PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 59 (73), 1996, 31–38

A PROOF OF THE HARDY–LITTLEWOOD THEOREM ON FRACTIONAL INTEGRATION AND A GENERALIZATION

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Communicated by Miroljib Jevtić

Abstract. We give an elementary proof of the Hardy-Littlewood theorem and extend it to a class of smooth functions on the unit ball. As a special case we prove the validity of this theorem in the class of polyharmonic function.

1. Introduction

Let B denote the unit ball in a fixed euclidean space E. The integral means of order p, $0 < p \leq \infty$, of a continuous, complexvalued function f on B are defined by

$$M_p(f,r) = \left\{ \int_{\partial B} |f(ry)|^p d\sigma(y) \right\}^{l/p} \qquad (0 \leq r < 1),$$

where $d\sigma$ is the normalized surface measure on B. The fractional integral of order $s > 0$ is defined by

$$I^s f(x) = \frac{1}{\Gamma(s)} \int_0^1 \left(\log \frac{1}{t} \right)^{s-1} f(tx) dt \qquad (x \in B).$$

The well known theorem of Hardy and Littlewood [**5**, **6**, **7**] states that if f is a harmonic function on B, $\dim E = 2$, then for $a > s > 0$ the following conditions are equivalent:

I. $$M_p(f,r) = O(l-r)^{-a}$$

II. $$M_p(I^s f,r) = O(l-r)^{s-a}$$

AMS Subject Classification (1991): Primary 30D55, 31B30

Key Words and Phrases: Fractional integral, Integral means, Polyharmonic functions

There are many proofs and extensions, in various directions, of this result, and we refer to [**4**] for information and references concerning harmonic functions in the disc. In the case of the ball, Coifman and Rochberg [**2**] deduced the equivalence I $\Leftrightarrow$ II ($p = \infty$) for the integer values of s from their representation theorem. For extensions in other directions see, for example, [**9**, **14**] and references given there.

The most of the existing proofs are either very long or depend very much on the specific properties of harmonic functions such as the existence of reproducing kernels or power series expansions. In this paper we give an elementary and short proof, which enables us to extend the Hardy–Littlewood theorem to wider classes of functions as well as to prove the validity of implication II $\Rightarrow$ I without the restriction $a > s$. Before stating our result we write the equivalence I $\Leftrightarrow$ II as one implication.

For a function $f \in C^\infty(B)$ let

$$D^1 f = f + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} \qquad (n = \dim E)$$

and $D^m f = D^1 D^{m-1} f$ (m an integer ≥ 2). For an arbitrary $s > 0$ we choose an integer $m > s$ and define D^s by

$$D^s f = I^{m-s} D^m f.$$

If f_k is a homogeneous polynomial of degree k, then $I^s f_k = (k+1)^{-s} f_k$ and $D^s f_k = (k+1)^s f_k$, which shows that $I^s D^s f = D^s I^s f = f$ for all $f \in C^\infty(B)$. We put

$$I^s f = D^{-s} f \ \ (s > 0), \quad I^0 f = f \qquad (f \in C^\infty(B)).$$

Then $\{I^s : -\infty < s < \infty\}$ acts on $C^\infty(B)$ as a group of linear operators and is isomorphic to the additive group of real numbers. Therefore the Hardy–Littlewood theorem can be stated in the following form.

THEOREM 1. *Let $-\infty < s < \infty$ and $a > s$. For a function f harmonic in B, the condition I implies the condition II.*

It should be noted that Theorem 1, for $s < 0$, states somewhat more than the Hardy–Littlewood theorem because we do not assume that $a > 0$. Of course, the condition I for $a < 0$ has sense only when $p < 1$ since otherwise $M_p(f, r)$ increases with r. The case $a = 0$, $s < 0$ is also due to Hardy and Littlewood.

Our extension of Theorem 1 contains the case of polyharmonic functions, and then for all $p > 0$ and $-\infty < a < \infty$ there are nontrivial functions satisfying the condition I.

The case $p = \infty$ is simple and will not be considered in the sequel. The main ingredient in proving Theorem 1 for $0 < p < \infty$ is a lemma of Hardy and Littlewood [**5**] and Fefferman and Stein [**3**] on subharmonic behaviour of $|f|^p$. We shall use a generalization proved in [**10**] of this lemma (Lemma 2).

2. Results

Let HC^1 denote the subclass of $C^1(B)$ consisting of those f for which there is a constant K such that

$$|\nabla f(x)| \le K\varrho^{-1}\sup\{|f(z)|: |z-x| < \varrho\} \tag{1}$$

whenever $0 < \varrho < 1 - |x|$ and $x \in B$.

The class HC^∞ is the subclass of HC^1 consisting of those functions whose partial derivatives of all orders belong to HC^1.

Let H_m, where m is a positive integer, be the class of the functions f for which $\Delta^m f = 0$. Here Δ stands for the ordinary Laplacian. Each $f \in H_m$ satisfies (1) with $K = Cnm$, where C is an absolute constant. This can be deduced from the inequality

$$|\Delta f(z)| \le (m-1)^2\varrho^{-2}\sup_{B_\varrho(z)}|f| \qquad (f \in H_m)$$

(see [**8**] and [**11**]) and the inequality

$$|\nabla f(x)| \le n\varrho^{-1}\sup_{B_\varrho(x)}|f| + \varrho\sup_{B_\varrho(x)}|\Delta f| \qquad (f \in C^2),$$

which follows from the representation of f via Poisson's integral and Green's function. Here $B_\varrho(x)$ denotes the euclidean ball centered at x and of radius ϱ.

Since the partial derivatives of $f \in H_m$ are in H_m, we see that $H_m \subset HC^\infty$. An elementary proof of a more general result will appear in [**11**].

Let $L_p(a)$ $(0 < p \le \infty, -\infty < a < \infty)$ denote the class of all continuous function f on B satisfying the condition I. If B is the unit ball in the complex space, then $L_p(a)$, for $p > 0$ and $a < 0$, does not contain nontrivial holomorphic functions. The same holds in the harmonic case for $p \ge 1$. However, if $p < 1$, then the Poisson kernel belongs to $L_p(a)$ for some $a < 0$. (See [**12**] and [**13**] for a detailed discussion of this phenomenon.) Passing to polyharmonic functions we have that for all $p > 0$ and $a < 0$ there are nontrivial functions in $L_p(a)$. This follows from the fact that if f is harmonic and m an integer > 0, then $(1-|x|^2)^m f(x)$ is in H_{m+1}, which is a special case of the Almansi representation theorem [**1**].

THEOREM 2. *Let $0 < p \le \infty$, $-\infty < s < \infty$ and $a > s$. If $f \in HC^\infty \cap L_p(a)$, then $I^s f \in L_p(a-s)$.*

It should be observed, however, that we do not state the validity of implication $II \Rightarrow I$ in the class HC^∞. But since the group $\{I^s\}$ preserves the class H_m, the case $s < 0$ of Theorem 2 yields the following.

COROLLARY 1. *Let $0 < p \le \infty$, $a > 0$ and $s > 0$. If f is polyharmonic and $I^s f \in L_p(a-s)$, then $f \in L_p(a)$.*

We will deduce Theorem 2 from the following three propositions.

Proposition 1. *Let g be a nonnegative continuous function on B such that $g(ry)$ increases with $r, 0 < r < 1$, for all $y \in \partial B$. If $g \in L_p(a)$ and $a > s > 0$, then $I^s g \in L_p(a-s)$.*

The proof is in Section 3.

For a continuous function f on B let

$$f^*(x) = \sup\{|f(tx)|: 0 < t < 1\} \qquad (x \in B).$$

The well known Complex Maximal Theorem of Hardy and Littlewood and Proposition 1 yield the holomorphic version of Theorem 2 for $s > 0$. Of course this deep theorem of Hardy and Littlewood cannot be applied in the general case, and we replace it by a simpler fact.

Proposition 2. *Let $a > 0$ and $f \in HC^1 \cap L_p(a)$. Then $f^* \in L_p(a)$.*

In order to prove Theorem 2 for $s < 0$ we use the following proposition which improves Theorem 2 in the case where s is a negative integer, and without appealing to the hypothesis $a > s$.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index, i.e., an n-tuple of nonnegative integers. We denote by $\partial^\alpha f$ the corresponding partial derivative of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Proposition 3. *If $f \in HC^\infty \cap L_p(a)$, $-\infty < a < \infty$, then $\partial^\alpha f \in L_p(a + |\alpha|)$.*

The proofs of Propositions 2 and 3 are in Section 4.

Proof of Theorem 2. Let $f \in HC^\infty \cap L_p(a)$. If $a > s > 0$, then $f^* \in L_p(a)$, by Proposition 2, and hence $I^s(f^*) \in L_p(a-s)$, by Proposition 1, and this solves the case $s > 0$.

Let $s < 0$, $a > s$. Choose an integer $k > -s$. Then, by Proposition 3, $\partial^\alpha f \in L_p(a+k)$ for $|\alpha| \le k$. Since $a + k > 0$ and $\partial^\alpha f \in HC^1$, we can apply Proposition 2 to conclude that $(\partial^\alpha f)^* \in L_p(a+k)$ for $|\alpha| \le k$. Hence, by Proposition 1, applied to $g = (\partial^\alpha f)^*$, we find that $I^{k+s}(|\partial^\alpha f|) \in L_p(a-s)$. (Observe that $0 < k+s < a+k$.) And since

$$|D^k f| \le C \sum_{|\alpha| \le k} |\partial^\alpha f| \qquad (C = \text{const} > 0) \tag{2}$$

we have that $I^{k+s}(|D^k f|) \in L_p(a-s)$. This completes the proof because $|I^s f| = |I^{k+s} D^k f| \le I^{k+s}(|D^k f|)$. $\square$

Remark 1. By (2),

$$f^* = (I^k D^k f)^* \le I^k (D^k f)^* \le C \sum_{|\alpha| \le k} (\partial^\alpha f)^* =: S_k(f).$$

On the other hand, by Proposition 1 and 2, if $f \in HC^\infty \cap L_p(a)$, $a > 0$, then $S_k(f) \in L_p(a)$.

Remark 2. Propositions 1 and 2 yield a converse to Proposition 3 for $a > 0$. Let $f \in HC^\infty$ and k a positive integer. Let $\partial^\alpha f \in L_p(a+k)$, $a > 0$, for all α with $|\alpha| = k$. Using first Proposition 2 and then Proposition 1, we see that $I^k(|\partial^\alpha f|) \in L_p(a)$. Then $f \in L_p(a)$, by Taylor's formula.

3. Proof of Proposition 1

LEMMA 1. *Let $0 < p < \infty$, $q = \min(p,1)$ and $0 < s < 1$. If g is as in Proposition 1, then there is a constant $C = C(p,s)$ such that*

$$M_p^q(I^s g, (1+r)/2) - M_p^q(I^s g, r) \le C(1-r)^{qs} M_p^q(g, (1+r)/2) \quad (0 < r < 1). \tag{3}$$

Proof. It suffices to prove that

$$I^s g(\varrho y) - I^s g(ry) \le C(1-r)^s g(\varrho y) \qquad (y \in \partial B), \tag{4}$$

$\varrho = (1+r)/2$. Then, if $p > 1$, we use Minkowski's inequality in continuous form to deduce (3) from (4). If $p \le 1$, we combine (4) with the inequality $u^p - v^p \le (u-v)^p$ $(u > v > 0)$ and integrate the resulting inequality over ∂B.

In order to prove (4) we write $I^s g(ry)$, for a fixed $y \in \partial B$, as $\Gamma(s)(I_0(r)+J(r))$, where

$$I_0(r) = \int_r^1 \left(\log\frac{1}{t}\right)^{s-1} g(try)\,dt,$$

$$J(r) = \int_0^r \left(\log\frac{1}{t}\right)^{s-1} g(try)\,dt = \frac{1}{r}\int_0^{r^2} \left(\log\frac{r}{t}\right)^{s-1} g(ty)\,dt.$$

It is easily seen that $I_0(\varrho) \le s^{-1}(1-\varrho)^s g(\varrho y)$. From the first expression for $J(r)$ it follows that $J(r)$ increases with r, and hence $dJ/dr \ge 0$. Then we use the Leibnitz rule to find dJ/dr from the second expresion. We obtain

$$\left|\frac{dJ}{dr}\right| = \frac{dJ}{dr} \le 2\left(\log\frac{1}{r}\right)^{s-1} g(ry).$$

The proof is completed by Lagrange's theorem.

Proof of Proposition 1. Let $g \in L_p(a)$, $a > s > 0$. We may assume that $s < 1$ because of the relation $I^{s+k}g = I^s I^k g$ $(s,k > 0)$ and the fact that $I^k g(ry)$ increases with r. Let $s < 1$, $q = \min(p,1)$ and $r_j = 1 - 2^{-j}$ $(j \ge 0)$. It follows from the hypotheses and (3) that

$$M_p^q(I^s g, r_j) - M_p^q(I^s g, r_{j-1}) \le C2^{j(a-s)q} \qquad (j \ge 1)$$

for some constant C. Hence by summation

$$M_p(I^s g, r_j) = O(2^{j(a-s)}),$$

which implies $I^s g \in L_p(a-s)$ because $M_p(I^s g, r)$ increases with r. □

4. Proofs of Propositions 2 and 3

LEMMA 2 [**10**]. *If $f \in HC^1$ and $p > 0$, then there is a constant $C = C(K,p,n)$ such that*

$$|f(x)|^p \le C\varrho^{-n} \int\limits_{B_\varrho(x)} |f|^p dV \tag{5}$$

whenever $B_\varrho(x) \subset B$.

LEMMA 3. *If $a \ge 0$ and $f \in HC^1 \cap L_p(a)$, then $(|f| + |\nabla f|)^* \in L_p(a+1)$.*

Proof. Let $g = (|f| + |\nabla f|)^p$ and $h = |f|^p$. Then, by (1) and (5),

$$g(x) \le C\varrho^{-n-p} \int\limits_{B_\varrho(x)} h\, dV \tag{6}$$

for some C independent of ϱ, x. Let $r_j = 1 - 2^{-j}$ and

$$g_j(y) = \sup\{g(ry) : r_{j-1} < r < r_j\} \qquad (y \in \partial B).$$

If $r_{j-1} < r < r_j$ and $\varrho = r_{j+1} - r_j$, then

$$B_\varrho(ry) \subset D_j := \{z : |z - r_{j-1}y| < r_{j+1} - r_{j-1}\},$$

whence, by (6),

$$g_j(y) \le C 2^{j(p+n)} \int\limits_{D_j} h\, dV.$$

Replace y by Uy, where U is an orthogonal transformation, then apply the change $z \to Uz$ to get

$$g_j(Uy) \le C 2^{j(n+p)} \int\limits_{D_j(y)} h(Uz)\, dV(z).$$

Integrating this inequality with respect to the Haar measure on the orthogonal group, we get

$$\|g_j\|_1 \le C 2^{j(n+p)} \int\limits_{D_j(y)} M_1(h, |z|)\, dV(z).$$

Here we used the relations

$$\int h(Uz)\,dU = M_1(h,|z|),$$
$$\int g_j(Uy)\,dU = \|g_j\|_{L^1(\partial B)} =: \|g_j\|_1.$$

It follows that

$$\|g_j\|_1 \le C2^{jp}\sup\{M_1(h,r) : r_{j-2} < r < r_{j+1}\} \qquad (j \ge 2). \tag{7}$$

If $f \in L_p(a)$, this implies

$$M_1(g^*, r_j) - M_1(g^*, r_{j-1}) \le \|g_j\|_1 \le C2^{jp(1+a)}$$

for some constant C. If in addition $1 + a > 0$, then

$$M_1(g^*, r_j) = O(2^{jp(1+a)}),$$

which proves the lemma. □

Proof of Proposition 2. Let $f \in HC^1 \cap L_p(a)$, $a > 0$. Then $(D^1 f)^* \in L_p(a+1)$ because of Lemma 3 and the inequality $|D^1 f| \le |f| + |\nabla f|$. Hence $I^1(D^1 f)^* \in L_p(a)$, by Proposition 1, and hence $f^* \in L_p(a)$ because $f^* = (I^1 D^1 f)^* \le I^1(D^1 f)^*$. □

Proof of Proposition 3. Let $f \in HC^\infty \cap L_p(a)$, $-\infty < a < \infty$. It suffices to prove that $\partial^\alpha f \in L_p(a+1)$ for $|\alpha| = 1$, then replace f by $\partial^\alpha f$ and so on. It follows from the proof of Lemma 3 that inequality (7) is independent of a. If $r_{j-1} \le r \le r_j$ $(j > 2)$, and $a < 0$, then (7) gives

$$M_p(|\nabla f|, r) \le C2^j(1 - r_{j-2})^{-a} \le C(1-r)^{-a-1}.$$

For $a \ge 0$ Lemma 3 gives a stronger conclusion. □

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(Received 26 05 1995)