PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 46 (60), 1989, 25–32

ON SUMS INVOLVING RECIPROCALS OF CERTAIN LARGE ADDITIVE FUNCTIONS (II)

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Abstract. Let $\beta(n)=\sum_{p|n}p$ and $B(n)=\sum_{p^\alpha\|n}\alpha p$. Let $p(n)$ denote the largest prime factor of an integer $n\geq 2$. In the present paper we sharpen the asymptotic formula for the sum $\sum_{2\leq n\leq x} B(n)/\beta(n)$ and we derive an asymptotic formula for the sum $\sum_{2\leq n\leq x}(B(n)-\beta(n))/p(n)$.

1. Introduction and statement of results

Let $\beta(n)=\sum_{p|n}p$ and $B(n)=\sum_{p^\alpha\|n}\alpha p$.

In [**2**] it was proved that

$$\sum_{2\leq n\leq x} B(n)/\beta(n)=x+O\big(x\exp(-c_1(\log x\log_2 x)^{1/2})\big),\quad c_1>0, \tag{1.1}$$

and

$$\sum_{2\leq n\leq x} \beta(n)/B(n)=x+O\big(x\exp(-c_2(\log x\log_2 x)^{1/2})\big),\quad c_2>0. \tag{1.2}$$

The above results were slightly sharpened in [**6**]. Let us define $p(n)$ as the largest prime factor of $n\geq 2$, and $p(1)=1$. In [**3**] it was proved that

$$\sum_{n\leq x}1/p(n)=x\delta(x)\Big(1+O((\log_2 x/\log x)^{1/2})\Big),$$

where

$$\delta(x)=\int_2^\infty \rho\left(\frac{\log x}{\log t}\right)t^{-2}\,dt.$$

Here $\rho(u)$ is the so-called "Dickman function", which is the solution of the differential-difference equation $u\rho'(u)+\rho(u-1)=0$, $(u>1)$, with the initial

AMS Subject Classification (1980): Primary 10 H 15, (10 H 25) condition $\rho(u) = 1$, $(0 \le u \le 1)$, $\rho(u)$ continuous at $u = 1$. An approximation to $\rho(u)$ in terms of elementary functions is

$$\rho(u) = \exp\{-u(\log u + \log_2 u - 1 + \log_2 u/\log u + O(1/\log u))\}, \tag{1.3}$$

where $\log_2 u = \log\log u$. The asymptotic formula (1.3) was established by Hua [**5**] and de Bruijn [**1**], independently.

In [**8**], we proved that

$$\sum_{2\le n\le x} 1/\beta(n) = \left(D + O(\log_3^2 x/\log_2 x)\right) \sum_{n\le x} 1/p(n), \tag{1.4}$$

where $1/2 < D < 1$ is an absolute constant.

One of the aimes of the present paper is to provide sharpenings of (1.1) and (1.2). The results are contained in the following theorem.

Theorem 1.

$$\sum_{2\le n\le x} \frac{B(n)}{\beta(n)} = x + \frac{1}{2} D \log x \left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right) \sum_{n\le x} \frac{1}{p(n)}, \tag{1.5}$$

and

$$\sum_{2\le n\le x} \frac{\beta(n)}{B(n)} = x - \frac{1}{2} D \log x \left(1 + O\left(\frac{\log_3^2 x}{\log_2 x}\right)\right) \sum_{n\le x} \frac{1}{p(n)}, \tag{1.6}$$

where D is the same as in (1.4).

Moreover, in [**9**] we proved that

$$\sum_{2\le n\le x} \frac{1}{p^r(n)} = x \exp\left\{-(2r \log x \log_2 x)^{1/2} \left(1 + \frac{\sqrt{r}}{2\sqrt{2}} \frac{\log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)\right\}, \tag{1.7}$$

and

$$\sum_{2\le n\le x} \frac{B(n) - \beta(n)}{p(n)} = x \exp\left\{-(2r \log x \log_2 x)^{1/2} \left(1 + \frac{\sqrt{r}}{2\sqrt{2}} \frac{\log_3 x}{\log_2 x} + O\left(\frac{1}{\log_2 x}\right)\right)\right\}, \tag{1.8}$$

where $r > 0$ is arbitrary but fixed.

Another aim of the present paper is to provide sharpenings of (1.8). The result is:

Theorem 2.

$$\sum_{2\le n\le x} \frac{B(n) - \beta(n)}{p(n)} = \frac{1}{2} \log x \left(1 + O\left(\frac{\log_3 x}{\log_2 x}\right)\right) \sum_{n\le x} \frac{1}{p(n)}.$$

By Theorem 1 and 2, we have

$$\sum_{2\le n\le x} \frac{B(n)-\beta(n)}{\beta(n)} \sim D \sum_{n\le x} \frac{B(n)-\beta(n)}{p(n)}.$$

It seems interesting to compare the sums involving reciprocals of $\beta(n)$ with the sums involving reciprocals of $p(n)$ [**8**]:

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} \sim D \sum_{n\le x} \frac{1}{p(n)}, \qquad \sum_{2\le n\le x} \frac{\omega(n)}{\beta(n)} \sim D \sum_{n\le x} \frac{\omega(n)}{p(n)},$$

and

$$\sum_{2\le n\le x} \frac{\Omega(n)-\omega(n)}{\beta(n)} \sim D \sum_{n\le x} \frac{\Omega(n)-\omega(n)}{p(n)},$$

where $\Omega(n)$ and $\omega(n)$ denote respectively the number of prime factors of n counted with and without multiplicities.

2. The necessary lemmas

Lemma 1 [**7**]. *Let*

$$L_1 = \exp\left[\left(\frac{1}{2}\log x \log_2 x\right)^{1/2}\left(1 - 2\frac{\log_3 x}{\log_2 x}\right)\right], \qquad \textit{and}$$

$$L_2 = \exp\left[\left(\frac{1}{2}\log x \log_2 x\right)^{1/2}\left(1 + 2\frac{\log_3 x}{\log_2 x}\right)\right].$$

Then we have

$$\sum_{n\le x} \frac{1}{p(n)} = \sum_{L_1<p\le L_2} \frac{1}{p}\Psi\left(\frac{x}{p}, p\right)\left(1 + O(\log^{-A} x)\right),$$

where $A > 0$ is arbitrary but fixed, and $\Psi(x,y)$ denotes the number of positive integers not exceeding x, all of whose prime factors do not exceed y.

Lemma 2 [**4**]. *For any fixed $\varepsilon > 0$ and $x \ge 3$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x$, we have uniformly*

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \qquad u = \frac{\log x}{\log y}.$$

Lemma 3 [**8**]. *For any fixed $\varepsilon > 0$ and $1 \le d \le y$, $\exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x^{1/2}$ we have uniformly*

$$\Psi(x/d,y) = \Psi(x,y)d^{-\beta}\left(1 + O(1/u) + \left(\frac{\log(u+1)}{\log y}\right)\right), \tag{2.1}$$

where

$$\beta = \beta(x, y) = 1 - \frac{\xi(\log x/\log y)}{\log y};$$

here $\xi(u)$ *denotes the positive solution of the equation*

$$e^{\xi} = u\xi + 1, \qquad (u > 1), \tag{2.2}$$

and satisfies

$$\xi(u) = \log u + O(\log_2(u+2)), \quad u \to \infty. \tag{2.3}$$

Lemma 4 [**8**]. *For any fixed* $\varepsilon > 0$ *and*

$$1 \le d \le y, \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x,$$

we have uniformly

$$\Psi(x/d, y) \ll \Psi(x, y) d^{-\beta}.$$

3. Proofs of the Theorems

We shall only give a detailed proof of Theorem 1, since Theorem 2 may be obtained in a similar and simpler way.

By the definition of $B(n)$ and $\beta(n)$ we have (p, q denote primes):

$$\begin{aligned} W(x) := \sum_{2 \le n \le x} \frac{B(n) - \beta(n)}{\beta(n)} &= \sum_{q^\alpha \le x} (\alpha - 1) q \sum_{2 \le n \le x,\, q^\alpha \| n} \frac{1}{\beta(n)} \\ &= \sum_{q^\alpha \le x} (\alpha - 1) q \sum_{q < p_1 \le x/q^\alpha} \sum_{m_1 \le x/q^\alpha p_1,\, p(m_1) \le p_1,\, (q, m_1) = 1} \frac{1}{q + \beta(m_1 p_1)} \\ &\quad + O\Big(\sum_{q^\alpha \le x} (\alpha - 1) \Psi(x q^{-\alpha}, q) \Big) = W_1 + O(W_2), \quad \text{say.} \end{aligned} \tag{3.1}$$

It is evident that

$$W_1 = \sum_{p_1 \le x} \sum_{\substack{q^\alpha \le x/p_1, \\ q < p_1}} (\alpha - 1) q \sum_{\substack{m_1 \le x/q^\alpha p_1 \\ p(m_1) \le p_1,\, (q, m_1) = 1}} \frac{1}{q + \beta(m_1 p_1)}.$$

We may write

$$W_1 = \sum_{p_1 \le z_1} + \sum_{z_1 < p_1 \le L_1} + \sum_{L_1 < p_1 \le L_2} + \sum_{L_2 < p_1 \le x} = W_3 + W_4 + W_5 + W_6, \tag{3.2}$$

where $z_1 = \exp\{(1/10)(\log x \log_2 x)^{1/2}\}$, and L_1 and L_2 are defined in Lemma 1. Let $R = (\log x \log_3 x / \log_2 x) \sum_{n \le x} 1/p(n)$; we have

$$W_3 \le \sum_{p_1 \le z_1} \frac{1}{p_1} \sum_{q^\alpha \le x/p_1,\, q < p_1} (\alpha - 1) q \Psi(x q^{-\alpha} p_1^{-1}, p_1)$$

$$\ll \log^2 x\Big(\sum_{p_1\le z_1}\frac{1}{p_1}\Big)\sum_{q\le z_1} q\Psi(xq^{-2},z_1) \tag{3.3}$$
$$\ll x\exp\{-4(\log x\log_2 x)^{1/2}\}\ll R,$$

since by Lemma 2 and (1.3) we have $\Psi(xq^{-2},z_1)\ll xq^{-2}\exp\{-4.5(\log x\log_2 x)^{1/2}\}$. Using Lemma 4 we have

$$\begin{aligned}W_4 &\le \sum_{z_1<p_1\le L_1}\frac{1}{p_1}\sum_{q^\alpha\le x/p_1,\,q<p_1}(\alpha-1)q\Psi(xq^{-\alpha}p_1^{-1},p_1)\\ &\ll \sum_{z_1<p_1\le L_1}\frac{1}{p_1}\Psi(x/p_1,p_1)\sum_{q^\alpha\le x/p_1,\,q<p_1}(\alpha-1)q^{1-\alpha(1-\delta')}\\ &\ll \sum_{z_1<p_1\le L_1}\frac{1}{p_1}\Psi(x/p_1,p_1)\sum_{q<p_1}q^{-1+2\delta'},\end{aligned}$$

where $\delta'=(\log p_1)^{-1}\xi\big(\log(x/p_1)/\log p_1\big)$. By (2.2) and (2.3) we have

$$q^{2\delta'}\le\exp\Big(2\xi\Big(\frac{\log(x/p_1)}{\log p_1}\Big)\Big)\ll\log x\log_2 x,$$

for $z_1<p_1\le L_1$. Therefore using Lemma 1 we obtain:

$$W_4\ll\log^2 x\sum_{z_1<p_1\le L_1}\frac{1}{p_1}\Psi(x/p_1,p_1)\ll R. \tag{3.4}$$

Similarly we have $W_6\ll R$.

Now we come to the estimation of W_5 in (3.2). We consider separately the cases $p(m_1)<p_1$ and $p(m_1)=p_1$ and obtain

$$\begin{aligned}W_5 &= \sum_{L_1<p_1\le L_2}\ \sum_{q^\alpha\le x/p_1,\,q<p_1}(\alpha-1)q\\ &\quad\times\sum_{m_1\le x/q^\alpha p_1,\,p(m_1)<p_1,\,(q,m_1)=1}\frac{1}{q+p_1+\beta(m_1)}\\ &\quad+O\Big(\sum_{L_1<p_1\le L_2}\ \sum_{q^\alpha\le x/p_1,\,q<p_1}\frac{(\alpha-1)q}{p_1}\Psi(xq^{-\alpha}p_1^{-2},p_1)\Big).\end{aligned}\tag{3.5}$$

Denoting by W_5' the main term on the right-hand side of (3.5) we may write

$$W_5'=\sum_{L_1<p(m_1)<p_1,\,p(m_1)||m_1}+\sum_{L_1<p(m_1)<p_1,\,p^2(m_1)||m_1}+\sum_{p(m_1)\le L_1}.$$

Then we have

$$W_5'=\sum_{L_1<p_1\le L_2}\ \sum_{L_1<p_2<p_1}\ \sum_{\substack{q^\alpha\le x/p_1p_2\\ q<p_1,\,q\ne p_2}}(\alpha-1)q\sum_{\substack{m_2\le x/q^\alpha p_1p_2\\ p(m_2)<p_2,\,(q,m_2)=1}}\frac{1}{q+p_1+p_2+\beta(m_2)}$$

$$+O\Big(\sum_{L_1<p_1\le L_2}\ \sum_{L_1<p_2<p_1}\ \sum_{q^\alpha\le x/p_1p_2,\, q<p_1}(\alpha-1)qp_1^{-1}\Psi(x/q^\alpha p_1p_2^2,p_2)\Big)$$
$$+O\Big(\sum_{L_1<p_1\le L_2}\ \sum_{p_2\le L_1}\ \sum_{q^\alpha\le x/p_1p_2,\, q<p_1}(\alpha-1)qp_1^{-1}\Psi(x/q^\alpha p_1p_2,p_2)\Big).$$

Proceeding as before, we obtain

$$W_5=W_5''+O\Big(\sum_{j=1}^{s}W_{7j}\Big)+O\Big(\sum_{j=2}^{s}W_{8j}\Big)+O\Big(\sum_{j=1}^{s-1}W_{9j}\Big), \tag{3.6}$$

where

$$W_5''=\sum_{p_1,\dots,p_s}\ \sum_{\substack{q^\alpha\le x/p_1\dots p_s\\ q<p_1}}(\alpha-1)q\sum_{\substack{m_s\le x/q^\alpha p_1\dots p_s\\ p(m_s)<p_s,\,(q,m_s)=1}}\frac{1}{q+p_1+\cdots+p_s+\beta(m_s)}, \tag{3.7}$$

where the ranges of summation in the above sums $p_1,\dots,p_s$ are $L_1<p_1\le L_2$, $L_1<p_2<p_1$, …, $L_1<p_s<p_{s-1}$, and $s\le\log_3 x$ is a large number which will be chosen later and

$$W_{7j}=\sum_{p_1,\dots,p_j}\ \sum_{q^\alpha\le x/p_1\dots p_j,\, q<p_1}\frac{(\alpha-1)q}{p_1}\Psi(x/q^\alpha p_1\dots p_{j-1}p_j^2,p_j), \tag{3.8}$$

$$W_{8j}=\sum_{p_1,\dots,p_{j-1}}\ \sum_{p_j\le L_1}\ \sum_{q^\alpha\le x/p_1\dots p_j,\, q<p_1}\frac{(\alpha-1)q}{p_1}\Psi(x/q^\alpha p_1\dots p_j,p_j), \tag{3.9}$$

$$W_{9j}=\sum_{p_1,\dots,p_j}\ \sum_{q^\alpha\le x/p_1\dots p_j,\, q<p_1}\frac{(\alpha-1)q}{p_1}\Psi(x/q^{\alpha+1}p_1\dots p_j,q). \tag{3.10}$$

Since

$$\frac{1}{q+p_1+\cdots+p_s+\beta(m_s)}=\frac{1}{p_1+\cdots+p_s}+O(qp_1^{-2})+O(p_1^{-2}\beta(m_s)),$$

and

$$\sum_{\substack{m_s\le x/q^\alpha p_1\dots p_s\\ p(m_s)<p_s,\,(q,m_s)=1}}1=\Psi\Big(\frac{x}{q^\alpha p_1\dots p_s},p_s\Big)-\Psi\Big(\frac{x}{q^{\alpha+1}p_1\dots p_s},p_s\Big)+O(W_{7s}),$$

we have further

$$W_5''=\sum_{p_1,\dots,p_s}\ \sum_{\substack{q^\alpha\le x/p_1\dots p_s\\ q<p_1,\,\alpha\ge2}}\frac{q}{p_1+\cdots+p_s}\Psi(x/q^\alpha p_1\dots p_s,p_s)$$

$$+O\Big(\sum_{p_1,\dots,p_s}\sum_{q^\alpha\le x/p_1\dots p_s,\,q<p_1}\frac{(\alpha-1)q^2}{p_1^2}\Psi(x/q^\alpha p_1\dots p_s,p_s)\Big)$$
$$+O\Big(\sum_{p_1,\dots,p_s}\sum_{q^\alpha\le x/p_1\dots p_s,\,q<p_1}\frac{(\alpha-1)q}{p_1^2}\sum_{\substack{m_s\le x/q^\alpha p_1\dots p_s\\ p(m_s)<p_s,\,(q,m_s)=1}}\beta(m_s)\Big)$$
$$+O(W_{7s})=W_{10}+O(W_{11})+O(W_{12})+O(W_{7s}),\quad \text{say.}$$

We estimate first W_{10}. We consider separately the cases $\alpha=2$ and $\alpha\ge 3$ and by using Lemmas 3 and 4 we obtain

$$W_{10}=\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}\sum_{q<p_1}q^{-1+2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big)$$
$$+O\Bigg(\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}\sum_{\substack{q^\alpha\le x/p_1\dots p_s\\ q<p_1,\,\alpha\ge 3}}q^{1-\alpha(1-\delta_1)}\Bigg),$$

where

$$\delta_i=\frac{1}{\log p_s}\xi\Big(\frac{\log(x/p_i\dots p_s)}{\log p_s}\Big),\qquad i=1,2,\dots,s.$$

Using partial summation and the prime number theorem we have

$$\sum_{q<p_1}q^{-1+2\delta_1}=\int_{e^{1/\delta_1}}^{p_1}z^{-1+2\delta_1}\log^{-1}z\,dz\big(1+O(\log_3 x/\log_2 x)\big)$$
$$=(\log_2 x)^{-1}p_1^{2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big).$$

Similarly

$$\sum_{q^\alpha\le x/p_1\dots p_s,\,q<p_1,\,\alpha\ge 3}q^{1-\alpha(1-\delta_1)}\ll 1.$$

Therefore we obtain

$$W_{10}=\frac{1}{\log_2 x}\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}p_1^{2\delta_1}\big(1+O(\log_3 x/\log_2 x)\big).$$

By (4.13) of [**8**] we have $p^{\delta_i}=p^{\delta}(1+O(\log_3 x/\log_2 x))$ for $L_1<p\le L_2$, where $\delta=\delta_s$. Moreover by (4.6), (4.16), (4.18) and (4.31) of [**8**], we have

$$\sum_{p_1,\dots,p_s}\frac{\Psi(x/p_1\dots p_s,p_s)}{p_1+\dots+p_s}=\Big(D+O\Big(\frac{\log_3^2 x}{\log_2 x}\Big)\Big)\sum_{L_1<p\le L_2}\frac{\Psi(x/p,p)}{p}.$$

Similarly, in a way analogous to the above we have for W_{10}

(3.12)
$$W_{10}=D(\log_2 x)^{-1}\big(1+O(\log_3^2 x/\log_2 x)\big)\sum_{L_1<p\le L_2}p^{-1+2\delta}\Psi(xp^{-1},p)$$
$$=\frac12 D\log x\big(1+O(\log_3^2 x/\log_2 x)\big)\sum_{n\le x}\frac{1}{p(n)}.$$

Now we come to the estimation of W_{12} in (3.11). By the definition of $\beta(m)$ and Lemma 4 we have

$$W_{12} \le \sum_{p_1,\dots,p_s} \sum_{q^\alpha \le x/p_1\dots p_s,\, q<p_1} \frac{(\alpha-1)q}{p_1^2} \sum_{p<p_s} p\Psi(x/q^\alpha p_1\dots p_s p, p_s)$$
$$\ll \sum_{p_1,\dots,p_s} \sum_{p<p_s} \frac{p}{p_1^2}\Psi(x/p_1\dots p_s p, p_s) p_1^{2\delta_1}(\log_2 x)^{-1}.$$

By (4.19) of [**8**]

$$\sum_{p_1,\dots,p_s} \sum_{p<p_s} \frac{p}{p_1^2}\Psi(x/p_1\dots p_s p, p_s) \ll \frac{1}{\log_2 x} \sum_{L_1<p\le L_2} \frac{\Psi(x/p,p)}{p} \ll R.$$

Similarly

$$W_{12} \ll R. \tag{3.13}$$

Similarly we have also

$$W_{11}, W_{7j}, W_{8j}, W_{9j} \ll R. \tag{3.14}$$

By putting (3.12)–(3.14) into (3.11) and (3.11), (3.14) into (3.6) and finally (3.3), (3.4) and (3.6) into (3.2) we get

$$W_1 = \frac{1}{2} D \log x\big(1 + O(\log_3^2 x/\log_2 x)\big) \sum_{n\le x} \frac{1}{p(n)}.$$

Moreover, it is easy to prove that

$$W_2 \ll R,$$

which completes the proof of Theorem 1.

REFERENCES

[**1**] N. G. de Bruijn, *The asymptotic behaviour of a function occurring in the theory of primes*, J. Indian Math. Soc. (N.S.) **15** (1951), 25–32.

[**2**] J.-M. De Koninck, P. Erdős and A. Ivić, *Reciprocals of large additive functions*, Canad. Math. Bull. **24** (1981), 225–231.

[**3**] P. Erdős, A. Ivić and C. Pomerance, *On sums involving reciprocals of the largest prime factor of an integer*, Glasnik Matematički **21** (**41**) (1986), 283–300.

[**4**] A. Hildebrand, *On the number of positive integers $\le x$ and free of prime factors $> y$*, J. Number Theory **22** (1986), 289–307.

[**5**] L. G. Hua, *Estimation of an integral*, Scientia Sinica **2** (1951), 393–402, (in Chinese).

[**6**] A. Ivić and C. Pomerance, *Estimates of certain sums involving the largest prime factor of an integer*, in: Coll. Math. Soc. J. Bolyai **34**, *Topics in Classical Number Theory*, North-Holland, Amsterdam, 1984, 769–789.

[**7**] A. Ivić, *On some estimates involving the number of prime divisors of an integer*, Acta Arith. **49** (1987), 21–33.

[**8**] T. Z. Xuan, *On sums involving reciprocals of certain large additive functions*, Publ. Inst. Math. (Belgrade) **41** (**55**) (1989), 41–55.

[**9**] T. Z. Xuan, *Sums of certain large additive functions*, J. Beijing Normal Univ. (N.S.) 1984, No **2**, 11–18, (in Chinese).

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(Received 18 01 1989)