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ON WEAK CONVERGENCE TO THE FIXED POINT OF A GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAP

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Abstract. Opial's type of convergence theorem [**3**] is extended to the case of a generalized asymptotically nonexpansive map in uniformly convex Banach space having a weak duality mapping. Bose's result would follow as a corollary to Theorem 3.1 of present work.

1. Introduction. Bose [**1**] gave a result on asymptotically nonexpansive and asymptotically regular map which in fact extended Opail's convergence theorem [**3**]. We give another generalization of Opail's result by introducing a new type of generalized asymptotically nonexpansive mapping. Suppose K is a nonempty closed bounded subset of a Banach space X. A mapping $T : K \to K$ is called asymptotically nonexpansive (see[1]) if for each x, $y \in K$,

$$(*) \qquad \|T^i x - T^j y\| \le k_i \|x - y\|, \quad i = 1, 2, 3, \dots$$

where $\{k_i\}$ is a fixed sequence of positive reals such that $k_i \to 1$ as $i \to \infty$. Existence of fixed points of such a mapping, when X is uniformly convex has been proved by Goebel and Kirk [**2**]. In Section 2 we recall some basic definitions and introduce generalized asymptotically nonexpansive and generalized asymptotically regular mapping. Also we recall the definition given by Kirk on asymptotically central set of a sequence. Some results on such a sequence are stated without proof. Our main results are given in Section 3.

2. *Definition*. A mapping $T : K \to K$ is called *generalized asymptotically nonexpansive if*,

$$(2.1) \qquad \|x_i - y_i\| \le k_i \|x - y\|$$

for x, $y \in K$, where x_i is defined by Mann-type iterations, and

$$x_i = \lambda x_{i-1} + (1 - \lambda) T x_{i-1}, \quad i = 1, 2, 3, \dots o \le \lambda < 1$$

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where $x_0 = x,\ \{k_i\}$ is a sequence of real numbers such that $k_i \to 1$ as $i \to \infty$. T is *generalized asymptotically regular* if for any $x = x_0$ in k,

$$x_i - x_{i+1} \to 0 \ \text{ as } \ t \to \infty.$$

The mapping T is said to be demiclosed if for any sequence $x_n \in K,\ x_n \to x_0$ (weakly), $Tx_n \to y_0 \ \Rightarrow \ Tx_0 = x_0$. The modulus of convexity of X is a function $\delta : [0,2] \to [0,1]$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|x+y\|/2 : \|x\| \leq 1,\ \|x-y\| \geq \varepsilon\}.$$

It is known that δ is a nondecreasing function and continuous on $[0.\ 2]$. It is also known that

$$\|x\| \leq \delta,\ \|y\| \leq \delta,$$

$$(**) \qquad x - y\| \geq \varepsilon \ \Rightarrow \|x+y\|2 \leq (1 - \delta(\varepsilon/d))d$$

Opial [**3**] has shown that in an uniformly convex Banach space having weakly continuous duality mapping (or in a Hilbert space) if a sequence $\{x_n\}$ converges weakly to x_0 then

$$(2.2) \qquad \liminf_n \|x_n - x_0\| < \lim_n \|x_n - x\|,\ \forall\, x \neq x_0.$$

Remark. Observe that the definitions (*) and (2.1) are independent of each other.

3. Let K be a nonempty bounded closed convex subset of a reflexive Banach space X and let $\{x_n\}$ be any sequence in K. Following Kirk and Edelstein (see [**1**] we define

$$r(x) = \limsup_n \|x_n - x\|,\ \ x \in X.$$

This r is acontinuous function of X into reals [**1**].

Let $\rho = \rho(\{x_n\}) = \inf\{r(x) : x \in K\}$ and $C_0 = \{x \in K : r(x) = \rho\}$. ρ is called the asymptotic radius of $\{x_n\}$ in K and C_0 is the asymptotically central set of $\{x_n\}$ in K. C_0 is a singleton if X is uniformly convex. In that case it is called the asymptotic center.

Let $B_n(r)$ denote the closed ball of radius r centered at x_n and define

$$C_\varepsilon = \cup_{j\geq 1}(\cap_{n\geq j} B_n(\rho + \varepsilon))$$

PROPOSITION 3.1. *$C_0 = \cap_{\varepsilon>0}(K \cap \overline{C_\varepsilon})$ and is a nonempty closed convex subset of K.*

PROPOSITION 3.2. *If the space is uniformly convex then C_0 is a singleton.*

As a consequence of Proposition 3.2 we derive the following lemma.

Lemma 3.1. *Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space having weakly continuous duality mapping. If a sequence $\{x_n\} \subset K$ converges weakly to a point x_0 then x_0 is the asymptotic centre of $\{x_n\}$ in K.*

Lemma 3.2. *Let K and X be as in Lemma 3.1 and let $T : K \to K$ be a generalized asymptotically nonexpansive mapping. Suppose x_0 is the asymptotic centre of the sequence $\{x_n\}$ for some x in K. If the weak limit ε_0 of the subsequence $\{x_{n_i}\}$ is a fixed point of T, then it must coincide with x_0.*

Proof. Let ρ and ρ' be the asymptotic radii respectively of $\{x_n\}$ and $\{x_{n_i}\}$. Clearly $\rho' \leq \rho$. Since $\{x_{n_i}\}$ converges weakly to ξ_o, by lemma 1, ξ_0 must ne the asymptotic centre of $\{x_{n_i}\}$ in K, so given $\xi > 0$, we can choose an integer i_0 such that $\|\xi_0 - \{x_{n_i0}\}\| \leq \rho' + \varepsilon/2$. Since ξ_0 is a fixed point of T, we get $\xi_{0_j} = \xi_0$, and since T is generalized asymptotically nonexpansive, we can choose an integer J such that,

$$\begin{aligned}\|\xi_0 - x_{n_{i0+j}}\| = \|\xi_{0_j} - x_{n_{i0+j}}\| &\leq k_j \|\xi_0 - x_{n_{i0}}\| \\ &\leq k_j(\rho' + \varepsilon/2) \leq \rho' + \varepsilon \leq \rho + \varepsilon, \text{ for all } j \geq J.\end{aligned}$$

It follows therefore that $\lim_n \sup \|\xi_0 - x_n\| = \rho$ and, x_0 being the unique point with this property, we have $x_0 = \xi_0$.

Our main convergence theorem goes as follows.

Theorem 3.1. *Let X be a uniformly convex Banach space having weakly continuous duality mapping and K a nonempty closed bounded convex subset of X. Suppose T is a continuous generalized asymptotically nonexpansive mapping, and generalized asymptotically regular. Then for any $x \in K$, the sequence $\{x_n\}$ converges wakly to a point of T.*

Proof. We will show that the generalized asymptotic regularity of T makes every weak cluster point of $\{x_n\}$ a fixed point of T. In view of Lemma 3.1 this would mean that all the weak cluster points of $\{x_n\}$ coincide with the asymptotic centre x_0 of $\{x_n\}$ in K (which is fixed point) and would complete the proof.

Let us suppose that the subsequence $\{x_{n_i}\}$ converges weakly to ξ_0. Then, by Lemma 3.1, ξ_0 is the asymptotic centre of $\{x_{n_i}\}$ in K. Let the asymptotic radius be ρ. By generalized asymptotic regularity of T.

$$x_{n_{i+1}} - x_{n_i} \to 0 \text{ as } i \to \infty.$$

Since $\{x_{n_i}\}$ converges weakly to ξ_0, this implies $\{x_{n_{i+1}}\}$ converges weakly to ξ_0. It follows in the same way that for any integer $r, \{x_{n_{i+r}}\}$ converges weakly tp ξ_0. Thus all these sequence have the same asymptotic centre ξ_0 in K. We now claim that all these sequences have the same asymptotic radius ρ.

We have

$$\begin{aligned}\|\xi_0 - x_{n_{i+1}}\| - \|\xi_0 - x_{n_i}\| \leq& \|(\xi_0 - x_{n_{i+1}}) - (\xi_0 - x_{n_i}\| \\ \leq& \|x_{n_{i+1}} - x_{n_i}\| \to 0 \text{ as } i \to \infty\end{aligned}$$

by generalized asymptotic regularity of T. Thus

$$\limsup_i \|\xi_0 - x_{n_{i+1}}\| = \limsup_i \|\xi_0 - x_{n_i}\| = \rho$$

and our claim follows.

We now prove that ξ_0 is a fixed point of T. For this it sufficies to show that $\xi_{0_j} \to \xi_0$ as $j \to \infty$. Indeed

$$\begin{aligned}(1-\lambda)\|T\xi_{0_j} - \xi_0\| =& \|(1-\lambda)T\xi_{0_j} - (1-\lambda)\xi_0\| \\ =& \|\xi_{0_{j+1}} - \lambda\xi_{0_j} - (1-\lambda)\xi_0\| \to 0\end{aligned}$$

as $j \to \infty$, since $\xi_{0_j} \to \xi_0$ as $j \to \infty$. Thus $T\xi_{0_j} \to \xi_0$ as $j \to \infty$ and since T is continuous, it follows that ξ_0 is a fixed point of T.

Let us suppose now that ξ_{0_j} does not converge to ξ_0. Then there is a $d > 0$ and a sequence $\{j_m\}$ of integers such that

$$\|\xi_0 - \xi_{0_j}\| \geq 0 \text{ for all } m.$$

By uniform convexity of the space, we may choose an $\varepsilon > 0$ such that

$$(\rho+\varepsilon)[1-\delta(d/(\rho+\varepsilon))] < \rho.$$

Since all the sequences $\{x_{n_{i+r}}\}_{i=1}^{\infty} r = 0, 1, 2, 3, \dots$, have the same asymptotic centre ξ_0 and same asymptotic radius ρ, there exist integers $I = I(r)$ such that

(1) $$\|\xi_0 - x_{n_{i+r}}\| \leq \rho + \varepsilon \text{ for all } i \geq I(r).$$

We have for any m

(2) $$\|\xi_{0_{j_m}} - x_{n_i+j_m}\| \leq k_{j_m}\|\xi_0 - x_{n_i}\| \leq k_{j_m}(\rho+\varepsilon/2) \text{ for } i \geq I(0),$$

We choose an integer M such that (as $k_j \to 1$ as $j \to \infty$)$k_{j_m}(\rho+\varepsilon/2) \leq \rho+\varepsilon$ for all $m \geq M$, so that we have

(3) $$\|\xi_{0_{j_m}} - x_{n_i+j_m}\| \leq \rho+\varepsilon \text{ for all } i \geq I(0) \text{ and all } m \geq M$$

and from (1) we have

(4) $$\|\xi_0 - x_{n_i+j_m}\| \leq \rho+\varepsilon \text{ for all } i \geq I(j_m),$$

since $\|\xi_0 - \xi_{0_{j_M}}\| \geq d$, (3) and (4) yield

$$\|(\xi_0 - \xi_{0_{j_M}})/2 - x_{n_{i+j_M}}\| \leq (\rho+\varepsilon)[1-\delta(d/\rho+\varepsilon)] < \rho$$

for all $i \geq \max\{I(0),\ I(j_m)\}$. This contradicts the fact that the sequence

$$\{x_{n_i+j_M}\}_{i=1}^{\infty}$$

has asymptotic radius ρ in K and so completes the proof.

Remark 1. The existence proof for a fixed point of a continuous generalized asymptotically nonexpansive mapping can be given in the same fashion as in the case of an asymptotically nonexpansive mapping (see Joshi and Bose [**2**, Theorem 4.2.20, p. 111]).

Remark 2. Theorem 3.1 implies the corresponding result of Bose [**1**] by taking $\lambda = 0$ in the definition of generalized asymptotic nonexpansiveness given at the beginning of Section 2.

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