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EXPANSIONS OF THE KUREPA FUNCTION

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Dedicated to the memory of Professor Đ. Kurepa

Abstract. The Taylor series expansions of the Kurepa function $K(a+z)$, $a \geq 0$, and numerical determination of their coefficients $b_\nu(a)$ for $a=0$ and $a=1$ are given. An asymptotic behaviour of $b_\nu(a)$ as well as that $|b_\nu(a)/b_{\nu+1}(a)| \sim a+1$, when $\nu \to \infty$, are shown. Using this fact, a transformation of series with much faster convergence is done. Numerical values of coefficients in such a transformed series for $a=0$ and $a=1$ are given with 30 decimal digits. Also, the Chebyshev expansions of $K(1+z)$ and $1/K(1+z)$ are obtained.

1. Introduction

In 1971 Professor Đ. Kurepa (see [**8**–**9**]) defined so-called left factorial as

$$!n = 0! + 1! + \cdots + (n-1)!$$

and extended it to the complex plane

$$K(z) = \int_0^\infty \frac{t^z - 1}{t - 1}\, e^{-t}\, dt \qquad (\operatorname{Re} z > 0). \tag{1.1}$$

This function can be extended analytically to the hole complex plane by

$$K(z) = K(z+1) - \Gamma(z+1), \tag{1.2}$$

where $\Gamma(z)$ is the gamma function defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad (\operatorname{Re} z > 0) \quad \text{and} \quad z\Gamma(z) = \Gamma(z+1).$$

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Kurepa [**9**] proved that $K(z)$ is a meromorphic function with simple poles at the points $z_k = -k$ $(k \in \mathbb{N} \setminus \{2\})$. Slavić [**17**] found the representation

$$K(z) = -\frac{\pi}{e} \cot \pi z + \frac{1}{e} \left(\sum_{n=1}^{\infty} \frac{1}{n!n} + \gamma \right) + \sum_{n=0}^{\infty} \Gamma(z-n),$$

where γ is Euler's constant. These formulas were mentioned also in the book [**11**]. A number of problems and hypotheses, especially in number theory, were posed by Kurepa and then considered by several mathematicians. For details and a complete list of references see a recent survey written by Ivić and Mijajlović [**6**].

Evaluation of $K(z)$ for some specific z in $(0, 1)$, using quadrature formulas with relatively small accuracy, was done by Slavić and the author of this paper (see [**9**]).

In this paper we give power series expansions of the Kurepa function $K(a+z)$, $a \geq 0$, and determine numerical values of their coefficients $b_\nu(a)$ for $a = 0$ and $a = 1$. Using an asymptotic behaviour of $b_\nu(a)$, when $\nu \to \infty$, we give a transformation of series with much faster convergence. Also, we obtain the Chebyshev expansions for $K(1+z)$ and $1/K(1+z)$. For similar expansions of the gamma function see e.g. Davis [**3**], Luke [**10**], Fransén and Wrigge [**4**], and Bohman and Fröberg [**1**].

2. Power series expansions

There are many investigations about the gamma function $\Gamma(z)$, including numerical calculations. Some old references can be found in Luke [**10**]. In 1980 Fransén and Wrigge [**4**] determined numerical values of the coefficients in the Taylor series expansion

$$\Gamma(a+x)^m = \sum_{k=0}^{\infty} g_k(m, a) x^k$$

for certain values of m and a and used these values to calculate $\Gamma(p/q)$ $(p, q = 1(1)10;\ p < q)$ with a high precision. Recently, Bohman and Fröberg [**1**] have given power series expansions of the form

$$\Gamma(n+1+z) = n!(1 + d_1 z + d_2 z^2 + \cdots) \qquad (n = 2, 3, 4, 10)$$

and

$$(-1)^n n! \Gamma(-n+z) = \frac{n}{1-z} - \frac{1}{(n+1)(1+z)} + \frac{1}{z} \left(1 + f_1 z + f_2 z^2 + \cdots\right)$$

for $n = 0$ 1, 2, 10.

In this section we investigate the Kurepa function in a similar way.

Let $a \geq 0$. Differentiating (1.1) we find

$$K^{(\nu)}(a) = \int_0^{\infty} \frac{t^a \log^\nu t}{t-1} e^{-t} \, dt \qquad (\nu \geq 1), \tag{2.1}$$

so that the Taylor series expansion is given by

$$K(a+z) = b_0(a) + b_1(a)z + b_2(a)z^2 + b_3(a)z^3 + \cdots\,, \tag{2.2}$$

where

$$b_0(a) = K(a), \qquad b_\nu(a) = \frac{1}{\nu!}\,K^{(\nu)}(a) \qquad (\nu \geq 1). \tag{2.3}$$

Taking a to be an integer, we have that $b_0(a) = !a$. For example, $b_0(0) = 0$, $b_0(1) = 1$, $b_0(2) = 2$, $b_0(3) = 4$, etc.

The following theorem gives an asymptotic behaviour of $b_\nu(a)$.

THEOREM 2.1. *For every $a > 0$, $b_\nu(a) \to 0$, when $\nu \to \infty$. If $a = 0$ we have*

$$\lim_{\nu\to\infty} b_{2\nu}(0) = -1, \qquad \lim_{\nu\to\infty} b_{2\nu+1}(0) = 1.$$

Precisely,

$$b_\nu(a) \sim \frac{(-1)^{\nu+1}}{(a+1)^{\nu+1}}, \qquad \textit{when}\ \ \nu \to \infty.$$

Proof. We start with the identity

$$K(a) = R_1(a) + R_2(a), \tag{2.4}$$

where

$$R_1(a) = \int_0^1 \frac{t^a - 1}{t-1}\,e^{-t}\,dt \quad \text{and} \quad \int_1^\infty \frac{t^a - 1}{t-1}\,e^{-t}\,dt.$$

Differentiating (2.4) ν times with respect to a, we obtain

$$b_\nu(a) = \frac{K^{(\nu)}(a)}{\nu!} = -\frac{1}{\nu!}\int_0^1 \frac{t^a \log^\nu t}{1-t} \sum_{k=0}^\infty \frac{(-1)^k t^k}{k!}\,dt + \frac{1}{\nu!}\,R_2^{(\nu)}(a), \tag{2.5}$$

i.e.,

$$b_\nu(a) = -\frac{1}{\nu!}\sum_{k=0}^\infty \frac{(-1)^k}{k!}\int_0^1 \frac{t^{k+a}\log^\nu t}{1-t}\,dt + \frac{1}{\nu!}\,R_2^{(\nu)}(a). \tag{2.6}$$

Using the Karamata inequality [**7**] (cf. [**14**, p. 272]),

$$\frac{\log t}{t-1} \leq \frac{1}{\sqrt{t}} \qquad (t \geq 1),$$

for the last term on the right in (2.6), we have

$$\begin{aligned} 0 \leq \frac{1}{\nu!}\,R_2^{(\nu)}(a) &= \frac{1}{\nu!}\int_1^\infty \frac{t^a \log^\nu t}{t-1}\,e^{-t}\,dt \\ &\leq \frac{1}{\nu!}\int_1^\infty \frac{(t-1)^{\nu-1}}{t^{\nu/2-a}}\,e^{-t}\,dt \\ &\leq \frac{1}{\nu! e}\int_0^\infty \frac{z^{\nu-1}}{(z+1)^{\nu/2-a}}\,e^{-z}\,dz, \end{aligned}$$

i.e.,

$$\frac{1}{\nu!}\,R_2^{(\nu)}(a) < \frac{1}{\nu!e}\int_0^\infty z^{\nu/2+a-1}e^{-z}\,dz = \frac{1}{\nu!e}\,\Gamma\Big(\frac{\nu}{2}+a\Big) \to 0,$$

when $\nu \to \infty$.

On the other side, we have that (cf. [**16**, p. 491])

$$\int_0^1 \frac{t^{k+a}\log^\nu t}{1-t}\,dt = (-1)^\nu \nu!\zeta(\nu+1, k+a+1),$$

where $\zeta(z,\alpha) = \sum\limits_{m=0}^{\infty}(\alpha+m)^{-z}$ is the generalized Riemman function (the Hurwitz Zeta function).

Therefore,

$$b_r(a) = (-1)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\,\zeta(\nu+1, k+a+1) + \frac{1}{\nu!}\,R_2^{(\nu)}(a). \tag{2.7}$$

When $\nu \to \infty$, we see that

$$\zeta(\nu+1, q+1) = \frac{1}{(q+1)^{\nu+1}} + \sum_{m=1}^{\infty}\frac{1}{(q+1+m)^{\nu+1}} \to \begin{cases} 0, & \text{if } q > 0,\\ 1, & \text{if } q = 0.\end{cases}$$

Thus, when $\nu \to \infty$ all terms in the right-hand side of (2.7) approaches zero except the term $(-1)^{\nu+1}$ in the case when $a = 0$ $(k = 0,\ q = k + a = 0)$. The last statement of the theorem follows immediately from (2.7). □

From this theorem we obtain:

Corollary 2.2. *For the coefficients* $b_\nu(0)$ *in the series* (2.2) *we have that* $b_\nu(0) = (-1)^{\nu+1}(1+\varepsilon_\nu)$, *where* $\lim\limits_{\nu\to\infty}\varepsilon_\nu = 0$.

Remark 2.1. We note that the radius of convergence of the series (2.2) increases as a increases, because of $|b_\nu(a)/b_{\nu+1}(a)| \sim a + 1$, when $\nu \to \infty$.

The following theorem gives an expansion with a faster convergence.

Theorem 2.3. *For* $|z| < 1$ *we have*

$$K(z) = \frac{1}{1+z}\sum_{\nu=1}^{\infty}\alpha_\nu z^\nu, \tag{2.8}$$

where $\alpha_\nu = b_\nu(0) + b_{\nu-1}(0) = (-1)^{\nu-1}\Delta\varepsilon_{\nu-1}$, $\nu \ge 1$. *The coefficients* $b_\nu(0)$ *and* ε_ν *are given in Corollary* 2.2.

Proof. Applying the Euler-Abel transformation (cf. Milovanović [**12**, pp. 49–50]) to (2.2), with $a = 0$, we obtain (2.8). □

Remark 2.2. The series (2.8) converges also for $z = 1$. Notice that $\sum\limits_{\nu=1}^{\infty}\alpha_\nu = 2$.

Similar to Theorem 2.3 we can state and prove the following more general result:

THEOREM 2.4. *For* $|z| < a+1$ *we have*

$$K(a+z) = \frac{1}{a+1+z} \sum_{\nu=0}^{\infty} \beta_\nu(a) z^\nu, \tag{2.9}$$

where

$$\beta_0(a) = (a+1)b_0(a), \quad \beta_\nu(a) = (a+1)b_{\nu+1}(a) + b_\nu(a), \quad \nu \geq 1. \tag{2.10}$$

Proof. Put $K(a+z) = b_0(a) + \bigl(z/(a+1)\bigr)g(z)$, where

$$g(z) = \sum_{\nu=1}^{\infty} (a+1)^\nu b_\nu(a) \Bigl(\frac{z}{a+1}\Bigr)^{\nu-1}.$$

Since

$$\Bigl(1 + \frac{z}{a+1}\Bigr) g(z) = \sum_{\nu=0}^{\infty} \bigl\{(a+1)b_{\nu+1}(a) + b_\nu(a)\bigr\} z^\nu - b_0(a),$$

we obtain (2.9), where the coefficients $\beta_\nu(a)$ are given by (2.10). This is, in fact, the Euler-Abel transformation of the series (2.2). □

Remark 2.3. Suppose that $a \geq 1$. For $z = 1$ and $z = -1$, (2.9) gives

$$\sum_{\nu=0}^{\infty} \beta_\nu(a) = (a+2)K(a+1) \quad \text{and} \quad \sum_{\nu=0}^{\infty} (-1)^\nu \beta_\nu(a) = aK(a-1),$$

respectively. If $a = 1$, we have that

$$\sum_{\nu=0}^{\infty} \beta_\nu(1) = 6 \quad \text{and} \quad \sum_{\nu=0}^{\infty} (-1)^\nu \beta_\nu(1) = 0.$$

Numerical calculation of $b_\nu(a)$ for $\nu \geq 1$ is not so easy, because of singular integrand in (2.1). Some Gaussian type of quadrature formulas, e.g. Gauss-Laguerre formulas or, after changing variable, Gauss-Einstein formulas on $(0, \infty)$ (cf. Gautschi and Milovanović [**5**]) show a slowly convergence. Using the fact that for an integral of an analytic function over $(-\infty, \infty)$ the trapezoidal rule with an equal mesh size is asymptotically optimal among formulas with the same density of sampling points (see Takahasi and Mori [**18**]), we will transform our integrals to ones over $(-\infty, \infty)$, taking $t = \exp(\pi \sinh x)$, and then apply the trapezoidal quadrature formula

$$I = \int_{-\infty}^{\infty} g(x)\, dx \approx I_h = h \sum_{n=-\infty}^{\infty} g(nh) \tag{2.11}$$

For a such kind of quadratures obtained by variable transformation, especially for the DE-rule, see a survey written by Mori [**15**].

TABLE 2.1: The coefficients in the series (2.8) and (2.9) when $a = 1$

ν	$\beta_\nu(0)$	$\beta_\nu(1)$
0	0.00000 00000 00000 00000 00000 00000	2.00000 00000 00000 00000 00000 00000
1	1.43220 57346 53224 41481 10310 06215	2.70998 01395 03383 10840 90378 32265
2	0.46888 45450 62632 23498 89857 40662	0.90645 96812 26452 30535 12196 88028
3	0.06386 42753 51364 60958 84436 58189	0.26514 75834 59521 37636 12078 98719
4	0.02916 62163 10236 51587 58769 27935	0.08712 40652 66430 32799 74653 52816
5	0.00384 29068 21880 78393 78013 20684	0.02344 30113 03591 62453 71921 03613
6	0.00183 20459 86444 14916 92261 41329	0.00610 50968 58694 42506 90720 59033
7	0.00009 73017 96964 77704 35972 03248	0.00137 03067 97499 58760 40919 95658
8	0.00010 88168 64034 75461 36273 53524	0.00029 98495 87774 13386 75427 15149
9	-0.00000 83124 80989 14474 80756 83316	0.00005 71279 77327 82762 62249 42045
10	0.00000 76093 36747 52906 90149 05470	0.00001 09812 77997 28649 23721 64909
11	-0.00000 16730 50119 56080 52965 01901	0.00000 17875 37686 00232 08394 11611
12	0.00000 06860 95605 33952 08525 32143	0.00000 03177 80645 24735 74359 66125
13	-0.00000 02068 17935 17460 02890 90873	0.00000 00427 02999 64944 38064 00163
14	0.00000 00715 53267 36586 97321 77786	0.00000 00078 08364 44191 12066 23204
15	-0.00000 00233 56503 79198 66091 37138	0.00000 00007 17143 23915 78700 00764
16	0.00000 00078 18902 04675 37681 86066	0.00000 00001 87654 45234 13299 33159
17	-0.00000 00025 94278 60137 72193 76381	0.00000 00000 01586 36642 25452 36344
18	0.00000 00008 64332 38788 78858 80196	0.00000 00000 05714 87184 82397 51814
19	-0.00000 00002 87697 66083 26943 17995	-0.00000 00000 00680 75693 40826 49802
20	0.00000 00000 95832 46989 44643 89904	0.00000 00000 00256 93394 02012 50700
21	-0.00000 00000 31923 54355 14763 58582	-0.00000 00000 00054 25128 23079 80583
22	0.00000 00000 10636 48853 89104 25473	0.00000 00000 00014 61950 44184 99559
23	-0.00000 00000 03544 28133 05353 85378	-0.00000 00000 00003 53643 92780 66646
24	0.00000 00000 01181 12896 81039 46306	0.00000 00000 00000 89493 51613 20739
25	-0.00000 00000 00393 63482 92231 22875	-0.00000 00000 00000 22239 88173 56536
26	0.00000 00000 00131 19298 21915 64240	0.00000 00000 00000 05567 76999 51411
27	-0.00000 00000 00043 72633 98654 72079	-0.00000 00000 00000 01390 20407 68723
28	0.00000 00000 00014 57428 48201 29110	0.00000 00000 00000 00347 51114 79725
29	-0.00000 00000 00004 85780 47285 87821	-0.00000 00000 00000 00086 84201 06978
30	0.00000 00000 00001 61919 57473 07582	0.00000 00000 00000 00021 70582 21355
31	-0.00000 00000 00000 53971 38024 88922	-0.00000 00000 00000 00005 42530 90710
32	0.00000 00000 00000 17990 00746 96841	0.00000 00000 00000 00001 35611 61567
33	-0.00000 00000 00000 05996 55604 63516	-0.00000 00000 00000 00000 33898 54027
34	0.00000 00000 00000 01998 82374 65548	0.00000 00000 00000 00000 08473 77501
35	-0.00000 00000 00000 00666 26751 66870	-0.00000 00000 00000 00000 02118 27096
36	0.00000 00000 00000 00222 08740 61997	0.00000 00000 00000 00000 00529 53328
37	-0.00000 00000 00000 00074 02869 39615	-0.00000 00000 00000 00000 00132 37643
38	0.00000 00000 00000 00024 67612 09747	0.00000 00000 00000 00000 00033 09273
39	-0.00000 00000 00000 00008 22534 60745	-0.00000 00000 00000 00000 00008 27291

40	0.00000 00000 00000 00002 74177 51295	0.00000 00000 00000 00000 00002 06817
41	-0.00000 00000 00000 00000 91392 33194	-0.00000 00000 00000 00000 00000 51703
42	0.00000 00000 00000 00000 30464 06756	0.00000 00000 00000 00000 00000 12926
43	-0.00000 00000 00000 00000 10154 67841	-0.00000 00000 00000 00000 00000 03231
44	0.00000 00000 00000 00000 03384 89011	0.00000 00000 00000 00000 00000 00808
45	-0.00000 00000 00000 00000 01128 29603	-0.00000 00000 00000 00000 00000 00202

Thus, coefficients given by (2.3) and (2.1) reduce to

$$b_\nu(a) = \frac{\pi^{\nu+1}}{2\nu!}\int_{-\infty}^{\infty} \frac{\sinh^\nu x \cosh x \exp\left((a+\frac{1}{2})\pi \sinh x - e^{\pi \sinh x}\right)}{\sinh\left(\frac{\pi}{2}\sinh x\right)}\,dx \qquad (\nu \geq 1).$$

Similarly,

$$\alpha_\nu = \frac{\pi^{\nu+1}}{2(\nu+1)!}\int_{-\infty}^{\infty} \frac{\sinh^\nu x \cosh x \exp\left(\frac{1}{2}\pi \sinh x - e^{\pi \sinh x}\right)}{\sinh\left(\frac{\pi}{2}\sinh x\right)}\left[\pi \sinh x + \nu + 1\right]\,dx$$

for $\nu \geq 1$.

Using (2.11) in Q-arthmetic on the MICROVAX 3400 computer (machine precision $\approx 1.93 \times 10^{-34}$), we computed the coefficients $b_\nu(a)$ and $\beta_\nu(a)$ ($\nu \leq 60$) for $a = 0$ and $a = 1$. Here, we give in Table 2.1 the coefficients $\beta_\nu(0)$ and $\beta_\nu(1)$ only for $\nu = 0(1)45$ with 30D.

3. Chebyshev expansions

Let $T_k^*(z)$ be the shifted Chebyshev polynomial of the first kind of degree k given by $T_k^*(z) = T_k(2z-1)$, $T_k(x) = \cos k\theta$ ($x = \cos\theta$). Numerical values of the coefficients in the Chebyshev expansions for the gamma function

$$\Gamma(z+1) = \sum_{k=0}^{\infty} \gamma_k T_k^*(z) \quad \text{and} \quad \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} \delta_k T_k^*(z) \qquad (0 \leq z \leq 1),$$

were given by Clenshaw [**2**] to 20D (see also Luke [**10**, p. 13]).

In this section we obtain similar expansions for the Kurepa function. Namely, we determine the coefficients c_k and d_k in the Chebyshev expansions

$$K(z+1) = \sum_{k=0}^{\infty} c_k T_k^*(z) \quad \text{and} \quad \frac{1}{K(z+1)} = \sum_{k=0}^{\infty} d_k T_k^*(z) \qquad (0 \leq z \leq 1). \quad (3.1)$$

Taking $x = 2z - 1$, we can reduce our problem to the interval $(-1, 1)$ and then use the least square approximation, with respect to the Chebyshev weight $w(x) = (1-x^2)^{-1/2}$, in order to find the coefficients c_k and d_k. Thus,

$$c_k = \frac{\left(K((x+3)/2), T_k\right)}{\|T_k\|^2}, \qquad d_k = \frac{\left(1/K((x+3)/2), T_k\right)}{\|T_k\|^2} \qquad (k \geq 0),$$

where the inner product and the norm are given by

$$(f,g)=\int_{-1}^{1} f(x)\overline{g(x)}w(x)\,dx, \qquad \|f\|^2=(f,f),$$

respectively.

TABLE 3.1: The coefficients c_k and d_k in series (3.1)

k	c_k	d_k
0	1.47404 93837 23786 91003 52009 68753	0.71933 88033 78564 80199 26920 51918
1	0.49590 32661 94158 22700 15501 19905	-0.24486 53611 27056 06741 80872 17436
2	0.02579 30869 03070 45788 53165 66164	0.02976 94905 09647 98622 36085 35374
3	0.00407 08812 31791 61386 34001 49176	-0.00498 33867 56921 33739 51852 46148
4	0.00015 73577 18751 30746 82134 28808	0.00086 57497 58509 74532 38298 56153
5	0.00002 57037 05826 92036 26292 93518	-0.00014 67971 82657 38722 42886 97797
6	0.00000 01775 04779 26512 29793 59549	0.00002 51924 09034 18335 36671 90502
7	0.00000 01478 12089 72264 27761 10704	-0.00000 43238 54800 40900 43392 52848
8	-0.00000 00057 65834 77355 97860 46608	0.00000 07414 54647 70219 98837 74381
9	0.00000 00010 46615 98880 68659 86996	-0.00000 01272 20052 30389 19464 41242
10	-0.00000 00000 83632 76388 29507 65862	0.00000 00218 27577 03559 46944 44963
11	0.00000 00000 09422 94336 75632 53320	-0.00000 00037 44928 13250 51616 37824
12	-0.00000 00000 00910 90916 80081 86592	0.00000 00006 42530 56969 56371 69126
13	0.00000 00000 00093 62274 78249 23554	-0.00000 00001 10240 75381 57048 29851
14	-0.00000 00000 00009 39687 44816 46570	0.00000 00000 18914 30308 43243 17659
15	0.00000 00000 00000 95140 34153 77730	-0.00000 00000 03245 18210 36461 92610
16	-0.00000 00000 00000 09603 61612 70033	0.00000 00000 00556 78518 95627 89374
17	0.00000 00000 00000 00970 39299 20617	-0.00000 00000 00095 52923 18136 84543
18	-0.00000 00000 00000 00098 02159 05061	0.00000 00000 00016 39022 51658 48302
19	0.00000 00000 00000 00009 90238 99413	-0.00000 00000 00002 81211 80450 47223
20	-0.00000 00000 00000 00001 00033 64229	0.00000 00000 00000 48248 31777 95406
21	0.00000 00000 00000 00000 10105 45983	-0.00000 00000 00000 08278 10261 20992
22	-0.00000 00000 00000 00000 01020 85778	0.00000 00000 00000 01420 29786 64079
23	0.00000 00000 00000 00000 00103 12756	-0.00000 00000 00000 00243 68458 86448
24	-0.00000 00000 00000 00000 00010 41800	0.00000 00000 00000 00041 80966 55298
25	0.00000 00000 00000 00000 00001 05243	-0.00000 00000 00000 00007 17340 45282
26	-0.00000 00000 00000 00000 00000 10632	0.00000 00000 00000 00001 23076 16403
27	0.00000 00000 00000 00000 00000 01074	-0.00000 00000 00000 00000 21116 53134
28	-0.00000 00000 00000 00000 00000 00108	0.00000 00000 00000 00000 03623 02400
29	0.00000 00000 00000 00000 00000 00011	-0.00000 00000 00000 00000 00621 61264
30	-0.00000 00000 00000 00000 00000 00001	0.00000 00000 00000 00000 00106 65187

Since $\|T_0\|^2=\pi$ and $\|T_k\|^2=\pi/2$ $(k\ge 1)$, we have

$$c_0=\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}K\Bigl(\frac{x+3}{2}\Bigr)\,dx, \quad c_k=\frac{2}{\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}K\Bigl(\frac{x+3}{2}\Bigr)T_k(x)\,dx,$$

where $k \geq 1$. Similarly,

$$d_0 = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{1}{K\left(\dfrac{x+3}{2}\right)} \, dx, \quad d_k = \frac{2}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \frac{T_k(x)}{K\left(\dfrac{x+3}{2}\right)} \, dx \quad (k \geq 1).$$

For numerical determination of these coefficients, we apply the N-point Gauss-Chebyshev quadrature formula (see **[13]**)

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x) \, dx = \frac{\pi}{N} \sum_{k=1}^{N} f\left(x_k^{(N)}\right) + R_N(f),$$

where $x_k^{(N)}$, $k = 1, \ldots, N$, are the zeros of N-th Chebyshev polynomial $T_N(x)$, i.e., $x_k^{(N)} = \cos((2k-1)\frac{\pi}{2n}$. This formula is exact for all algebraic polynomials of degree at most $2N-1$, i.e., $R_N(\mathcal{P}_{2N-1}) \equiv 0$. To compute values of the Kurepa function we use the power series expansion from Section 2. In order to obtain numerical values to 30D of coefficients c_k and d_k for $k \leq 30$, the Gaussian formula (3.1) needs $N = 40$ and $N = 35$ points, respectively. Numerical values are displayed in Table 3.1.

REFERENCES

[1] J. Bohman and C.-E. Fröberg, *The Γ-function revisited: power series expansions and real-imaginary zero lines*, Math. Comp. **58** (1992), 315–322.

[2] C.W. Clenshaw, *Chebyshev Series for Mathematical Functions*, Nat. Phys. Lab. Math. Tables, vol. V, H.M. Stationery Office, London, 1962.

[3] H.T. Davis, *Tables of the Higher Mathematical Functions*, vol. I, Principia Press, Bloomington, IN, 1933.

[4] A. Fransén and S. Wrigge, *High-precision values of the gamma function and some related coefficients*, Math. Comp. **34** (1980), 553–566.

[5] W. Gautschi and G.V. Milovanović, *Gaussian quadrature involving Einstein and Fermi functions with an application to summation of series*, Math. Comp. **44** (1985), 177–190.

[6] A. Ivić and Ž. Mijajlović, *On Kurepa problems in number theory*, These Publications.

[7] J. Karamata, *Sur quelques problèmes posés par Ramanujan*, J. Indian Math. Soc. (N.S.) **24** (1960), 343–365.

[8] Đ. Kurepa, *On the left factorial function* !*n*, Math. Balkanica **1** (1971), 147–153.

[9] Đ. Kurepa, *Left factorial function in complex domain*, Math. Balkanica **3** (1973), 297–307.

[10] Y.L. Luke, *Mathematical Functions and Their Approximations*, Mir, Moscow, 1980 (Russian).

[11] O.I. Marichev, *Handbook of Integral Transformation of Higher Transcendental Functions: Theory and Algorithmic Tables*, Ellis Horwood Ltd., Chichester, 1983

[12] G.V. Milovanović, *Numerical Analysis*, vol. I, Naučna Knjiga, Beograd, 1991 (Serbian).

[13] G.V. Milovanović, *Numerical Analysis*, vol. II, Naučna Knjiga, Beograd, 1991 (Serbian).

[14] D.S. Mitrinović and P.M. Vasić, *Analytic Inequalities*, Springer-Verlag, Berlin – Heidelberg – New York, 1970.

[15] M. Mori, *Quadrature formulas obtained by variable transformation and DE-rule*, J. Comput. Appl. Math. **12&13** (1985), 119–130.

[16] A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev, *Intergrals and Series: Elementary Functions*, Nauka, Moscow, 1981 (Russian).

[17] D.V. Slavić, *On the left factorial function of the complex argument*, Math. Balkanica **3** (1973), 472–477.

[18] H. Takashasi and M. Mori, *Error estimation in the numerical integration of analytic functions*, Report Computer Centre University of Tokyo **3** (1970), 41–108.

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