PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 51 (65), 1992, 71–76

BEST COAPPROXIMATION IN METRIC SPACES

T. D. Narang

Abstract. As a counterpart to best approximation, the concept of best coapproximation in normed linear spaces was introduced by C. Franchetti and M. Furi [**2**] in 1972. This study was subsequently pursued in normed linear spaces and Hilbert spaces by H. Behrens, L. Hetzelt, P.L. Papini, Geetha S. Rao, Ivan Singer, U. Westphal, the author and a few others (see e.g. [**3**], [**5**], [**7**]). In this paper we discuss best coapproximation in metric spaces there by generalizing some of the results proved in [**3**], [**7**] and [**8**].

The main object of the theory of best approximation is the solution to the problem: Given a subset G of a metric space (X, d) and an element $x \in X$, find an element $g_0 \in G$ such that

$$d(x, g_0) \leq d(x, g) \qquad \text{for every} \quad g \in G. \tag{1}$$

The set of all such elements $g_0 \in G$ (if any), called the set of elements of best approximation of x by elements of G, is denoted by $P_G(x)$. Clearly,

$$P_G(x) = \Big[\bigcap_{g \in G} B(x, d(x, g)) \Big] \cap G,$$

where $B(x, d(x, g))$ denotes the closed ball in X with centre x and radius $d(x, g)$.

Another kind of approximation, called best coapproximation was introduced by Franchetti and Furi [**2**], who considered those elements $g_0 \in G$ satisfying

$$d(g_0, g) \leq d(x, g) \qquad \text{for every } g \in G. \tag{2}$$

The set of all such $g_0 \in G$ (if any) is denoted by $R_G(x)$. Clearly,

$$R_G(x) = \Big[\bigcap_{g \in G} B(g, d(x, g)) \Big] \cap G.$$

Many results in the theory of best approximation are available in metric spaces (see e.g. Singer [**9**]). In this paper we discuss best coapproximation in metric spaces. We start with recalling a few definitions.

AMS Subject Classification (1980): Primary 41 A 65; Secondary 41 A 50

*The author is thankful to U.G.C. for financial support.

Definition 1. A metric space (X, d) is said to be *convex* (or *M-convex* [**1**]) if for every x, y in X, such that $x \neq y$, there exists a z in X different from x and y such that $d(x, z) + d(z, y) = d(x, y)$. Such a point z is said to be *between* x and y.

Definition 2. A subset G of a metric space (X, d) is said to be *metrically convex* or *convex in the sense of Menger* (see [**9**]) or *M-convex* (see [**1**], also [**4**]) if for every x, y in G, $x \neq y$, there exists a point z in G between x and y.

Definition 3. Given a subset G of a metric space (X, d) and $x \in X$, an element $g_0 \in G$ satisfying (1) is called a *best approximation* to x in G, and satisfying (2) it is called a *best coapproximation* to x in G. The set G is said to be *proximinal* or an *existence set of best approximation* (respectively, *coproximinal* or an *existence set of best coapproximation*) if $P_G(x)$ (respectively, $R_G(x)$) is nonempty for every x in X. It is said to be *Chebyshev* (respectively, *co-Chebyshev*) if $P_G(x)$ (respectively, $R_G(x)$) is exactly a singleton for every x in X.

In the theory of best approximation, it is well known (see e.g. [**9**]) that a proximinal set in a metric space is a closed set. It was remarked in [**1**] that in general, proximinal sets, or Chebyshev sets, are not M-convex and it is not known whether in a H Hilbert space, every Chebyshev set is M-convex. For existence sets of best coapproximation, we can however, prove the following:

THEOREM 1. *In a metric space, an existence set of best coapproximation is closed and it is M-convex if the metric space is M-convex.*

Proof. Let G be an existence set of best coapproximation in a metric space (X, d). Then the set

$$R_G(x) = \{g_0 \in G : d(g_0, g) \leq d(x, g) \text{ for every } g \in G\}$$

is nonempty for every $x \in X$.

Let $p \in \overline{G} \setminus G$ and $g_0 \in R_G(p)$. Then there exists a sequence $\langle g_n \rangle$ in G such that $\langle g_n \rangle \to p$ and $d(g_0, g) \leq d(p, g)$ for every $g \in G$, and so $d(g_0, g_n) \leq d(p, g_n)$ for every n. In the limiting case, this implies that $\langle g_n \rangle \to g_0$, and so $p = g_0 \in G$. Consequently, G is closed.

Now, suppose that the metric space is M-convex and $g_1, g_2 \in G$, $g_1 \neq g_2$. Let $x \in X$ be between g_1 and g_2 i.e. $d(g_1, x) + d(x, g_2) = d(g_1, g_2)$. Let $g_0 \in R_G(x)$. We show that $g_0 \in G$ is between g_1 and g_2. Consider

$$d(g_1, g_2) \leq d(g_1, g_0) + d(g_0, g_2) \leq d(g_1, x) + d(x, g_2) = d(g_1, g_2),$$

so, $d(g_1, g_0) + d(g_0, g_2) = d(g_1, g_2)$. Hence G is M-convex.

Remark 1. In finite dimensional normed linear spaces, this result was proved in [**3**].

Definition 4. A mapping $u : X \to 2^Y$, where X and Y are metric spaces and 2^Y denotes the collection of all subsets of Y, is said to be:

(a) *Upper (K)-semicontinuous* if the relations $\lim_{n\to\infty} x_n = x$, $y_n \in u(x_n)$, $\lim y_n = y$ imply $y \in u(x)$,

(b) *upper semicontinuous* if the set $\{x \in X : u(x) \cap H \neq \varnothing\}$ is closed for each closed $N \subset Y$.

For a subset A of a metric space (X,d), we denote by $R_G(A)$ the set $\bigcup_{x\in A} R_G(x)$. If $D(R_G) = \{x \in X : R_G(x) \neq \varnothing\}$, we define the mapping $R_G : D(R_G) \to 2^G$ by $x \mapsto R_G(x)$. In general, $D(R_G) \neq X$ and the mapping R_G is multivalued on $D(R_G) \setminus G$, but we always have $G \subset D(R_G)$, $R_G(x) = \{x\}$ for every $x \in G$, and the restriction of the mapping R_G to G is single-valued. We have $D(R_G) = X$ if G is coproximinal and is single-valued on X if G is co-Chebyshev. This map R_G, which takes each point $x \in D(R_G)$ to those points of G that are best coapproximations to x, is called the *metric coprojection.*

For closed subsets G we have:

THEOREM 2. *If G is a closed subset of a metric space (X,d), then*:

(a) *for each $x \in X$, the set $R_G(x)$ is closed,*

(b) *the metric coprojection R_G is upper (K)-semicontinuous on $D(R_G)$, and*

(c) *for each compact subset A of $D(R_G)$, the subset $R_G(A)$ is closed in C.*

Proof. (a) Let g^* be a limit point of $R_G(x)$. Then there exists a sequence $\langle g_n\rangle$ in $R_G(x)$, such that $\langle g_n\rangle \to g^*$. Now:

$$\begin{aligned} g_n \in R_G(x) &\Longrightarrow d(g_n, g) \le d(x, y) \quad \text{for every } g \in G \\ &\Longrightarrow \lim_{n\to\infty} d(g_n, g) \le d(x, g) \quad \text{for every } g \in G \\ &\Longrightarrow d(g^*, g) \le d(x, g) \quad \text{for every } g \in G \\ &\Longrightarrow g^* \in R_G(x) \quad \text{as } g^* \in G \text{ by the closedness of } G. \end{aligned}$$

Hence, $R_G(x)$ is closed.

(b) Let $\lim_{n\to\infty} x_n = x$, $x_n \in R_G(x_n)$ and $\lim g_n = g^*$. To show $g^* \in R_G(x)$ we have:

$$\begin{aligned} g_n \in R_G(x_n) &\Longrightarrow d(g_n, g) \le d(x_n, g) \quad \text{for every } g \in G \\ &\Longrightarrow \lim_{n\to\infty} d(g_n, g) \le \lim_{n\to\infty} d(x_n, g) \quad \text{for every } g \in G \\ &\Longrightarrow d(g^*, g) \le d(x, g) \quad \text{for every } g \in G \\ &\Longrightarrow g^* \in R_G(x) \quad \text{as } g^* \in G \text{ by the closedness of } G. \end{aligned}$$

Hence R_G is upper-(K)-semicontinuous.

(c) Let g_0 be a limit point of $R_G(A)$. Then there exists a sequence $\langle g_n\rangle$ in $R_G(A)$ such that $\langle g_n\rangle \to g_0$. Since $g_n \in R_G(A)$, there exists an $x_n \in A$ such that $g_n \in R_G(x_n)$. The compactness of the set A implies the existence of a subsequence $\langle x_{n_i}\rangle \to x_0 \in A$. Since $g_{n_i} \in R_G(x_{n_i})$, $d(g_{n_i}, g) \le d(x_{n_i}, g)$ for every $g \in G$. In the limiting case this implies $d(g_0, g) \le d(x_0, g)$ for every $g \in G$, i.e. $g_0 \in R_G(x_0) \subset R_G(A)$. Hence $R_G(A)$ is closed.

Since for an existence sets of best coapproximation G, $D(R_G) = X$, by using Theorem 1, we have:

COROLLARY. *If G is an existence set of best coapproximation, then $R_G(x)$ is closed and $R_G : X \to 2^G$ is upper (K)-semicontinuous.*

This is analogous to the following result in the theory of best approximation (see [**9**]): *If G is an existence set of best approximation, then $P_G(x)$ is closed and $P_G : X \to 2^G$ is upper (K)-semicontinuous.*

Definition 5. A subset G of a metric space (X, d) is said to be *boundedly compact* if every bounded sequence in G has a subsequence converging to an element of X.

It is easy to observe that for each $x \in X$, $R_G(x)$ is a bounded subset of G. For boundedly compact closed subsets G of X, we have:

THEOREM 3. *If G is a boundedly compact closed subset G of a metric space (X, d), then*:

(a) *for each $x \in X$, $R_G(x)$ is compact,*

(b) *R_G is upper semicontinuous on $D(R_G)$, and*

(c) *for each compact subset A of $D(R_G)$, the subset $R_G(A)$ is compact in G.*

Proof. (a) Let $\langle g_n \rangle$ be any sequence in $R_G(x)$. Then $d(g_n, g) \leq d(x, g)$ for every $g \in G$. Since $R_G(x)$ is bounded, $\langle g_n \rangle$ is a bounded sequence in G, and so there is a subsequence $\langle g_{n_i} \rangle$ of $\langle g_n \rangle$ such that $\langle g_{n_i} \rangle \to g_0 \in G$, as G is also closed. Consider

$$d(g_0, g) = \lim_i d(g_{n_i}, g) \leq d(x, g)$$

for every $g \in G$. So $g_0 \in R_G(x)$, which implies that $R_G(x)$ is compact.

(b) Let N be a closed subset of G and $B = \{x \in D(R_G) : R_G(x) \cap N \neq \varnothing\}$. To show that B is closed, let x be a limit point of B. Then there exists a sequence $\langle x_n \rangle$ in B such that $x_n \to x$. Now, $x_n \in B$ implies that there exists a $g_n \in R_G(x_n) \cap N$, and so $d(g_n, g) \leq d(x_n, g)$ for every $g \in G$. Since G is boundedly compact and $\langle g_n \rangle$ is bounded, there exists a subsequence $\langle g_{n_i} \rangle$ of $\langle g_n \rangle$ such that $\langle g_{n_i} \rangle \to g_0 \in G$ (as G is closed), and so $d(g_{n_i}, g) \leq d(x_{n_i}, g)$ for every $g \in G$ implies $d(g_0, g) \leq d(x, g)$ for every $g \in G$. Therefore $g_0 \in R_G(x) \cap N$, i.e. $x \in B$, and so B is closed, which implies that R_G is upper semicontinuous.

(c) Let $\langle g_n \rangle$ be a sequence in $R_G(A)$. Then there exists an $x_n \in A$ such that $g_n \in R_G(x_n)$, and so $d(g_n, g) \leq d(x_n, g)$, $g \in G$. Since G is boundedly compact and $\langle g_n \rangle$ is bounded, there exists a subsequence $\langle g_{n_i} \rangle$ of $\langle g_n \rangle$ such that $g_{n_i} \to g_0 \in G$ (as G is closed). Since $x_{n_i} \in A$, the compactness of A implies the existence of a subsequence $\langle x_{n_{i_m}} \rangle \to x \in A$. Now, $g_{n_{i_m}} \in R_G(x_{n_{i_m}})$ implies $d(g_{n_{i_m}}, g) \leq d(x_{n_{i_m}}, g)$ for every $g \in G$, which in the limiting case implies $d(g_0, g) \leq d(x, g)$ for every $g \in G$, i.e. $g_0 \in R_G(x) \subset R_G(A)$. Hence, $R_G(A)$ is compact.

Since for single-valued maps upper semicontinuity and continuity are equivalent, we have:

COROLLARY. *If G is a boundedly compact co-Chebyshev subset of a metric space (X, d), then R_G is continuous on X.*

The following theorem gives another condition under which R_G is upper semicontinuous.

THEOREM 4. *If G is a closed subset of a metric space (X,d) such that, for every compact subset A of $D(R_G)$, the subset $R_G(A)$ is compact, then R_G is upper semicontinuous on $D(R_G)$.*

Proof. Let N be a closed subset of G and $B = \{x \in D(R_G) : R_G(x) \cap N \neq \varnothing\}$. To show that B is closed, let x_0 be a limit point of B. Then there exists a sequence $\langle x_n \rangle$ in B such that $x_n \to x_0$. Now, $x_n \in B$ implies that there exists a $g_n \in R_G(x_n) \cap N$, $n = 1, 2, \dots$. Let $A = \{x_0, x_1, x_2, \dots\}$. Then A is a compact subset of $D(R_G)$, and so $R_G(A)$ is compact in G. Since $g_n \in R_G(A)$, there is a subsequence $\langle g_{n_i} \rangle$ of $\langle g_n \rangle$ such that $g_{n_i} \to g_0 \in R_G(A) \cap N$. Now $g_{n_i} \in R_G(x_{n_i})$ implies $d(g_{n_i}, g) \leq d(x_{n_i}, g)$ for every $g \in G$. In the limiting case, this gives $d(g_0, g) \leq d(x_0, g)$ for every $g \in G$. Therefore, $g_0 \in R_G(x_0) \cap N$, i.e. $x_0 \in B$, and so B is closed. Hence R_G is upper semicontinuous.

Remark 2. For linear subspaces G of normed linear spaces, Theorems 3 and 4 were proved in [**7**].

Remark 3. From Theorem 3(c), we also get the upper semicontinuity of R_G for boundedly compact closed sets G from Theorem 4.

Definition 6. For a metric space (X,d), a set-valued mapping T on X is said to be *quasi-nonexpansive* if $\operatorname{dist}(Tx, p) \leq d(x,p)$ for every $x \in X$ and every fixed point p of T (i.e. for $p \in T(p)$).

The following theorem shows that existence sets of best coapproximation are fixed point sets of quasi-nonexpansive mappings. For finite dimensional normed linear spaces, this result was observed in [**3**].

THEOREM 5. *An existence set of best coapproximation G in a metric space (X,d) is a fixed point set of the quasi-nonexpansive mapping R_G.*

Proof. Let $x \in X$. Then there exists a $g_0 \in G$ such that $g_0 \in R_G(x)$, i.e. $d(g_0, g) \leq d(x, g)$ for every $g \in G$. This gives: $\operatorname{dist}(R_G(x), g) \leq d(x,g)$ for every $g \in G$. Since $R_G(g) = g$, R_G is quasi-nonexpansive.

Let $A = \{x \in X : x \in R_G(x)\}$. We claim that $G = A$, i.e. G is a fixed point set of the quasi-nonexpansive mapping R_G.

Let $g \in G$; then $g \in R_G(g)$, i.e. $g \in A$, and so $G \subset A$. Now suppose $x \in A$, i.e. $x \in R_G(x) \subset G$, i.e. $x \in G$, and so $A \subset G$. Hence $G = A$.

We say that g_0 belongs *strongly* to $R_G(x)$ if $x \notin G$ and there exists an $r > 0$ $(r \leq 1)$ such that $d(g_0, g) + r d(g_0, x) \leq d(x, g)$ for every $g \in G$.

The following theorem gives an upper bound for the radius of the set of strong coapproximation in metric spaces. For normed linear spaces, this result was proved by P. L. Papini in [**8**].

THEOREM 6. *If G is a subset of a metric space (X,d) and $x \in X$, then the radius of the set of elements that belong strongly to $R_G(x)$ for a given r is not larger than $(1-r)D$, where $D = \operatorname{dist}(x, G)$.*

Proof. Let $\varepsilon > 0$ be arbitrary. Choose $x_\varepsilon \in G$ such that $d(x, x_\varepsilon) < D + \varepsilon$. Let g_0 belong strongly to $R_G(x)$, i.e. there exists an r, $0 < r \le 1$, such that $d(g_0, g) + rd(g_0, x) \le d(x, g)$ for every $g \in G$. Taking $g = x_\varepsilon$, we obtain

$$d(g_0, x_\varepsilon) \le d(x, x_\varepsilon) - rd(g_0, x) < D + \varepsilon - rD = (1 - r)D + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we get $d(g_0, x_\varepsilon) \le (1 - r)D$.

Remark 5. If $R_{S,G}(x)$ is the set of all those elements of G that belong strongly to $R_G(x)$, the mapping $R_{S,G} : x \mapsto R_{S,G}(x)$ defined on $D(R_{S,G}) = \{x \in X : R_{S,G}(x) \neq \varnothing\}$ is called the *strong best coapproximation map*. For linear subspaces G of normed linear spaces, Theorem 2 and some other results were proved for $R_{S,G}$ in [**6**], but all the proofs appear to be incorrect as they don't take into account the fact that r in the definition of $R_{S,G}(x)$ may depend upon x.

Remark 6. As a counterpart to the notion of "sun" available in the literature (first introduced by N. V. Efimov and S. B. Stechkin in the theory of best approximation in normed linear spaces, see e.g. [**9**]), we may introduce as follows the notion of "cosun" in metric spaces.

An existence set of best coapproximation G in a metric space (X, d) is called a *cosun* if for every $x \in X$ there is at least one $g \in R_G(x)$ that is an element of best coapproximation of each point on the ray through x originating from g, i.e. on $\overrightarrow{gx}$ (we may recall that a point $z \in X$ is said to be on the ray $\overrightarrow{xy}$ if either z is between x and y or y is between x and z).

In the theory of best coapproximation, cosuns have been discussed in finite dimensional normed linear spaces by L. Hetzelt in [**3**]. It would be interesting to study cosuns in metric spaces.

REFERENCES

[**1**] D. Chatterjee, *M-convexity and best approximation*, Publ. Inst. Math. (Beograd) (N.S.) **28** (**42**) (1980), 43–50.

[**2**] C. Franchetti and M. Furi, *Some characteristic properties of real Hilbert spaces*, Rev. Roumaine Math. Pures Appl. **17** (1972), 1045–1048.

[**3**] L. Hetzelt, *On suns and cosuns in finite dimensional normed real vector spaces*, Acta Math. Hungar. **45** (1985), 53–68.

[**4**] T. D. Narang, *Some remarks on M-convexity and best approximation*, Publ. Inst. Math. (Beograd) (N.S.) **37** (**51**) (1985), 85–88.

[**5**] T. D. Narang, *On best coapproximation in normed linear spaces*, Rocky Mountain J. Math. (to appear).

[**6**] Rao, Geetha S. and S. Elumalai, *Semicontinuity properties of the strong best coapproximation operator*, Indian J. Pure Appl. Math. **16** (1985), 257–270.

[**7**] Rao, Getha S. and S. Muthukumar, *Some continuity properties of the coapproximation operator*, Mathematics Today **5** (1987), 37–48.

[**8**] P.L. Papini, *Approximation and strong approximation in normed spaces via tangent functionals*, J. Approx. Theory **22** (1978), 111–118.

[**9**] Ivan Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces*, Springer-Verlag, 1970.

Department of Mathematics,
Guru Nanak Dev University,
Amritsar – 143005, India.

(Received 17 05 1991)