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ISOMORPHISMS OF REAL CLOSED FIELDS

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Abstract. We prove that a non-archimedean real closed field is not characterized by its order type, answering a question of Erdös, Gillman and Henriksen [**9**]. We also provide an alternative proof to their another long lasting question.

Introduction. In their famous paper [**10**], Erdös, Gillman and Henriksen obtained a nice results about η_1-ordered real closed fields, asked a few questions, which stayed open for more than twenty years. Even not explicitly mentioned in that form, these questions are about the existence of saturated models of this theory. One of the most interesting ones is the question 5.4 asking if the statement that all η_1-ordered real closed fields of size c could be proved without CH. This problem was negatively solved by Dow [**8**], who proved that the statement is equivalent to CH. In his proof he used ultrapowers. In section 2 we give another proof of this theorem, constructing our examples using Hahn diagrams and the results of Alling [**1**, **2**, **3**]. Let us just note that similar result was proved by van Douwen and van Mill for Parovichenko spaces, or dually (as it was proved by Perović) for ω_1-saturated atomless Boolean algebras of size c.

Another question, asked in the same paper, was if the order type of a real closed field determines its isomorphism type. In section 3 we provide an example answering this question negatively. We announced this result in [**17**]. Meanwhile a similar result was obtained by Alling and Kuhlmann [**4**]. To say the least, we do not think that it was obtained independently from our result.

Background. Here we recall some definitions and facts, mostly following [**10**].

$\mathcal{Q}$ is $\{(x_\alpha)_{\alpha<\omega 1} \mid x_\alpha \in \{0,1\}, \{\alpha \mid x_\alpha = 1\}$ has maximum$\}$, ordered lexicographically.

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Let $\mathcal{G} = (G, +, 0, \leq)$ be a totally ordered, divisible Abelian group. The absolute value of an element $a \in G$ is defined as $|a| = \max\{a, -a\}$. An o-subgroup $\mathcal{H}$ of $\mathcal{G}$ is convex if for every $b \in H, a \in G, |a| \leq |b| \Leftrightarrow a \in H$. The principal convex subgroup generated by $a \in G$ is $V(a) = \{b \in G|\, |b| \leq n|a|$ for some $n \in \omega\}$. $\mathcal{G}$ is Archimedean if all the principal convex subgroups generated by nonzero elements coincide. The maximal convex subgroup not containing a is $V^-(a) = \{b \in G|\ n|b| < |a|$ for every $n \in \omega\}$. $V^-(a)$ is a subgroup of $V(a)$ and $V(a)/V^-(a)$ is an Archimedean group. $\{V(a)|a \in G\}$ ordered by inclusion is called the value set of $\mathcal{G}$. Let T be a totally ordered set, G a totally ordered Abelian group, and let $f \in G^T$. The set $s(f) = \{t \in T|f(t) \neq 0\}$ is called the support of f. $G\{T\} = \{f \in G^T|s(f)$ is anti-well ordered$\}$ is an Abelian group with pointwise addition and totally ordered if we define that an element is positive if its maximal nonzero coordinate is positive. If G is an Archimedean group, then T is the value set of $G\{T\}$.

Let $\mathcal{F} = (F, +, \cdot, 0, 1, -, ^{-1}, \leq)$ be a totally ordered field. Let Π be the value set of its additive group. We also define the addition in Π as $V(a) + V(b) = V(ab)$. The ordered group $(\Pi, +, V(1), \leq)$ is called the value group of the field $\mathcal{F}$. For every $a \in F, V(a)/V(a)^- \cong V(1)/V(1)^-$ is called the residue field of $\mathcal{F}$. Let F be a totally ordered field and G be a totally ordered Abelian group. $K = F\{G\}$ becomes a totally ordered field under multiplication: For $a, b \in K$, $ab(g) = \sum_{x \in G} a(x)b(g-x)$. This is so called the field of formal power series.

$F\{G\}_\alpha$ will denote the o-subfield of $F\{G\}$ consisting of sequences with the support of cardinality $< \aleph_\alpha$.

Let α be an ordinal. An ordered field $\mathcal{F}$ is η_α-ordered if $(F, \leq)$ is an η_α-set i.e. for every two subsets $H, K \subset F$ such that $|H| + |K| < \aleph_\alpha$ and $H < K$, there exists $a \in F$ such that $H < a < K$. We will now recall the definition of α-maximal fields introduced by I. Kaplansky.

Definition. Let $\mathcal{F}$ be an ordered field with the valuation V, and let $\mathcal{G}$ be its value group. Let also α be an ordinal. For a sequence $\{a_\beta|\beta < \alpha\}$ we will say that it is pseudo-convergent if for every $\delta < \gamma < \beta < \alpha$ we have $V(a_\beta - a_\gamma) < V(a_\gamma - a_\delta)$. We will also say that an element $a \in F$ is a pseudo-limit of that pseudo-convergent sequence if $V(a - a_\alpha) = V(a_{\beta+1} - a_\beta)$.

Definition. Let $\aleph_\alpha$ be the α-th infinite cardinal. We will say that $\mathcal{F}$ is α-maximal if every pseudo-convergent sequence of length less then $\aleph_\alpha$ has a pseudo-limit. $\mathcal{F}$ is maximal if it is α-maximal for $|F| = \aleph_\alpha$.

We will now recall a few facts from [**3**]:

THEOREM 1.1. (called Main Theorem in [**3**]) *Let G be a totally ordered Abelian group and let α be a nonzero ordinal number. G is an η_α-set iff*

(1) *its factors are conditionally complete,*

(2) *its value set is an η_α-set, and*

(3) *it is α-maximal.*

THEOREM 1.2. (called Main Corollary in [**3**]) *Let $\mathcal{F}$ be an ordered field with the valuation V and the value group $\mathcal{G}$. The following three conditions are necessary and sufficient for $\mathcal{F}$ to be η_α-ordered:*

(i) *Its residue field is isomorphic to $\mathcal{R}$.*

(ii) *Its value group $\mathcal{G}$ is an η_α-ordered group.*

(iii) *It is α-maximal.*

Real-closed fields are fields where every positive element is a square, and every polynomial (of one variable) of odd degree has a zero. Since being positive is describable as being a square, o-isomorphisms and isomorphisms coincide.

The following theorem appears in [**3**] as Corollary 3.2:

THEOREM 1.3. *Let F be a real-closed field, H a totally ordered Abelian divisible group, and α a nonzero ordinal number. Then $F\{H\}_\alpha$ is a real-closed field.*

2. Are all η_1-ordered real closed fields isomorphic?

LEMMA 2.1. *If two totally ordered Abelian groups are o-isomorphic, then their value sets are similar.*

Proof. If $\varphi : G \to H$ is an o-isomorphism, then $\phi(V(a)) = V(\varphi(a))$ defines an isotone mapping of the value sets.

Similarly we prove the following:

LEMMA 2.2. *Let G, H be totally ordered Abelian groups, and T atotally ordered set. If $G\{T\} \cong H\{T\}$, then $G \cong H$.*

LEMMA 2.3. *If two totally ordered fields are o-isomorphic, then their value groups are o-isomorphic.*

THEOREM 2.4. *The following are equivalent:*

(i) CH*;*

(ii) *Every two η_1 real-closed fields of size c are isomorphic;*

(iii) *Every two divisible η_1 totally ordered Abelian groups of size c are o-isomorphic.;*

(iv) *Every two η_1 totally ordered sets of size c are similar.*

Proof. (i) $\Leftrightarrow$ (ii) was proved by Erdös, Gillman and Henriksen [**9**], (i) $\Leftrightarrow$ (ii) by Alling [**1**], (i) $\Leftrightarrow$ (iv) by Hausdorff. Nowdays these proofs are straightforward since in the presence of CH all these models become saturated models of complete theories, hence they are isomorphic.

(iv) $\Leftrightarrow$ (i) (Gillman) Actually $\neg$ (i) $\Leftrightarrow$ $\neg$(iv) was proved. If $c \geq \omega_2$, then $\omega_2 \times \mathcal{Q}$ and $\mathcal{Q}$ are both of size c, η_1-totally-ordered and not similar.

$\neg$(iv) $\Leftrightarrow$ $\neg$(iii) Let $(L_1, \leq)$ and $(L_2, \leq)$ be two nonsimilar η_1 totally ordered sets of size c. Let $G_1 = R\{L_1\}_1$ and $G_2 = R\{L_2\}_1$ be their appropriate Hahn groups, consisting of sequences with countable support. They are divisible since the additive group of reals is divisible. By Theorem 0.1 they are η_1-sets. Since their value sets are not similar, G_1 and G_2 are not o-isomorphic. Hence $\neg$(iii) is proved.

$\neg$(iv) $\Leftrightarrow$ $\neg$(ii) Let $(L_1, \leq)$ and $(L_2, \leq)$ be two nonsimilar η_1 totally ordered sets of size c. By Theorems 0.2 and 0.3 $R\{R\{L_1\}_1\}_1$ and $R\{R\{L_2\}_1\}_1$ are η_1 totally ordered real-closed fields of size c. Since the value sets of their additive groups are nonsimilar, they are not isomorphic (Lemma 1.2). Hence $\neg$(ii).

The equivalence i) $\Rightarrow$ ii) answers the question 5.4 from [**9**] negatively. The authors asked whether the isomorphism of real-closed fields that are η_1 sets of size c could be proved without CH. Since we proved that this statement is equivalent to CH the answer is obviously: No.

2. Real closed fields and their types of order

In [**9**] was asked the following question (5.1): Is a nondenumerable real-closed field—in particular if it is non-archimedean-characterized by its type of order as an ordered set? We answer the question negatively giving the examples of two non-isomorphic, nondenumerable, nonarchimedean real-closed fields ordered similarly.

PROPOSITION 3.1. *Let $s, t \in R$ be independent transcendentals over Q. Then:*

(i) *The ordered fields $Q, Q(s), Q(t)$ and their real closures are all similar*

(ii) *There exist no isomorphism from $\mathcal{R}(Q(s))$ into $\mathcal{R}(Q(t))$, actually there is no o-isomorphism between their additive groups.*

Proof. (See [**12**], 13.C] (i) They are all η_0 sets.

(ii) Any such an isomorphism would preserve the order, and mapping rationals identically on themself it would be the identity, contrary to the assumption that s and t are independent.

Example 3.2: Let $F_1 = \mathcal{R}(Q(\pi)), F_2 = \mathcal{R}(Q(e))$ be the smallest real-closed fields containing π and e respectively, and G_1 and G_2 be their additive groups. Let T be a totally ordered set of size 2^c. $H_1 = G_1\{T\}$ and $H_2 = G_2\{T\}$ are totally ordered divisible groups of size 2^c. They are obviously similarly ordered and not o-isomorphic. By a theorem of Krull [**13**], $R\{H_1\}$ and $R\{H_2\}$ are real-closed fields. Since H_1 and H_2 are similarly ordered, then $R\{H_1\}$ and $R\{H_2\}$ are also similarly ordered. On the other hand they are nonisomorphic since their value groups are not o-isomorphic. Finally they are of size 2^c, hence they cannot be embedded in reals, so they are non-archimedean.

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