

## ITERATION OF ULTRAPRODUCTS OF LOCALLY CONVEX SPACES

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**Abstract.** Let  $\{E_{ij}: (i, j) \in I \times J\}$  be a family of locally convex Hausdorff spaces. We study the relation between the spaces  $P_{(i,j) \in I \times J}(E_{ij})$  and  $P_{j \in J}(P_{i \in I}(E_{ij}))$ . In particular, we prove that the ultraproduct of an ultraproduct is again an ultraproduct.

In the following,  $\mathbf{N}$  will denote the set of all positive integers,  $\mu$  the first uncountable measurable cardinal,  $|X|$  the cardinal of the set  $X$ ,  $I$  and  $J$  sets of indices such that  $|I| \geq \mu$  and  $|J| \geq \mu$ ,  $\mathcal{F}$  and  $\mathcal{G}$  countably complete ultrafilters on  $I$  and  $J$  respectively and  $\{E_{ij}: (i, j) \in I \times J\}$  a family of locally convex Hausdorff spaces.

An ultrafilter  $\mathcal{F}$  is said to be countably complete if for each sequence  $\{A_n\}_{n \in \mathbf{N}} \subset \mathcal{F}$  such that  $\bigcup_{n \in \mathbf{N}} A_n \in \mathcal{F}$  there exists  $m \in \mathbf{N}$  such that  $A_m \in \mathcal{F}$  (or equivalently, if  $A_n \notin \mathcal{F}, \forall n \in \mathbf{N}$  then  $\bigcup_{n \in \mathbf{N}} A_n \notin \mathcal{F}$ ).

In [2] the ultraproduct is defined of a family  $\{E_i: i \in I\}$  of locally convex Hausdorff spaces such that  $|I| \geq \mu$ , with respect to a countably complete ultrafilter  $\mathcal{F}$  on  $I$ , whose construction we recall again for the sake of completeness: consider the equivalence relation in the product space  $\prod_{i \in I} E_i$  given by:

$$\{x_i: i \in I\} =_{\mathcal{F}} \{y_i: i \in I\} \iff \{i \in I: x_i = y_i\} \in \mathcal{F}$$

Let  $p_i$  be an arbitrary continuous seminorm on the space  $E_i$ , ( $i \in I$ ), and  $[\{x_i: i \in I\}]$  the equivalence classes of the quotient space. We call the ultraproduct of the family  $\{E_i: i \in I\}$  with respect to the ultrafilter  $\mathcal{F}$  the above mentioned quotient space endowed with the topology defined by the family of seminorms:

$$\bar{p}([\{x_i: i \in I\}]) = \lim_{\mathcal{F}} p_i(x_i)$$

This limit always exists because  $I = \bigcup_{n \in \mathbf{N}} \{i \in I: p_i(x_i) \in [n, n+1]\} \in \mathcal{F}$ , and since  $\mathcal{F}$  is countably complete there exists  $m \in \mathbf{N}$  such that  $\{i \in I: p_i(x_i) \in$

$[m, m + 1] \in \mathcal{F}$ . Furthermore, it is easy to prove that  $\bar{p}$  is a seminorm. We denote the ultraproduct space by  $P_{i \in I}(E_i)$ . Since  $E_i$ ,  $i \in I$ , are Hausdorff spaces, we obtain that the ultraproduct space is also a Hausdorff space. If all the spaces  $E_i$  ( $i \in I$ ) are equal to a certain space  $E$ , then we refer to their ultraproduct with respect to  $\mathcal{F}$  as to the ultrapower of  $E$  with respect to  $\mathcal{F}$  which we shall denote by  $P_{\mathcal{F}}(E)$ .

Finally, we denote by  $\mathcal{F} \times \mathcal{G}$ , the family of all subsets  $K \subset I \times J$  such that

$$\{j \in J: \{i \in I: (i, j) \in K\} \in \mathcal{F}\} \in \mathcal{G}$$

It is an ultrafilter on  $I \times J$  [1, page 156]. We prove that it is countably complete. Let  $\{A_n\}_{n \in \mathbf{N}} \subset \mathcal{F} \times \mathcal{G}$  be an arbitrary sequence. It follows that  $\{j \in J: \{i \in I: (i, j) \in A_n\} \in \mathcal{F}\} \in \mathcal{G}$ , for all  $n \in \mathbf{N}$ . Since  $\{i \in I: (i, j) \in \bigcup_{n \in \mathbf{N}} A_n\} = \bigcup_{n \in \mathbf{N}} \{i \in I: (i, j) \in A_n\} \in \mathcal{F}$ , because  $\mathcal{F}$  is countably complete, then

$$\{j \in J: \{i \in I: (i, j) \in \bigcup_{n \in \mathbf{N}} A_n\} \in \mathcal{F}\} = \bigcup_{n \in \mathbf{N}} \{j \in J: \{i \in I: (i, j) \in A_n\} \in \mathcal{F}\},$$

which belongs to  $\mathcal{G}$ , because it is countably complete.

Now, we prove the iteration theorem:

**THEOREM.** *Let  $\{E_{ij}: (i, j) \in I \times J\}$  be a family of locally convex Hausdorff spaces,  $|I| \geq \mu$ ,  $|J| \geq \mu$  and let  $\mathcal{F}$ ,  $\mathcal{G}$  be countably complete ultrafilters on  $I$  and  $J$  respectively. Then the ultraproduct spaces  $P_{(i,j) \in I \times J}(E_{ij})$  and  $P_{j \in J}(P_{i \in I}(E_{ij}))$  are isomorphic.*

*Proof.* Let  $\tilde{x} = [\{x_{ij}: (i, j) \in I \times J\}] \in P_{(i,j) \in I \times J}(E_{ij})$  and let  $\{x_{ij}: (i, j) \in I \times J\}$  be an element of this class. Denote by  $\pi_j(\tilde{x})$  the class in the space  $P_{i \in I}(E_{ij})$ , corresponding to the element  $\{x_{ij}: i \in I\} \in \prod_{i \in I} E_{ij}$ , and by  $[\pi_j(\tilde{x})]$  the class in the space  $P_{j \in J}(P_{i \in I}(E_{ij}))$  associated to the element  $\{\pi_j(\tilde{x}): j \in J\} \in \prod_{j \in J}(P_{i \in I}(E_{ij}))$ .

We prove that the mapping

$$\pi: P_{(i,j) \in I \times J}(E_{ij}) \longrightarrow P_{j \in J}(P_{i \in I}(E_{ij})); \quad \pi(\tilde{x}) = [\pi_j(\tilde{x})]$$

is the desired isomorphism.

Clearly  $\pi$  is linear and onto.  $\pi$  is one to one: let  $\pi(\tilde{x})$  be the null class of the ultraproduct  $P_{j \in J}(P_{i \in I}(E_{ij}))$ . So  $\{j \in J: \pi_j(\tilde{x}) = 0\} \in \mathcal{G}$ , that is to say,

$$\{j \in J: \{i \in I: x_{ij} = 0\} \in \mathcal{F}\} \in \mathcal{G} \quad (1)$$

If we call  $K = \{(i, j) \in I \times J: x_{ij} = 0\}$ , then, by (1),  $K \in \mathcal{F} \times \mathcal{G}$  and so  $\tilde{x} = 0$ .

$\pi$  is bicontinuous: the proof of the continuity of  $\pi$  and  $\pi^{-1}$  is similar, we only see that of  $\pi$ . Let  $\bar{p} = \lim_{\mathcal{G}} \bar{p}_j$  be a continuous seminorm on  $P_{j \in J}(P_{i \in I}(E_{ij}))$ , where  $\bar{p}_j = \lim_{\mathcal{F}} p_{ij}$ . Let  $[\pi_j(\tilde{x})]$  be an arbitrary element of the iterated ultraproduct,  $\bar{q} = \lim_{\mathcal{F} \times \mathcal{G}} p_{ij}$  continuous seminorm on  $P_{(i,j) \in I \times J}(E_{ij})$  and  $\alpha = \lim_{\mathcal{F} \times \mathcal{G}} p_{ij}(x_{ij})$ . This means that

$$\forall \varepsilon > 0, \quad K_\varepsilon = \{(i, j) \in I \times J: |p_{ij}(x_{ij}) - \alpha| < \varepsilon\} \in \mathcal{F} \times \mathcal{G}$$

Therefore, for all  $j \in J_0 \in \mathcal{G}$ , we have  $\{i \in I: |p_{ij}(x_{ij}) - \alpha| < \varepsilon\} \in \mathcal{F}$ , where  $J_0 = \{j \in J: \{i \in I: (i, j) \in K_\varepsilon\} \in \mathcal{F}\}$ .

Then  $|\bar{p}_j(\{x_{ij}: i \in I\}) - \alpha| \leq \varepsilon, \forall j \in J_0$ . It follows that  $\lim_{\mathcal{G}} \lim_{\mathcal{F}} p_{ij}(x_{ij}) = \alpha$ . So  $\bar{p}(\pi(\tilde{x})) \leq \bar{q}(\tilde{x})$  and  $\pi$  is continuous, as required.

**COROLLARY.** *Let  $\mathcal{F}, \mathcal{G}$  be countably complete ultrafilters on  $I$  and  $J$  respectively and let  $\{E_i: i \in I\}$  be a family of locally convex Hausdorff spaces. Then the ultrapower  $P_{\mathcal{G}}(P_{i \in I}(E_i))$  of the ultraproduct  $P_{i \in I}(E_i)$  is isomorphic to the ultraproduct  $P_{(i,j) \in I \times J}(E_{ij})$  where  $E_{ij} = E_i$ , for all  $j \in J$ . In particular the ultrapower of an ultrapower is itself an ultrapower.*

*Proof.* For each  $j \in J$ ,  $P_{i \in I}(E_{ij})$  is isomorphic to  $P_{i \in I}(E_i)$ .

*Remarks.* 1. In fact we have proved that for all  $\bar{p}$ , continuous seminorm on the iterated ultraproduct  $P_{j \in J}(P_{i \in I}(E_{ij}))$ , there exists  $\bar{q}$ , continuous seminorm on  $P_{(i,j) \in I \times J}(E_{ij})$ , such that  $\bar{p}(\pi(\tilde{x})) = \bar{q}(\tilde{x})$  (and reciprocally).

2. A result of this type is known within the field of normed spaces ([3, page 101], [4, Proposition 2.1]).

#### REFERENCES

- [1] W. W. Comfort, S. Negrepointis, *The Theory of Ultrafilters*. Springer-Verlag, Berlin, 1974.
- [2] J. A. Facenda Aguirre, *Ultraproducts of locally convex spaces*, Rev. Roumaine Math. Pures Appl. **32** (1987), 99-106.
- [3] B. Sims, "Ultra"-techniques in Banach space theory", Queen's Papers in Pure and Applied Mathematics no. **60** (1982), Queen's University.
- [4] J. Stern, *Ultrapowers and local properties of Banach spaces*, Trans. Amer. Math. Soc. **240** (1978), 231-252.

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