) = \varphi(x+iy), \quad x \in E. \tag{iii$_y$}$$

If the Pick kernel $\mathcal{P}_y$ (i.e., the Pick matrix) of the interior problem (i)–(iii$_y$) is positive, then this problem has a solution (for ex. [**14**]). Let ψ_y denote a solution to this problem. Since for any $a \in H$, $a \neq 0$, the scalar functions

$$[\langle \psi_y(z)a, a\rangle_H - i\langle a, a\rangle_H]/[\langle \psi_y(z)a, a\rangle + i\langle a, a\rangle_H]$$

form a normal family, we can choose a sequence (y_n) of positive numbers tending to 0 and such that the sequence of functions $\langle \psi_n a, a\rangle_H$, $n \in \mathbf{N}$, ($\psi_n := \psi_{y_n}$) converges uniformly on compact subsets of $\mathbf{C}^+$, where the limit can be ∞ in the whole $\mathbf{C}^+$ or nowhere in $\mathbf{C}^+$. Starting with an orthonormal basis (e_j) in H, putting above e_j, $e_j \pm e_k$, $e_j \pm ie_k$ ($1 \le j, k \le \dim H$) instead of a, and passing several times to a subsequence, we can get a sequence (y_n) such that $y_n \to 0$, $n \to \infty$, and that $\langle \psi_n e_j, e_k\rangle_H$ converges uniformly on compact subsets of $\mathbf{C}^+$ as $n \to \infty$, for any j, k,

$1 \leq j,k \leq \dim H$. This implies that for any two vectors $a,b \in H$, $\langle \psi_n a, b\rangle_H$ converges uniformly on compact subsets of $\mathbf{C}^+$.

Denote the reproducing kernel of the space $H(\psi_n)$ by K_n, $n \in \mathbf{N}$. Thus

$$K_n(w,z) = \frac{1}{z-\bar{w}}[\psi_n(z) - \psi_n(w)^*], \quad z,w \in \mathbf{C}^+.$$

If $y > 0$ and $y_n \leq y$, then we have

$$K_n(x+iy, x+iy) \leq K_n(x+iy_n, x+iy_n) \tag{7}$$

(see [**13**, Remark 2]) and therefore

$$\begin{aligned} |\langle K_n(x+iy_n, x+iy)a,b\rangle_H|^2 &\leq \langle K_n(x+iy_n, x+iy_n)a, a\rangle_H \cdot \langle K_n(x+iy, x+iy)b, b\rangle_H \\ &\leq \langle K_n(x+iy_n, x+iy_n)a, a\rangle_H \cdot \langle K_n(x+iy_n, x+iy_n)b, b\rangle_H \\ &= \langle \varphi^1(x)a, a\rangle_H \cdot \langle \varphi^1(x)b, b\rangle_H, \quad x \in E, \ a,b \in H. \end{aligned}$$

Since

$$K_n(x+iy_n, x+iy) = \frac{1}{i(y+y_n)}[\psi_n(x+iy) - \varphi(x+iy_n)^*],$$

it follows

$$|\langle[\psi_n(x+iy) - \varphi(x+iy_n)^*]a, b\rangle_H|^2 \leq (y+y_n)^2 \langle \varphi^1(x)a, a\rangle_H \cdot \langle \varphi^1(x)b, b\rangle_H, \tag{8}$$

$x \in E$, $a,b \in H$. This shows that $\langle \psi_n(x+iy)a, b\rangle_H$ cannot tend to ∞, and, consequently, that ψ_n converges to a function $\psi : \mathbf{C}^+ \to L(H)$ satisfying (i) and (ii).

Let $n \to \infty$ in (8). Then we obtain

$$|\langle[\psi(x+iy) - \varphi(x)]a, b\rangle_H|^2 \leq y^2 \langle \varphi^1(x)a, a\rangle_H \langle \varphi^1(x)b, b\rangle_H,$$

which shows that

$$\lim_{y \downarrow 0} \psi(x+iy) = \varphi(x), \quad x \in E,$$

i.e., that ψ satisfies (iii).

Rewrite (7) in the form $K_n(x+iy, x+iy) \leq \varphi^1(x)$, and let $n \to \infty$: $K(x+iy, x+iy) \leq \varphi^1(x)$, $x \in E$. Since $K(x+iy, x+iy)$ is Hermitian nonincreasing in y [**13**, Remark 2], it follows that there exist the limits

$$\lim_{y \downarrow 0} K(x+iy, x+iy) =: K(x,x), \quad x \in E, \tag{9}$$

and that

$$K(x,x) \leq \varphi^1(x), \quad x \in E. \tag{10}$$

Since $K(x+iy,x+iy)=\frac{1}{y}\operatorname{Im}\psi(x+iy)$, the relations (9) and (10) show that (iv) is also satisfied.

Thus ψ is a solution to the problem $I(\varphi,\varphi^1)$.

Second case: $H_x\equiv H$, $x\in E$. Take an $\varepsilon>0$ and add εI to $\varphi^1(x)$, to obtain $\varphi^1_\varepsilon(x):=\varphi^1(x)+\varepsilon I$, $x\in E$. Denote the corresponding Pick kernel by $\mathcal{P}_\varepsilon$. It is clear that $\mathcal{P}_\varepsilon>0$.

According to the preceding case, the problem $I(\varphi_\varepsilon,\varphi^1_\varepsilon)$ has a solution ψ_ε, for each $\varepsilon>0$. Just as above, we can choose a sequence (ε_n) tending to 0 and such that $\psi_n:=\psi_{\varepsilon_n}$ converges uniformly on compact subsets of $\mathbf{C}^+$ to an operator function $\psi:\mathbf{C}^+\to L(H)$. Now $\langle\psi_n(z)a,a\rangle_H$ cannot tend to ∞, because of $\psi_n(x)=\varphi(x)$. Then ψ satisfies (i) and (ii).

Reasoning quite analogously as in the preceding case, we show that

$$\lim_{y\downarrow 0}\psi(x+iy)=\varphi(x)$$

and

$$\lim_{y\downarrow 0}\frac{1}{y}\operatorname{Im}\psi(x+iy)\le\varphi^1(x),\quad x\in E.$$

Thus ψ satisfies also (iii) and (iv) and therefore it is a solution to the problem $I(\varphi,\varphi^1)$.

Third case: $H_x\not\equiv H$ for some $x\in E$; for each equality of the form (1) in h it must be $a_v\in H_v$ for any $v\in E\setminus D$. Then we can extend the kernel $\mathcal{P}$ in the following way.

Let $v\in E$ be such that $H_v\not\equiv H$ and let a_0 be a nonzero vector in $H\ominus H_v$. Denote by H_v^e the linear span of $H_v\cup\{a_0\}$, set $\varphi_e(v)a_0:=\varphi(v)^*a_0$ and $\varphi_e(v)a:=\varphi(v)a$ for $a\in H_v$, and then extend $\varphi_e(v)$ to H_v^e by linearity. It is easily seen that then the operator $P_v^e\varphi_e(v)P_v^e$ is self-adjoint, where P_v^e denotes the orthogonal projection in H onto H_v^e.

Denote by h_v the subspace of h generated by functions of the form $\mathcal{P}(x,\cdot)a$, $x\in E\setminus\{v\}$, $a\in H_x$. Let F_0 be the h_v function for which

$$\langle F_0(x),a\rangle_H=\frac{1}{x-v}[\langle a_0,\varphi(x)a\rangle_H-\langle\varphi_e(v)a_0,a\rangle_H],\quad x\in E\setminus\{v\},\ a\in H_x.$$

To show that F_0 is well-defined, it is enough (and necessary) to check that the functional

$$f_0:\mathcal{P}(x,\cdot)a\to\langle F_0(x),a\rangle_H,\quad x\in E\setminus\{v\},\ a\in H_x,$$

can be extended to a linear functional on h_v (i.e., that F_0 gives the Riesz representation of a linear functional on h_v). The functional f_0 really can be linearly extended to h_v, since for any equality of the form (1) in h_v, i.e., such that $v\notin D$, it holds

$$\sum_{x\in D}\langle F_0(x),a_x\rangle_H=0.$$

Namely, the last equality is equivalent to the following:

$$\Big\langle a_0, \sum_{x\in D} \frac{1}{v-x}\varphi(x)a_x \Big\rangle_H - \langle \varphi_e(v)a_0, a_v\rangle_H = 0,$$

which is true since it must be $a_v \in H_v$ and

$$\varphi(v)a_v = \sum_{x\in D} \frac{1}{v-x}\varphi(x)a_x.$$

Now, set $\langle\varphi_e^1(v)a_0, a\rangle_H := \langle a_0, \varphi_e^1(v)a\rangle_H := \langle F_0(v), a\rangle_H$ for any vector $a \in H_v$, $\langle\varphi_e^1(v)a_0, a_0\rangle_H := \|F_0\|_h^2 + 1$, $\langle\varphi_e^1(v)a, b\rangle_H := \langle\varphi^1(v)a, b\rangle_H$ for $a, b \in H_v$, and extend $\langle\varphi_e^1(v)a, b\rangle_H$ to $H_v^e \times H_v^e$ by linearity and antilinearity. It is easy to see that then $\langle\varphi_e^1(v)a, b\rangle_H$ is an Hermitian bilinear form on $H_v^e \times H_v^e$ and that its restriction to $H_v \times H_v$ coincides with $\langle\varphi^1(v)a, b\rangle_H$.

Let further $\varphi_e(x) := \varphi(x)$ and $\varphi_e^1(x) := \varphi^1(x)$ for $x \in E \setminus \{v\}$, and let $\mathcal{P}_e$ be the kernel $\mathcal{P}(\varphi_e, \varphi_e^1; \cdot, \cdot)$.

The kernel $\mathcal{P}_e$ is nonnegative and has the absence-of-residues property. Really, nonnegativity follows from the fact that $P_t\mathcal{P}_e(x,t)P_x = \mathcal{P}(x,t)$, $x, t \in E$, that $\mathcal{P}$ is nonnegative, and that

$$P_t\mathcal{P}_e(v,t)a_0 = F_0(t), \quad t \in E, \tag{11}$$

and

$$\langle\mathcal{P}_e(v,v)a_0, a_0\rangle_H > \|F_0\|_h^2. \tag{12}$$

Denote by h_e the corresponding space $h(\varphi_e, \varphi_e^1)$. Then h is isomorphic to the functional Hilbert subspace $\tilde{h}$ of h_e whose reproducing kernel is $\mathcal{P}_e(x,t)P_x$, $x, t \in E$, and an isomorphism from $\tilde{h}$ onto h is the mapping $\Phi : F \to F_P$, $F \in \tilde{h}$, where $F_P(t) := P_tF(t)$, $t \in E$.

It follows from (11) and (12) that the function $\mathcal{P}_e(v,\cdot)a_0$ is not in $\tilde{h}$, since otherwise $\Phi\mathcal{P}_e(v,v)a_0 = F_0$ and $\langle\mathcal{P}_e(v,v)a_0, a_0\rangle_H = \|F_0\|_h^2$. This means that $\mathcal{P}_e(v,\cdot)a_0$ cannot be a linear combination of functions of the form $\mathcal{P}_e(x,\cdot)a$, $x \in E$, $a \in H$, so that there is no new equality of the form (1) in h_e, in comparison with those in h (more precisely, in $\tilde{h}$). It follows that $\mathcal{P}_e$ has the absence-of-residues property, since $\mathcal{P}$ has.

Evidently, every solution of the problem $I(\varphi_e, \varphi_e^1)$ is also a solution of the initial problem $I(\varphi, \varphi^1)$.

If $H_v^e \not\equiv H$ or $H_u \not\equiv H$ for some $u \in E \setminus \{v\}$, then $\mathcal{P}_e$ can be extended in the same way as $\mathcal{P}$ above. Continuing in this way, we come, in finitely many steps, to a problem of the type $I(\varphi, \varphi^1)$ for which $H_x \equiv H$, $x \in E$, and whose every solution is also a solution of the initial problem $I(\varphi, \varphi^1)$. Thus the case under consideration is reduced to the preceding one.

Fourth case: there is an equality of the form (1) in h such that $a_v \notin H_v$ for some $v \in E \setminus D$. Then we can extend the kernel $\mathcal{P}$ in the following way.

Starting with an equality (1) and a $v \in E \setminus D$ such that $a_v \notin H_v$, we set

$$\varphi_e(v)a_v := \sum_{x \in D} \frac{1}{v-x}\varphi(x)a_x.$$

The absence-of-residues property of $\mathcal{P}$ implies independence of the vector $\varphi_e(v)a_v$ on the relation (1), for fixed v and a_v.

To establish this, it is enough to show that for any equality of the form (1) such that $a_v = 0$ for some $v \in E \setminus D$ it must be

$$\sum_{x \in D} \frac{1}{v-x}\varphi(x)a_x = 0. \tag{13}$$

This is true whenever $\mathcal{P}$ has the absence-of-residues property, since (13) is the condition (3) in the case $a_v = 0$.

Consider the following h function:

$$F_0 := \sum_{x \in D} \frac{1}{v-x}\mathcal{P}(x,\cdot)a_x.$$

Compute its inner product with any function of the form $\mathcal{P}(u,\cdot)a$, $u \in E \setminus \{v\}$, $a \in H_u$:

$$\begin{aligned}
\langle F_0, \mathcal{P}(u,\cdot)a\rangle_h &= \sum_{x \in D} \frac{1}{v-x}\langle \mathcal{P}(x,u)a_x, a\rangle_H + \frac{1}{u-v}\sum_{x \in D}\langle \mathcal{P}(x,u)a_x, a\rangle_H \\
&= \sum_{x \in D} \frac{u-x}{(u-v)(v-x)}\langle \mathcal{P}(x,u)a_x, a\rangle_H \\
&= \frac{1}{u-v}\sum_{x \in D\setminus\{u\}} \frac{1}{v-x}[\langle a_x, \varphi(u)a\rangle_H - \langle \varphi(x)a_x, a\rangle_H] \\
&= \frac{1}{u-v}[\langle a_v, \varphi(u)a\rangle_H - \langle \varphi_e(v)a_v, a\rangle_H],
\end{aligned}$$

because of $\langle a_u, \varphi(u)a\rangle_H = \langle \varphi(u)a_u, a\rangle_H$.

Thus,

$$\langle F_0(u), a\rangle_H = \frac{1}{u-v}[\langle a_v, \varphi(u)a\rangle_H - \langle \varphi_e(v)a_v, a\rangle_H, \quad u \in E \setminus \{v\},\ a \in H_u.$$

Verify that $\langle \varphi_e(v)a_v, a\rangle_H = \langle a_v, \varphi(v)a\rangle_H$, $a \in H_v$. It follows from (1) that

$$\sum_{x \in D}\langle \mathcal{P}(x,v)a_x, a\rangle_H = 0,$$

i.e.,

$$\sum_{x\in D}\frac{1}{v-x}[\langle a_x,\varphi(v)a\rangle_H-\langle\varphi(x)a_x,a\rangle_H]=0,$$

which gives $\langle a_v,\varphi(v)a\rangle_H-\langle\varphi_e(v)a_v,a\rangle_H=0$, $a\in H_v$, and we are done.

As well, multiplying (1) by the function F_0 (innerly), we obtain $\langle\varphi_e(v)a_v,a_v\rangle_H = \langle a_v,\varphi_e(v)a_v\rangle_H$.

Denote by H_v^e the linear span of $H\cup\{a_v\}$, set $\varphi_e(v)a:=\varphi(v)a$ for $a\in H_v$, and extend $\varphi_e(v)$ to H_v^e by linearity. Evidently, then the operator $P_v^e\varphi_e(v)P_v^e$ is self-adjoint, where P_v^e denotes the orthogonal projection in H onto H_v^e.

Now, set $\langle\varphi_e^1(v)a_v,a\rangle_H:=\langle a_v,\varphi_e^1(v)a\rangle_H:=\langle F_0(v),a\rangle_H$ for any vector $a\in H_v$, $\langle\varphi_e^1(v)a_v,a_v\rangle_H:=\|F_0\|_h^2+1$, $\langle\varphi_e^1(v)a,b\rangle_H:=\langle\varphi^1(v)a,b\rangle_H$ for $a,b\in H_v$, and extend $\langle\varphi_e^1(v)a,b\rangle_H$ to $H_v^e\times H_v^e$ by linearity and antilinearity. It is easy to see that then $\langle\varphi_e^1(v)a,b\rangle_H$ is an Hermitian bilinear form on $H_v^e\times H_v^e$ and that its restriction to $H_v\times H_v$ coincides with $\langle\varphi_e^1(v)a,b\rangle_H$.

For $x\in E\setminus\{v\}$, let $\varphi_e(x):=\varphi(x)$ and $\varphi_e^1(x):=\varphi^1(x)$. Let $\mathcal{P}_e$ be the kernel $\mathcal{P}(\varphi_e,\varphi_e^1;\cdot,\cdot)$.

The kernel $\mathcal{P}_e$ is nonnegative and has the absence-of-residues property. This is proved in the same way as in the previous case.

Evidently, every solution of the problem $I(\varphi_e,\varphi_e^1)$ is also a solution of the initial problem $I(\varphi,\varphi^1)$.

If there exists some equality of the form (1) in h_e, such that $a_v\notin H_v$ for some $v\in E\setminus D$, then we can continue to extend the kernel in the same way as above. So we reduce, in finitely many steps, the problem $I(\varphi,\varphi^1)$ to a problem of the type $I(\varphi,\varphi^1)$ such that there exists no equality of the form (1) with $a_v\notin H_v$ for some $v\in E\setminus D$.

The proof is completed.

The interpolation problem (i)–(iii) considered in the sufficiency part of the above proof is obtained by a perturbation of the problem $I(\varphi,\varphi^1)$. That is the earlier mentioned perturbed interpolation problem.

We can easily extend our result to the case when the set E is infinite, in a quite natural way, using an adequate limit process. This can lead to a generalization of the classical Loewner theorem [**5**], where we prescribe the full matrix values of ψ and ψ' at points of an open subset of $\mathbf{R}$. That generalization will be published elsewhere.

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