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IDENTITY AND PERMUTATION

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Abstract. It is known that in the purely implicational fragment of the system $\mathbf{TW}_{\to}$ if both $(A \to B)$ and $(B \to A)$ are theorems, then A and B are the same formula (the Anderson-Belnap conjecture). This property is equivalent to NOID (no identity!): if the axiom-shema $(A \to A)$ is omitted from $\mathbf{TW}_{\to}$ and the system $\mathbf{TW}_{\to}$-ID is obtained, then there is no theorem of the form $(A \to A)$.

A Gentzen-style purely implicational system $\mathbf{J}$ is here constructed such that NOID holds for $\mathbf{J}$. NOID is proved to be equivalent to NOE: there no theorem of $\mathbf{J}$ of the form $((A \to A) \to B) \to B$, i.e., of the form of the characteristic axiom of the implicational system $\mathbf{E}_{\to}$ of entailment.

If $(p \to p)$ is adjoined to $\mathbf{J}$ as an axiom-schema (ID), then there are theorems $(A \to B)$ and $(B \to A)$ such that A and B are distinct formulas, which shows that for $\mathbf{J}$ the Anderson-Belnap conjecture is not equivalent to NOID.

The system $\mathbf{J}$+ID is equivalent to $\mathbf{RW}_{\to}$ of relevance logic.

Introduction

By $\mathbf{TW}_{\to}$ we understand the system of propositional relevance logic defined in the language with $\to$ as the sole connective, by the following axiom-schemata:

ID	$(A \to A)$
ASU	$((A \to B) \to ((B \to C) \to (A \to C)))$
APR	$((B \to C) \to ((A \to B) \to (A \to C)))$.

The only rule of $\mathbf{TW}_{\to}$ is modus ponens.

By $\mathbf{TW}_{\to}$–ID we understand the system obtained from $\mathbf{TW}_{\to}$ by deleting the schema ID.

It has been shown that the following propositions were equivalent (Dwyer-Powers theorem):

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if both $(A \to B)$ and $(B \to A)$ are provable in $\mathbf{TW}_{\to}$, then A and B are the same formula (Anderson-Belnap's conjecture)

For no formula A is $(A \to A)$ provable in $\mathbf{TW}_{\to}$–ID (NOID).

Anderson-Belnap's conjecture is about an interesting property. Let us write $A \equiv B$ iff both $(A \to B)$ and $(B \to A)$ are theorems of $\mathbf{TW}_{\to}$; then the axioms of $\mathbf{TW}_{\to}$ and modus ponens are sufficient to show that (a) $\equiv$ is an equivalence relation and (b) that it is a congruence with respect to $\to$. By Anderson-Belnap's conjecture (the antisymmetry of $\to$) this congruence is the smallest congruence relation i.e., equality. Thus, the identity of formulas in the language with $\to$ as the only connective can be characterized exclusively by logical means – by the theory $\mathbf{TW}_{\to}$ of implication.

NOID (and hence the Anderson-Belnap's conjecture) has been proved true (cf. [**2**], [**3**] and [**4**]).

The proof of NOID in [**3**] has been obtained for a proper extension **L** of $\mathbf{TW}_{\to}$–ID.

Let **S** and **S**' be theories of implication and let A-B and NOID be the following claims about **S** and **S**':

A-B if both $(A \to B)$ and $(B \to A)$ are provable in **S**, then A and B are the same formula,

and

NOID there is no theorem of **S**' of the form $(A \to A)$.

Obviously, if **S** = $\mathbf{TW}_{\to}$ and **S**' = $\mathbf{TW}_{\to}$–ID, then A-B and NOID are equivalent.

In this paper we shall develop a proper extension **J** of **L** and prove that

(1) NOID holds for **J** and A-B does not hold for **J**+ID;

(2) NOID is equivalent to the following proposition: $((A \to B) \to B)$ is a theorem of **J** iff so is A.

The non-equivalence of A-B and NOID for **J** and **J**+ID is due to permutation present in **J** in the form of the rule PERM.

The claim (2) is interesting because it shows that NOID cannot hold in any system containing as a theorem any form of the $\mathbf{E}_{\to}$ axiom

$$(((A \to A) \to B) \to B).$$

Also, (2) will enable us to prove that there are in **J** some restricted forms of contraction: any formula $((A \to (A \to B)) \to (A \to B))$ is a theorem of **J** iff so is A.

(3) We shall show than NOE can be extended to formulas of a certain type.

The system J

Some of the basic definitions given below are taken from [**3**].

Let p, q, r, ... stand for propositional variables. The letters A, B, C, ... range over the set of formulas. Instead of $(A \to B)$ we shall write (AB). Also, we omit parentheses, with the association to the left. Thus, ABC stands for $(AB)C$.

Let R, S, T, U, V, W, X, Y, Z, ... range over finite (possibly empty) sequences of formulas. If X consists of a single formula A, we shall write A for X. If X is empty, let $X.B$ denote B. If $X = \langle A_1, \dots, A_n \rangle$, $n \geq 1$, then $X.B$ denotes the formula

$$A_1 \to (A_2 \to \cdots \to (A_n \to B) \dots).$$

Notice that any formula is of the form $W.p$, for some W and a variable p. Very often we shall write $W_A.p$ for A, for any formula A.

By $\pi(X)$ we denote any permutation of X, and by $\pi(X).B$ we denote any formula $Y.B$ such that Y is a permutation of X.

Let $C.DE$ be a subformula of A; suppose that B is obtained from A by substitution of $D.CE$ for $C.DE$, at a single occurrence of $C.DE$ in A; then we shall say that B is obtained from A by the rule PERM. Let us write $A \sim B$ iff B can be obtained from A by a finite (possibly zero) number of applications of PERM. It is clear that $\sim$ is an equivalence relation. We shall write $X \sim Y$ iff Y can be obtained from a permutation Z of X by a finite (possibly zero) number of applications of PERM to some members of Z. For any A by A^* we shall denote any formula B such that $A \sim B$. Also, for any X by X^* we denote any Y such that $X \sim Y$. It is clear that $(\pi(X))^* \sim \pi(X^*)$.

The axioms of **J** are given by the following schema:

ASU $\quad \pi((AB)^*, Bp, A).p.$

The rules of **J** are:

JSU $\quad$ From $\pi(X, Y).p$ to infer $\pi(X^*, (Y^*.p)q).q$.

JPR $\quad$ From $\pi(X, B).p$ to infer $\pi(X^*, (AB)^*, A).p$.

JG $\quad$ From $\pi(X, Y).p$ and $\pi(Z, B).q$ to infer

$$\pi(X^*, Z^*, ((Y.p)B)^*).q.$$

The rule JG is to be understood as follows: if there are permutations V and W of the sequences X, Y and Z, B, respectively, such that $V.p$ and $W.q$, are derivable in **J**, so is $W'.q$, for any permutaion W' of the sequence $X^*, Z^*, ((Y.p)B)^*$. In a similar way we understand JSU and JPR.

We shall assume that derivations in **J** are given in forms of trees, with usual properties. The *weight* w of a node in a derivation, *derivability with weight* w and the *combined weight* are defined as in [**5**, p. 113]. By the *degree* of A (of X) we understand the number of occurrences of $\to$ in A (in X).

Let us define Ap^n as follows: $Ap^0 = A; Ap^{n+1} = (Ap^n)p$.

J is closed under modus ponens

We start with

THEOREM 1. *If A is derivable in* **J** *with weight w, so is A^*; if $X.p$ is derivable in* **J**, *so is $\pi(X).p$, for any permutation $\pi(X)$ of X.*

Proof. By an easy induction on the weight of A in a given derivation of A.

Theorem 1 shows that **J** is closed under PERM; it enables us to identify A and A^*, X and X^*, and X and $\pi(X)$ in derivations in **J**. In the sequel this identification is assumed.

THEOREM 2. *If* (a) $\pi(X, Y).p$ *is derivable in* **J**, *so is* (b) $\pi(X, \pi(Y.p, Z).q.Z).q$.

Proof. By JSU we obtain $\pi(X, (Y.p)q).q$ from (a); hence, (b) is obtained by using JPR.

Theorem 2 and JG show that **J** is closed under the following assertion rules:

ASS1 From A to infer ABB.
ASS2 From A and $\pi(X, B).p$ to infer $\pi(X, AB).p$.

THEOREM 3. (TRANSITIVITY, JTR) *If* (a) $\pi(X, Y).p$ *and* (b) $\pi(Y.p, Z).q$ *are derivable in* **J**, *so is* (c) $\pi(X, Z).q$.

Proof. Proceed by double induction. Suppose that (a) and (b) are derivable with combined weight w and that $Y.p$ is of degree d. Our induction hypotheses are:

Hyp 1 The theorem holds for any $Y'.p$ of degree $d' < d$ and any combined weight w;

Hyp 2 The theorem holds for $Y.p$ and any combined weight $w' < w$.

Case I (b) is an instance of ASU; hence, $\pi(Y.p, Z) \sim \pi(AB, Bq, A)$ for some A, B and q.

I.1 $Y.p \sim AB \sim \pi(A, W_B).p$, and $Y \sim \pi(A, W_B)$. From (a) we obtain (c) by using JSU.

I.2 $Y.p \sim Bq$; hence, $Y \sim B$ and $p \sim q$. From (a) we obtain (c) by using JPR.

I.3 $Y.p \sim A$; hence, by Theorem 2 we obtain (c).

Case II (b) is obtained by JSU from (b$_1$) $\pi(V, W).r$, where $\pi(V, (W.r)q) \sim \pi(Y.p, Z)$.

II.1 $V \sim \pi(V', Y.p)$; by (a), (b$_1$) and Hyp 2 $\pi(X, V', W).r$ is derivable; hence, by using JSU we obtain (c).

II.2 $(W.r)q \sim Y.p$ and $Z \sim V$; hence, $Y \sim W.r$ and $p \sim q$. By (a), (b$_1$) and Hyp 1, (c) is derived.

Case III (b) is obtained by JPR from (b$_1$) $\pi(V, B).q$, where $\pi(V, AB, A) \sim \pi(Y.p, Z)$.

III.1 $V \sim \pi(V', Y.p)$ and $Z \sim \pi(V', AB, A)$; by (a), (b_1), and Hyp 2, we obtain $\pi(X, V', B).q$, and then (c) by using JPR.

III.2 $AB \sim Y.p$. We have (a) $\pi(X, A).B$; hence, by (a), (b_1), and Hyp 1, (c) is derived.

III.3 $A \sim Y.p$. From (a) and (b_1) we obtain (c) by JG.

Case IV (b) is obtained by JG from (b_1) $\pi(U, V).r$ and (b_2) $\pi(W, A).q$, where

$$\pi(Y.p, Z) \sim \pi(U, W, (V.r)A).$$

IV.1 $U \sim \pi(U', Y.p)$ and $Z \sim \pi(U', W, (V.r)A)$. By (a), ($b_1$) and Hyp 2, $\pi(X, U', V).r$ is derivable. Hence (c), by using (b_2) and JG.

IV.2 $W \sim \pi(W', Y.p)$ and $Z \sim \pi(U, W', (V.r)A)$. Now $\pi(X, W', A).q$ is derivable by (a), (b_2) and Hyp 2; hence (c), by using (b_1) and JG.

IV.3 $(V.r)A \sim Y.p$ and $Z \sim \pi(U, W)$. It is clear that (a) is $\pi(X, V.r).A$. By (a), (b_2) and Hyp 1, (a') $\pi(X, W, V.r).q$ is derivable. Now by (b_1), (a'), and Hyp 1, (c) is derivable.

A trivial consequence of this theorem is

THEOREM 4 (MODUS PONENS, MP). *If A and AB are derivable in* **J**, *so is B.*

There is a Hilbert style formulation of **J**. Let **K** be the system with MP, PERM, ASS1 and the axiom-schema $\pi(AB, BC, A).C$.

THEOREM 5. **K** *and* **J** *are equivalent.*

Proof. It is obvious that **J** contains **K**.

The rules JTR and JPR are easily derivable in **K**, by using the axioms, MP and PERM. In the same way the rules JSU and JG are easily derivable provided that X is nonempty. The rule ASS1 plays the role of JSU when X is empty. Now by using ASS1, JPR and JTR we derive JG when X is empty (ASS2).

The system L

The system **L** is obtained from **J** by restricting JSU and JG: in JSU and JG X must not be empty. Let LSU and LG be JSU and JG, respectively, restricted in this way. In [**3**] it is assumed that **L** has a single propositional variable p.

The following theorems were proved in [**3**].

L_1 If A is derivable in **L** with weight w, so is A^*.
L_2 **L** is closed under the following transitivity rule:
from $\pi(X, A, Y).p$ and $\pi(Z, Y^*.p).p$ to infer $\pi(X^*, Z^*, A^*).p$.
L_3 **L** contains $\mathbf{TW}_{\rightarrow}$–ID.
L_4 There is no theorem of **L** of the form Ap.
L_5 There is no theorem of **L** of the form $\pi((\pi(X, Y).p)p^{2k}, Y^*).p$, $k \in \omega$.
L_6 There is no theorem of **L** of the form AA.

L_7 There is no theorem of **L** of the form ABB.
L_8 There is no theorem of **L** of the form $A.ABB$.
L_9 There is no theorem of **L** of the form $ABBA$.

$L_6 - L_9$ are consequences of L_5. We shall prove or disprove theorems about **J** analogous to $L_1 - L_9$ first.

Notice that **L** is not closed under MP. Let $A \sim pp.pp.pp$ and $B \sim (pp.pp)p.ppp$; AB is an instance of ASU. If **L** were closed under MP, applying MP to

$$\pi(AB, Bp, A).p$$

twice, Bpp would be obtained in **L**, contrary to L_4.

L is not closed under ASS1 either. Otherwise, App would be derivable, contrary to L_4.

That **L** is not closed under ASS2 can be seen as follows. Let $A \sim \pi(pp, pp, p).p$; by using A and ASS2, in **J** we derive B, $B \sim \pi(\pi(A, p).p, pp, p).p$. Let us show that B is not derivable in **L**.

B is not an instance of ASU.

If B is obtained by LSU from C, then $C \sim \pi(\pi(A, p).p, p).p \sim (A(pp))(pp)$, violating thus L_7.

If B is derivable by LPR from $\pi(X, F).p$, then

$$\pi(\pi(A, p).p, pp, p) \sim \pi(X, EF, E)$$

for some X, E, and F. It is clear that $E \sim p$.

If $F \sim Ap$, then $\pi(Ap, pp).p$ is derivable in **L**. But this is neither an axiom nor can it be obtained by LPR or LG. If it is obtained by LSU, then App is derivable, contrary to L_4.

If $F \sim p$, then $(A(pp))(pp)$ is derivable, contrary to L_7.

Suppose that B is derived by LG from $\pi(X, Y).p$ and $\pi(Z, E).p$; hence,

$$\pi(\pi(A, p).p, pp, p).p \sim \pi(X, (Y.p)E, Z).p.$$

If $(Y.p)E \sim pp$, then Y is empty, $E \sim p$ and $\pi(X, Z) \sim \pi(\pi(A, p).p, p)$. Now X is not p and Z is not empty (otherwise, pp is derivable). Hence, B is obtained from $(\pi(A, p).p)p$ and $\pi(p, p).p$, which is impossible.

If $(Y.p)E \sim \pi(A, p).p$, then $\pi(X, Z) \sim \pi(pp, p)$.

Let Z be empty; then B is obtained from $\pi(pp, p, Y).p$ and Ep, contrary to L_4.

Let $Z \sim pp$; then B is obtained from $\pi(Y, p).p$ and $\pi(E, pp).p$. Obviously, Y is not empty and E is not pp; hence, $E \sim Ap$ and $Y.p \sim p$ - a contradiction.

Let $Z \sim p$; then B is obtained from $\pi(pp, Y).p$ and $\pi(E, p).p$. Since Y cannot be empty, $Y.p \sim A$ and $E \sim pp$, contary to L_6.

This shows that **L** is not closed under ASS2.

Since JSU = LSU + ASS1 and JG = LG + ASS2, we have **J** = **L** + ASS1 + ASS2.

J is a proper extension of **L** and there is no theorem about **J** analogous either to L_4 or L_5 or L_7. However, theorems analogous to L_6, L_8, and L_9 still hold true.

No instance of AA is derivable in J

THEOREM 6. *$(X.p)p$ is derivable in* **J** *iff X is nonempty and any member of X is derivable in* **J**.

Proof. Let X be $\pi(A_1, \ldots, A_n)$, $n > 0$, and let $A_1, \ldots, A_n$ be derivable in **J**. By ASS1, $A_n pp$ is derivable; if $n > 1$, by using JG in the form of ASS2, we derive $(\pi(A_1, \ldots A_n).p)p$, i.e., $(X.p)p$.

Suppose that $(X.p)p$ is derivable. If X is empty, then pp is derivable; however, this is neither an axiom nor can it be obtained by any of the rules. Hence, X is nonempty.

Let $X \sim \pi(A_1, \ldots, A_n)$ and proceed by induction on the weight of the derivation of $(X.p)p$.

Obviously, $(X.p)p$ is neither an instance of ASU nor can it be obtained by JPR. If it is obtained from (a') by JSU, then $(X.p)p \sim (V.p)pp$ and (a') is $V.p$; hence, $X \sim V.p \sim A_1$, X is nonempty and A_1 is derivable in **J**.

If $(X.p)p$ is obtained from (a') and (a") by JG, then $X.p \sim \pi(U, W, (V.p)C)$, U and W are empty, $X \sim \pi(A_1, \ldots, A_n) \sim \pi(V.p, W_C)$ for some $A_1, \ldots, A_n$, and (a') and (a") are $V.p$ and $(W_C.p)p$, respectively. By induction hypothesis, all members of W_C, say $W_C \sim \pi(A_1, \ldots, A_{n-1})$, are derivable in **J**. Obviously, we can take $V.p \sim A_n$.

This completes the proof of the theorem.

Since **J** is (as **L**) closed under uniform substitution, to prove the main theorems of this paper it suffices to prove them under the assumption that there is only one variable in **J**, say p. Let $\mathbf{J}_1$ be **J** with just one variable p. In the sequel, if not stated otherwise, "derivable" means "derivable in $\mathbf{J}_1$".

THEOREM 7 (NOID). *There is no theorem of $\mathbf{J}_1$ of the form $\pi((X.p)p^{2k}, X).p$, $k \in \omega$.*

Proof. If there is a theorem of $\mathbf{J}_1$ of this form, then

Hyp 3 there is a formula (a) $\pi((X.p)p^{2k}, X).p$ of smallest degree derivable in $\mathbf{J}_1$.

Let us consider how (a) could have been obtained. We leave to the reader the verification that (a) cannot be an instance of ASU.

Case I (a) is obtained from (a') by JSU; hence,

$$\pi((X.p)p^{2k}, X) \sim \pi(Y, (Z.p)p)$$

for some Y and Z.

I.1 $Y \sim \pi(Y', (X.p)p^{2k})$ and $X \sim \pi(Y', (Z.p)p)$. Obviously, we have (a')

$$\pi(\pi((Y', (Z.p)p).p)p^{2k}, Y', Z).p.$$

If both Y' and Z are empty, then (a') is $pppp^{2k}p$; hence, pp is derivable by Theorem 6. This is impossible.

If Y' is empty and Z nonempty, then (a') is $\pi((Z.p)ppp^{2k}, Z).p$, contrary to Hyp 3.

Let Y' be nonempty and Z arbitrary. By using ASU and JPR we derive (b)

$$\pi(\pi(Y', Z).p, (Z.p)p, Y').p.$$

If $k > 0$, we use JSU to derive (c) $\pi(\pi(Y', Z).p, (\pi(Y', (Z.p)p).p)p^{2k-1}).p$. Hence, by (c), (a'), and JTR we derive $\pi(\pi(Y', Z).p, Y', Z).p$, contrary to Hyp 3.

I.2 $(X.p)p^{2k} \sim (Z.p)p$ and $X \sim Y$. If $k = 0$, then $X \sim Z.p$ and (a') is $\pi(Z.p, Z).p$, contrary to Hyp 3.

If $k > 0$, then $Z.p \sim (X.p)p^{2k-1}$ and $Z \sim (X.p)p^{2k-2}$. Hence, we have (a')

$$\pi((X.p)p^{2k-2}, X).p,$$

contrary to Hyp 3.

Case II (a) is obtained by JPR; hence, $\pi((X.p)p^{2k}, X) \sim \pi(Y, AB, A)$ for some Y, A and B.

II.1 $Y \sim \pi(Y', (X.p)p^{2k})$ and $X \sim \pi(Y', AB, A)$. Obviously, we have (a')

$$\pi((\pi(Y', AB, A).p)p^{2k}, Y', B).p.$$

Now $\pi(Bp, AB, A).p$ is an instance of ASU; hence, by JPR we obtain

$$\pi(\pi(Y', B).p, Y', AB, A).p$$

and then by using JSU we derive $\pi(\pi(Y', B).p, \pi(Y', AB, A).p)p).p$. If $k > 0$, by JSU we get $\pi(\pi(Y', B).p, \pi(Y', AB, A).p)p^{2k-1}).p$. Hence, using JTR and (a') we obtain $\pi(\pi(Y', B).p, Y', B).p$, contradicting thus Hyp 3.

II.2 $(X.p)p^{2k} \sim AB$ and $X \sim \pi(Y, A)$. Hence,

$$(X.p)p^{2k} \sim (\pi(Y, W_A.p).p)p^{2k} \sim \pi(W_A.p, W_B).p.$$

If $k > 0$, then W_B is empty and we have $B \sim p$, and $W_A.p \sim (\pi(Y, W_A.p).p)p^{2k-1}$; this is impossible.

Let $k = 0$; then $\pi(Y, A) \sim \pi(A, W_B)$ and $Y \sim W_B$. Thus, (a') is $\pi(W_B.p, W_B).p$, contrary to Hyp 3.

II.3 $(X.p)p^{2k} \sim A$ and $X \sim \pi(Y, AB)$; this is impossible.

Case III (a) is obtained by JG; hence, $\pi((X.p)p^{2k}, X) \sim \pi(Y, Z, (U.p)B)$ and both (a') $\pi(Y, U).p$ and (a") $\pi(Z, B).p$ are derivable.

III.1 $Y \sim \pi(Y', (X.p)p^{2k})$ and $X \sim \pi(Y', Z, (U.p)B)$; hence, (a') is

$$\pi((\pi(Y', Z, (U.p)B).p)p^{2k}, Y', U).p.$$

From (a") we obtain (b) $\pi(U.p, Z, (U.p)B).p$ by using JPR. If necessary, we apply JPR to obtain (c) $\pi(\pi(Y', U).p, Y', Z, (U.p)B).p$. If $k > 0$, by using JSU we derive (d)

$$\pi(\pi(Y', U).p, \pi(Y', Z, (U.p)B).p)p^{2k-1}).p.$$

Hence, by (d), (a'), and JTR we derive $\pi(\pi(Y', U).p, Y', U).p$, contrary to Hyp 3.

III.2 $Z \sim \pi(Z', (X.p)p^{2k})$ and $X \sim \pi(Y, Z', (U.p)B)$; hence, (a") is

$$\pi((\pi(Y, Z', (U.p)B.p)p^{2k}, Z', B).p.$$

On the other hand, from (a') we obtain (b) $\pi(Y, (U.p)B).B$, by Theorem 2. Hence, by using JSU we derive (c) $\pi(Bp, Y, (U.p)B).p$, and if Z' is nonempty, we derive (d)

$$\pi(\pi(Z', B).p, Y, Z', (U.p)B).p$$

by using JPR. Now if $k > 0$, we can use JSU to obtain (e)

$$\pi(\pi(Z', B).p, (\pi(Y, Z', (U.p)B).p)p^{2k-1}).p.$$

In any case we can use (e), (a"), and JTR to obtain $\pi(\pi(Z', B).p, Z', B).p$, contrary to Hyp 3.

III.3 $(X.p)p^{2k} \sim (U.p)B$ and $X \sim \pi(Y, Z)$. If $k > 0$, then $B \sim p$, $U.p \sim (X.p)p^{2k-1}$ and $U \sim (X.p)p^{2k-2}$. Obviously, we have (a') $\pi((\pi(Y, Z).p)p^{2k-2}, Y).p$ and (a") $\pi(Z, p).p$. Hence, Z is nonempty.

III.3.1 Let Y be empty; then (a') is $(Z.p)p^{2k-1}$. We derive

$$\text{(b)} \quad \pi(p, (Z.p)p^{2k-1}).p$$

by using (a") and JSU. Hence, by using (a'), (b), and MP we obtain pp, which is impossible.

III.3.2 Let Y be nonempty. By using (a") and JPR we derive

$$\text{(b)} \quad \pi(Y.p, Y, Z).p;$$

hence, by applying JSU to (b) we derive (c) $\pi(Y.p, (\pi(Y, Z).p)p^{2k-1}).p$, and hence $\pi(Y.p, Y).p$ is derivable by using (a'), (c), and JTR, contrary to Hyp 3.

Let $k = 0$ and $B \sim V.p$; then $X \sim \pi(U.p, V)$.

III.3.3 $Y \sim \pi(Y', U.p)$ and $V \sim \pi(Y', Z)$. We have

$$\text{(a')}\ \pi(Y', U.p, U).p \text{ and (a")}\ \pi(\pi(Y', Z).p, Z).p.$$

If Y' is empty, Hyp 3 is violated.

Let Y' be nonempty. If Z is empty, (a") becomes $(Y'.p)p$ and hence $\pi(U.p, U).p$ is obtained from (a') and (a") by JTR, contrary to Hyp 3.

Let Z be nonempty. By using JPR, from (a') we obtain

$$\pi(\pi(\pi(Z, U).p, U, Y', Z).p.$$

Hence, by using JTR and (a"), we obtain $\pi(\pi(Z, U).p, Z, U).p$, contrary to Hyp 3.

III.3.4 $Z \sim \pi(Z', U.p)$ and $V \sim \pi(Y, Z')$. We have

$$\text{(a')}\ \pi(Y, U).p \text{ and (a")}\ \pi(Z', U.p, \pi(Y, Z').p).p.$$

From (a') and (a") we obtain $\pi(\pi(Y, Z').p, Y, Z').p$, by using JTR, contrary to Hyp 3.

This completes the proof.

THEOREM 8. *There is no theorem of* **J** *of the form* AA.

THEOREM 9. *There is no theorem of* **J** *of the form $A.ABB$.*

THEOREM 10. *There is no theorem of* **J** *of the form $ABBA$.*

Theorems 8 – 10 are trivial consequences of NOID.

No istance of $AABB$ is derivable in J

THEOREM 11 (NOE). $\pi(\pi(X,Y).p,Y).p$ *is derivable in* $\mathbf{J}_1$ *iff* X *is nonempty and every member of* X *is derivable in* $\mathbf{J}_1$.

Proof. To prove the non-trivial part of the theorem, proceed by induction on the degree of $\pi(X,Y).p$. If Y is empty, we use Theorem 6. Let us accept the induction hypothesis

Hyp 4 The theorem holds for any $\pi(X',Y').p$ of degree smaller than the degree of $\pi(X,Y).p$.

Suppose that (a) $\pi(\pi(X,Y).p,Y).p$ is derivable in $\mathbf{J}_1$. By NOID, X is nonempty. The verification that (a) is not an instance of ASU is left to the reader.

Case I (a) is obtained by JSU from (a') $\pi(U,V).p$, where $\pi(\pi(X,Y).p,Y) \sim \pi(U,(V.p)p)$.

I.1 $(V.p)p \sim \pi(X,Y).p$ and $U \sim Y$; obviously, either X or Y is empty. But X is nonempty. If Y is empty, then by Theorem 6, $X \sim \pi(A_1,\dots,A_n)$ for some derivable $A_1,\dots,A_n$.

I.2 $Y \sim \pi((V.p)p,Y')$ and $U \sim \pi(\pi(X,(V.p)p,Y').p,Y')$. Obviously, (a) is obtained from (a') $\pi(\pi(X,(V.p)p,Y').p,V,Y').p$. Since X is nonempty, there is a member A of X. But as an instance of ASU we have $\pi(\pi(A,V).p,(V.p)p,A).p$. By using JPR we derive $\pi(\pi(X,V,Y').p,X,(V.p)p,Y').p$. Hence, by using JTR and (a') we obtain $\pi(\pi(X,V,Y').p,V,Y').p$. By Hyp 4, $X \sim \pi(A_1,\dots,A_n)$ for some $A_1,\dots,A_n$ and n, and $A_1,\dots,A_n$ are derivable in $\mathbf{J}_1$.

Case II (a) follows by JPR from (a') $\pi(U,D).p$, where $\pi(\pi(X,Y).p,Y) \sim \pi(U,CD,C)$.

II.1 $Y \sim \pi(CD,C,Y')$ and $U \sim \pi(\pi(X,CD,C,Y').p,Y')$. But

$$\pi(\pi(X,D,Y').p,X,CD,C,Y').p$$

is easily derivable in $\mathbf{J}_1$. Hence, by using JTR and (a'), so is

$$\pi(\pi(X,D,Y').p,D,Y').p.$$

Hence, by Hyp 4, $X \sim \pi(A_1,\dots,A_n)$ for some $A_1,\dots,A_n$ and n, and $A_1,\dots,A_n$ are derivable in $\mathbf{J}_1$.

II.2 $\pi(X,Y).p \sim CD$ and $Y \sim \pi(U,C)$; hence, $\pi(X,C,U).p \sim CD$. It is clear that $\pi(X,U) \sim W_D$. Now (a') is $\pi(\pi(X,U).p,U).p$ and by Hyp 4, $X \sim \pi(A_1,\dots,A_n)$ for some $A_1,\dots,A_n$ and n, and thus $A_1,\dots,A_n$ are derivable in $\mathbf{J}_1$.

II.3 $\pi(X,Y).p \sim C$ and $Y \sim \pi(CD,Y')$; this is impossible.

Case III (a) follows by JG from (a') $\pi(U,V).p$ and (a") $\pi(W,D).p$, where we have

$$\pi(\pi(X,Y).p,Y) \sim \pi(U,W,(V.p)D).$$

III.1 $\pi(X,Y).p \sim (V.p)D$ and $Y \sim \pi(U,W)$; hence,

$$\pi(X,U,W) \sim \pi(V.p,W_D).$$

III.1.1 $X \sim \pi(X',V.p)$, $W_D \sim \pi(X',U,W)$, and (a") is

$$\pi(\pi(X',U,W).p,W).p.$$

If U is empty, by Hyp 4 and (a"), $X' \sim \pi(A_1,\ldots,A_{n-1})$ for some derivable $A_1,\ldots,A_{n-1}$ and n. On the other hand, (a') is $V.p$ and we may take $V.p \sim A_n$.

If U is nonempty, from (a') we obtain $\pi((V.p)p,U).p$ and hence

$$\pi(\pi(X',V.p,W).p,X',U,W).p$$

by JPR. Now by using JTR and (a"), we obtain $\pi(\pi(X,W).p,W).p$. Hence, by Hyp 4, $X \sim \pi(A_1,\ldots,A_n)$ for some derivable $A_1,\ldots,A_n$.

III.1.2 $U \sim \pi(V.p,U')$ and $W_D \sim \pi(X,U',W)$. Obviously, (a') and (a") are

$$\pi(V.p,U',V).p \text{ and } \pi(W,\pi(X,U',W).p).p,$$

respectively. Hence, by using JPR and (a'), we easily derive

$$\pi(\pi(X,V,W).p,X,U',V,W).p.$$

Now by using (a") and JTR we get $\pi(\pi(X,V,W).p,V,W).p$ in $\mathbf{J}_1$. By Hyp 4 we have that for some derivable $A_1,\ldots,A_n$, $X \sim \pi(A_1,\ldots,A_n)$.

III.1.3 $W \sim \pi(V.p,W')$ and $W_D \sim \pi(X,U,W')$. Obviously, (a') and (a") are $\pi(U,V).p$ and $\pi(V.p,W',\pi(X,U,W').p).p$, respectively. By JTR, we derive $\pi(\pi(X,U,W').p,U,W').p$. By Hyp 4, $X \sim \pi(A_1,\ldots,A_n)$ and $A_1,\ldots,A_n$ for some derivable $A_1,\ldots,A_n$.

III.2 $U \sim \pi(\pi(X,Y).p,U')$ and $Y \sim \pi((V.p)D,U',W)$. Now (a') is

$$\pi((\pi(X,V.p)D,U',W).p,U',V).p.$$

By using (a") $\pi(W,D).p$ and JPR we derive

$$\pi(\pi(X,U',V).p,X,U',W,(V.p)D).p.$$

Hence, by using JTR and (a'), we obtain $\pi(\pi(X,U',V).p,U',V).p$. Hence, $X \sim \pi(A_1,\ldots,A_n)$, by Hyp 4, for some derivable $A_1,\ldots,A_n$.

III.3 $W \sim \pi(\pi(X,Y).p,W')$ and $Y \sim \pi((V.p)D,U,W')$. Now, obviously, (a") is

$$\pi(\pi(X,(V.p)D,U,W').p,W',D).p.$$

From (a') $\pi(U,V).p$, we obtain $\pi(U,(V.p)D).D$ by Theorem 2, and

$$\pi((V.p)D,Dp,U).p$$

by JSU. Now by repeatedly using JPR, we easily derive

$$\pi(\pi(X,W',D).p,X,(V.p)D,U,W').p,$$

and hence

$$\pi(\pi(X,W',D).p,W',D).p,$$

by using JTR and (a"). By Hyp 4, $X \sim \pi(A_1, \ldots, A_n)$ for some derivable $A_1, \ldots, A_n$.

This completes the proof of the theorem.

COROLLARY *There is no theorem of* **J** *of the form* $AABB$.

Proof. Suppose that there are A and B such that $AABB$ is derivable in **J**. Since **J** is closed under uniform substitution, there are A_1 and B_1 such that $A_1A_1B_1B_1$ is derivable in $\mathbf{J}_1$. By NOE, A_1A_1 is derivable in $\mathbf{J}_1$ and hence in **J**, contrary to NOID.

In fact, NOE is in **J** equivalent to NOID. For, suppose NOE and let A be a formula such that AA is derivable in **J**; then $AApp$ is derivable, contrary to NOE.

It is known that $AABB$ is a theorem of $\mathbf{E}_{\rightarrow}$; hence the name NOE.

A corollary of NOE concerning contraction and the Reirce Law is the following

THEOREM 12. $(A(AB))(AB)$ *is derivable in* **J** *iff so is* A*;* $ABAA$ *is derivable in* **J** *iff so is* AB.

NOE can be generalized to the following theorem.

THEOREM 13. (a) $\pi((\pi(X,Y).p)p^{2k},Y).p$ *is derivable iff* X *is nonenmpty and every member of* X *is derivable.*

Proof. If $k = 0$, the theorem is true by NOE.

Let $k > 0$ and proceed by induction on k. If Y is empty, we use Theorem 6.

Suppose that (a) is derivable in $\mathbf{J}_1$. By NOID, X is nonempty. The verification that (a) is not an instance of ASU is left to the reader.

Case I (a) is obtained by JSU from (a') $\pi(U,V).p$, where

$$\pi((\pi(X,Y).p)p^{2k},Y) \sim \pi(U,(V.p)p).$$

I.1 $(V.p)p \sim (\pi(X,Y).p)p^{2k}$ and $U \sim Y$; obviously, either X or Y is empty. But X is nonempty. If Y is empty, then by Theorem 6, $X \sim \pi(A_1, \ldots, A_n)$ for some derivable $A_1, \ldots, A_n$.

I.2 $Y \sim \pi((V.p)p, Y')$ and $U \sim \pi(\pi(X,(V.p)p,Y').p,Y')$. Obviously, (a) is obtained from (a') $\pi((\pi(X,(V.p)p,Y').p)p^{2k},V,Y').p$. Since X is nonempty, there is a member A of X. But as an instance of ASU we have $\pi(\pi(A,V).p,(V.p)p,A).p$. By using JPR we derive $\pi(\pi(X,V,Y').p,X,(V.p)p,Y').p$, and then by using JSU we obtain $\pi(\pi(X,V,Y').p,(\pi(X,(V.p)p,Y').p)p^{2k-1}).p$. Hence, by using JTR and (a') we obtain $\pi(\pi(X,V,Y').p,V,Y').p$. Now we use NOE to conclude that $X \sim \pi(A_1, \ldots, A_n)$ for some $A_1, \ldots, A_n$ derivable in $\mathbf{J}_1$.

Case II (a) follows by JPR from (a') $\pi(U,D).p$, where $\pi((\pi(X,Y).p)p^{2k},Y) \sim \pi(U,CD,C)$.

II.1 $Y \sim \pi(CD,C,Y')$ and $U \sim \pi((\pi(X,CD,C,Y').p)p^{2k},Y')$. But

$$\pi(\pi(X, D, Y').p, X, CD, C, Y').p$$

is easily derivable in $\mathbf{J}_1$. By JSU we derive

$$\pi(\pi(X, D, Y').p, (\pi(X, CD, C, Y').p)p^{2k-1}).p$$

Hence, by using JTR and (a'), we obtain $\pi(\pi(X, D, Y').p, D, Y').p$. Hence, $X \sim \pi(A_1, \ldots, A_n)$, by NOE, for some $A_1, \ldots, A_n$ and n, and $A_1, \ldots, A_n$ are derivable in $\mathbf{J}_1$.

II.2 $(\pi(X, Y).p)p^{2k} \sim CD$ and $Y \sim \pi(U, C)$; since $k > 0$, $D \sim p$ and Y is empty. The theorem follows by Theorem 6.

II.3 $(\pi(X, Y).p)p^{2k} \sim C$ and $Y \sim \pi(CD, Y')$; this is impossible.

Case III (a) follows by JG from (a') $\pi(U, V).p$ and (a") $\pi(W, D).p$, where we have

$$\pi((\pi(X, Y).p)p^{2k}, Y) \sim \pi(U, W, (V.p)D).$$

III.1 $(\pi(X, Y).p)p^{2k} \sim (V.p)D$ and $Y \sim \pi(U, W)$. Hence, $D \sim p$ and $V \sim (\pi(X, U, W).p)p^{2k-2}$. We have (a') $\pi(U, (\pi(X, U, W).p)p^{2k-2}).p$ and (a") $\pi(W, p).p$. By using JTR we derive

$$\pi((\pi(X, U, W).p)p^{2k-2}, U, W).p.$$

By induction hypothesis, $X \sim \pi(A_1, \ldots, A_n)$ for some derivable $A_1, \ldots, A_n$.

III.2 $U \sim \pi((\pi(X, Y).p)p^{2k}, U')$ and $Y \sim \pi((V.p)D, U', W)$. Now (a') is

$$\pi(((\pi(X, V.p)D, U', W).p)p^{2k}, U', V).p.$$

By (a") $\pi(W, D).p$ and JPR we derive $\pi(\pi(X, U', V).p, X, (V.p)D, U', W).p$, and then we use JSU to obtain $\pi(\pi(X, U', V).p, (\pi(X, (V.p)D, U', W).p)p^{2k-1}).p$. Hence, by using JTR and (a'), we obtain $\pi(\pi(X, U', V).p, U', V).p$. Hence, $X \sim \pi(A_1, \ldots, A_n)$, by NOE, for some derivable $A_1, \ldots, A_n$.

III.3 $W \sim \pi((\pi(X, Y).p)p^{2k}, W')$ and $Y \sim \pi((V.p)D, U, W')$. Now, obviously, (a") is

$$\pi((\pi(X, (V.p)D, U, W').p)p^{2k}, W', D).p.$$

From (a') $\pi(U, V).p$, we obtain $\pi(U, (V.p)D).D$ by Theorem 2, and

$$\pi((V.p)D, Dp, U).p$$

by JSU. Now by repeatedly using JPR and JSU, we easily derive

$$\pi(\pi(X, W', D).p, (\pi(X, (V.p)D, U, W').p)^{2k-1}).p,$$

and hence $\pi(\pi(X, W', D).p, W', D).p$, by using JTR and (a"). Now we use NOE to conclude that $X \sim \pi(A_1, \ldots, A_n)$ for some derivable $A_1, \ldots, A_n$.

This completes the proof of the theorem.

The difference between $\mathbf{L}$ and $\mathbf{J}_1$ is now clear: by L_5, there is no theorem of $\mathbf{L}$ of the form $\pi((\pi(X, Y).p)p^{2k}, Y).p$; by Theorem 13, $\pi((\pi(X, Y).p)p^{2k}, Y).p$ is derivable in $\mathbf{J}_1$ iff X is nonempty and every member of X is derivable in $\mathbf{J}_1$.

Two open problems

Let us adjoin to **J** the axiom-schema *pp*. It is easy to prove that ASU, JSU, and JPR are redundant. The system **J**+ID is equivalent to $\mathbf{RW}_{\to}$, defined by MP and the following axiom-schemata:

ID	AA
ASS	$A.ABB$
TR	$AB.BC.AC$

(the proof is omitted). It is then easy to show that A-B is not true for **J**+ID. From $A.ABB$, by Theorem 2 we obtain $ABB(AB).A.AB$. On the other hand, $A(AB).ABB.AB$ is an instance of ASU. Thus there are distinct formulas C and D such that both CD and DC are derivable in **J**+ID. It is therefore natural to raise the following two questions:

Question 1. Is there any proper extension **EX** of $\mathbf{TW}_{\to}$ such that A-B holds for **EX**?

Question 2. Is there any proper extension **EX** of **J** such that NOID holds for **EX**?

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