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FREDHOLM THEORY AND SEMILINEAR EQUATIONS WITHOUT RESONANCE INVOLVING NONCOMPACT PERTURBATIONS, I.

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Dedicated to Academician Đuro Kupera, on the occasion of his eightieth birthday, in gratitude.

1. Introduction. Nonlinear Fredholm theory began with the works of Lasota [9] and Lasota-Opial [10] for (multivalued) compact maps and has attracted the attention of many authors. Since then, extensions of the first Fredholm theorem and of the Fredholm alternative in a weaker form (i. e. without the dimension assertion) have been obtained for various classes of nonlinear maps, like compact, (set) condensing, of types (S) and (S_+), monotone and A-proper ones (cf. [3, 4, 5, 6, 18, 19, 23]). In contrast to the works of other authors, in [11-15] we began developing a Fredholm theory for (pseudo) A-proper type of maps that are asymptotically close to a suitable map (cf. (2.2)) and, in particular, have a *positive* quasinorm (cf. (2.2)).

The purpose of this paper is twofold. First, in Section 2, we prove a rather general extension of the first Fredholm theorem for equations of the form

$$Tx = f \qquad (x \in X,\ f \in Y) \tag{1.1}$$

where X and Y are normed, linear spaces and $T : X \to Y$ is either (pseudo) A-proper or a uniform limit of A-proper maps. When $T = A + N$ is pseudo A-proper with $A : D(A) \subset X \to Y$ linear and N nonlinear with quasinorm $N \geq 0$, we also prove a weaker form of the Fredholm alternative for semilinear equations

$$Ax + Nx = f \qquad (x \in D(A),\ f \in Y). \tag{1.2}$$

In case when $A + N$ is a continuous A-proper map, we prove a complete Fredholm alternative (Theorem 2.3). Second, in Section 3, using these results, we study the solvability of Eq. (1.2) with $\dim\ker(A) \leq \infty$ when there is no resonance at infinity.

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Moreover, the case of nonlinear A is also studied. Due to the generality of the A-proper like maps, the obtained results are applicable to many different classes of nonlinear maps mentioned above. We also note that, using a degree theory for multivalued maps, the results of this paper are also valid for multivalued maps T and N. Applications of the theory to integral and partial differential equations are given in Part II (this issue).

2. Fredholm theory. Let $\{E_n\}$ and $\{F_n\}$ be sequences of finite dimensional spaces and $\{V_n\}$ and $\{W_n\}$ be sequences of continuous linear maps with V_n mapping E_n into X injectively and W_n mapping Y onto E_n. Suppose that $\operatorname{dist}(x, V_nE_n) \to 0$ as $n \to \infty$ for each $x \in X$, $\dim X_n = \dim Y_n$ for each n and $\delta = \max ||Q_n|| < \infty$. Then $\Gamma = \{E_n, V_n; F_n, W_n\}$ is said to be an admissible scheme for (X, Y). In particular, let $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of X and Y respectively, and $P_n : X \to X_n$ and $Q_n : Y \to Y_n$ be linear projections onto X_n and Y_n with $P_nx \to x$ and $Q_ny \to y$ for each $x \in X$ and $y \in Y$. If $V_n = P_n|X_n = I_n$, then $\Gamma_0 = \{X_n, P_n; Y_n, Q_n\}$ is a projectionally complete scheme for (X, Y).

Let $D \subset X$, $T : D \to Y$ and $T_n \equiv W_nTY_n : D_n = V_n^{-1}(D) \to F_n$. Recall [21].

Definition 2.1. A map $T : D \to Y$ is A-proper (pseudo A-proper) w.r.t. Γ if T_n is continuous for each n and, whenever $\{V_{n_k}u_{n_k} | u_{n_k} \in D_{n_k}\}$ is bounded and $||T_{n_k}u_{n_k} - W_{n_k}f|| \to 0$ as $k \to \infty$ for some $f \in Y$, then some subsequence $V_{n_{k(i)}}u_{n_{k(i)}} \to x$ (there is an x, respectively) with $Tx = f$.

We say that the equation $Tx = f$ is feebly approximation (f. a.) solvable w.r.t. Γ if $T_nu_n = W_nf$ for some $u_n \in D_n$, $n \geq 1$, and some subsequence $V_{n_k}u_{n_k} \to x$ with $Tx = f$. The theory of (pseudo) A-proper maps is well developed and we refer to, e.g., [14–16, 21–23], where one can find also many examples of such maps.

Our first result is the following generalized first Fredholm theorem.

THEOREM 2.1. *Let A, $T : X \to Y$ be nonlinear maps such that*

(2.1) *There are an $n_0 \geq 1$ and a function $c : R^+ \to R^+$ such that $c(r) \to \infty$ as $r \to \infty$ and $||W_nAx_n|| \geq c(||x||)$ for $x \in V_n(E_n)$ and $n \geq n_0$.*

(2.2) *T is asymptotically close to A, i.e.*

$$|T - A| = \limsup_{||x|| \to \infty} \frac{||Tx - Ax||}{c(||x||)} < 1/\delta.$$

(2.3) *There is an $R > 0$ such that either A is odd on $X \backslash B(0, R)$ or, for each $r \geq R$, the Brouwer degree $deg(T_n + \mu G_n, B_n(0, r), 0) \neq 0$ for all large n, some bounded map $G : X \to Y$ and all $\mu \in (0, \mu_0)$ with μ_0 small. Then*

(a) *If T is A-proper w.r.t. Γ and $\mu = 0$ in (2.3), Eq. (1.1) is f.a. solvable for each $f \in Y$.*

(b) *If $T + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ and T satisfies condition (*) (i.e. whenever $Tx_n \to f$ with $\{x_n\}$ bounded, then $Tx = f$ for some x), then T is surjective, i.e. $T(X) = Y$.*

(c) *If T is pseudo A-proper w.r.t. Γ and $\mu = 0$ in* (2.3)*, then $T(X) = Y$.*

Proof. We shall first consider the case when A is odd on $X \backslash B(0, R)$ in (2.3). Then parts (a) and (c) have been proved in [11, 12] and [15], respectively. The validity of part (b) has been announced in [12, 15] (cf. also [14]) without proof and we shall prove it now using a finite dimensional antipodes theorem of Borsuk.

Let $f \in Y$ be fixed. Then, since the map $Bx = Tx - f$ has the same properties as T, it suffices to show that $Tx = 0$ is solvable. Let $\varepsilon > 0$ be such that $|T - A| + 2\varepsilon < 1/\delta$ and $r \geq R$ such that $c(r) \geq 1$ and $||Tx - Ax|| \leq (|T - A| + \varepsilon)\, c(||x||)$ for each $||x|| \geq r$. Since G is bounded, there is $\mu_1 \in (0, \mu_0)$ such that $\mu_1 ||Gx|| < \varepsilon$ for all $||x|| = r$. Then, for each $\mu \in (0, \mu_1)$ and $||x|| = r$, we have

$$||Tx + \alpha Gx - Ax|| \leq (|T - A| + 2\varepsilon)c(r) < c(r)/\delta.$$

Let $\mu \in (0, \mu_1)$ be fixed. Then, for each $n \geq 1$,

$$T_n(u) + \mu G_n(u) \neq \lambda(T_n(-u) + \mu G_n(-u)) \quad \text{for} \quad u \in \partial B_n(0, r),\ \lambda \in [0, 1]. \tag{2.4}$$

If not, then there would exist an $u_n \in \partial B_n(0, r)$ and $\lambda \in [0, 1]$ such that $(T_n + \mu G_n)(u_n) = \lambda(T_n + \mu G_n)(-u_n)$ for some n. Hence,

$$\frac{1}{1+\lambda}(A_n - T_n - \mu G_n)(u_n) + \frac{\lambda}{1+\lambda}(T_n + \mu G_n - A_n)(-u_n) = A_n u_n$$

and therefore

$$c(||V_n u_n||) \leq ||A_n u_n|| \leq \frac{\delta}{1+\lambda}||(T + \mu G - A)V_n u_n|| + \frac{\delta\lambda}{1+\lambda}||(T + \mu G - A)(-V_n u_n)|| < \\ < c(||V_n, u_n||),$$

a contradiction. Hence, (2.4) holds and consequently, for each $n \geq 1$ there is an $u_n \in \partial B_n(0, r)$ such that $T_n u_n + \mu G_n u_n = 0$ by the Borsuk antipodes theorem. Since $T + \mu G$ is A-proper, a subsequence $V_{n_k} u_{n_k} \to x \in \overline{B}(0, r)$ with $Tx + \mu Gx = 0$. Next, let $\mu_k \in (0, \mu_1)$, $\mu_k \to 0$ and $Tx_k + \mu_k G x_k = 0$ for some $x_k \in \overline{B}(0, r)$. Since G is bounded, $Tx_k \to 0$ and $Tx = 0$ for some $x \in X$ by condition (*).

Next, let us suppose in (2.3) that for each $r \geq R$ and $\mu \in [0, \mu_0]$, $\deg(T_n + \mu G_n, B_n(0, r), 0) \neq 0$ for all large n. When $\mu = 0$, this happens if, for example, T is odd on $X \backslash B(0, R)$ or if $(Tx, Kx) \geq 0$ for $||x|| \geq R$ and some additional conditions on $K : X \to Y^*$ and Γ (cf., e.g., [14, 21]). Part (a) has been proved in [12] in these special cases and, using similar arguments, we shall now give a unified proof of the parts (a)-(c).

Let $f \in Y$ be fixed and define $Bx = Tx - f$, $x \in X$. Then B satisfies (2.2) and let $\beta > 0$ be such that $|B - A| + 2\varepsilon < (1 - \beta)/\delta$. Then there is an $r > R$ such that $c(r) \geq \max\{1, 2\delta ||f||/\beta\}$ and $||Bx - Ax|| \leq (|B - A| + \varepsilon)c(||x||)$ for each $||x|| \geq r$. Let $\mu_1 \in (0, \mu_0)$ be such that $\mu_1 ||Bx|| < \varepsilon$ for all $||x|| = r$. Then, for each $\mu \in [0, \mu_1)$ and $||x|| = r$ we have

$$||(B + \mu G - A)x|| \leq ||Bx - Ax|| + \varepsilon > (|B - A| + 2\varepsilon)c(r) < (1 - \beta)c(r)/\delta.$$

Let $\mu \in [0, \mu_1)$ be fixed. Then, for $||x|| = r$,

$$
\begin{aligned}
(2.5) \qquad ||W_n(T + \mu G - A)x - tW_n f|| &\le ||W_n(T + \mu G - A)x - W_n f|| + ||W_n f|| \\
&\le \delta(|B - A| + 2\varepsilon)c(r) + c(r)\beta/2 < (1 - \beta/2)c(r).
\end{aligned}
$$

For $B_n = V_n^{-1}(B(0,r)) \subset E_n$ we have that $\overline{B} \subset V_n^{-1}(\overline{B}(0,r))$ and $\partial B_n \subset V_n^{-1}(\partial B(0,r))$. It follows from (2.1) and (2.5) that for each $\mu \in [0,\mu_1)$ fixed, each $u \in \partial B_n$, $n \ge 1$, and $t \in [0,1]$ we have that

$$
\begin{aligned}
||(T_n + \mu G_n) - tW_n f|| &\ge ||A_n u|| - ||(T_n + \mu G_n - A_n)u - tW_n f|| \\
&\ge c(||V_n u||) - (1 - \beta/2)c(||V_n u||) = \beta c(||V_n u||)/2 > 0.
\end{aligned}
$$

Hence, for each $\mu \in [0,\mu_1)$ fixed, $(T_n - \mu G_n)u \ne tW_n f$ for $u \in \partial B_n$, $t \in [0,1]$ and $n \ge 1$, and therefore the Brouwer degree $\deg(T_n + \mu G_n, B_n, W_n f) \ne 0$ for each $n \ge 1$.

Now, if $\mu = 0$, it follows that the equation $T_n u = W_n f$ is solvable in B_n for each n and the conclusion of (a) ((c), respectively) follows from the A-properness (pseudo A-properness, respectively) of T. In case (b) we have that for each $\mu \in [0,\mu_1)$ fixed the equation $T_n u + \mu G_n u = W_n f$ is solvable in B_n for each n, and therefore the equation $Tx + \mu Gx = f$ is solvable in $B(0,r)$. As before, the boundedness of G and condition (*) imply the solvability of $Tx = f$. □

The following special cases are useful in applications.

COROLLARY 2.1. *Let $T = A + N : X \to Y$, A satisfy* (2.1) *and*

$$
(2.6) \qquad |N| = \limsup_{||x|| \to \infty} \frac{||Nx||}{c(||x||)} < 1/\delta.
$$

Then the conclusions of Theorem 2.1 *hold.*

COROLLARY 2.2. *Let $T = A + N : X \to Y$ with $Q_n Ax = Ax$ for $x \in V_n E_n$ and*

$$
(2.7) \qquad ||Ax_n|| \to \infty \quad \text{as} \quad ||x_n|| \to \infty \quad \text{for} \quad x_n \in X;
$$

$$
(2.8) \qquad |N| = \limsup_{||x|| \to \infty} \frac{||Nx||}{||Ax||} < 1/\delta.
$$

Then the conclusions of Theorem 2.1 *hold.*

Proof. It follows from Corollary 2.1 by taking $c(||x||) = ||Ax||$ on X. □

Regarding condition (2.1), the following lemma is useful [cf. 12, 23].

LEMMA 2.1. *Let $A : X \to Y$ be A-proper at $f = 0$ w.r.t. Γ and α-positively homogeneous (i.e., $A(tx) = t^\alpha Ax$ for $x \in X$, $t > 0$ and some $a > 0$). Then, if $Ax = 0$ implies $x = 0$, there is a constant $c > 0$ and $n_0 > 1$ such that*

$$
(2.9) \qquad ||W_n Ax|| \ge c||x||^\alpha \quad \text{for} \quad x \in V_n(E_n),\ n \ge n_0
$$

Remark 2.1. Theorem 2.1 and Corollaries 2.1–2.2 are applicable to many classes of nonlinear maps and, in particular to (generalized) pseudo monotone ones from X to X^* (cf. [4]). This will be discussed in detail elsewhere.

Next, we shall prove a Fredholm alternative in a weaker form for maps of the form $T = A + N$, where A is a linear Fredholm map of index zero i.e., the kernel $X_0 = N(A)$ and cokernel of A are of the same finite dimension and the range $R(A)$ is closed. We have the direct sums $X = X_0 \oplus \tilde{X}$ and $Y = Y_0 \oplus \tilde{Y}$, $\tilde{Y} = R(A)$, and let $L : X_0 \to Y_0$ be a linear isomorphism and $P : X \to X_0$ be a linear projection onto X_0. Then $C = LP : X \to Y_0$ is completely continuous.

THEOREM 2.2. [17] (Fredholm alternative). *Let $A : V \subset X \to Y$ be a linear Fredholm map of index zero with $N(A) \neq \{0\}$ and A-proper w.r.t. Γ for (V, Y). Let $T : X \to Y$ be nonlinear and such that its range $R(T) \subset R(A)$ and $|T - A| < c/\delta$ for c sufficiently small. Suppose that either*

(a) *T satisfies condition (*) and $T + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ and some bounded map $G : X \to Y$; or*

(b) *$T + C : V \to Y$ is pseudo A-proper w.r.t. Γ.*

Then the equation $Tx = f$ is solvable if and only if $f \in R(A)\,(= N(A^*)^{\perp})$.

Proof. Since $A_1 = A + C$ is injective and A-proper w.r.t. Γ, there is a constant $c > 0$ such that (2.9) holds. Then $T_1 = T + C$ is such that $|T_1 - A_1| < c/\delta$. If (a) holds, then $T_1 + \mu G$ is A-proper w.r.t. Γ for each $\mu \in (0, \mu_0)$ by the compactness of C. In either case, the equation $T_1 x = f$ is solvable for each $f \in Y$ by Theorem 2.1. Moreover, if $f \in R(A)$ and $T_1 x = f$, then $Cx = f - Tx \in R(A)$ and consequently $Cx = 0$ and $Tx = f$. Conversely, if $Tx = f$ is solvable, then $f \in R(A)$ since $R(T) \subset R(A)$. □

Finally, we shall establish a complete extension of the classical Fredholm alternative for A-proper maps of the form $T = A + N$. Recall that the *covering dimension* of a normal topological space is equal to n, provided n is the smallest integer with the property that whenever U is an open covering of X, there exist a refinement U' of U, which also covers X, and no more than $n + 1$ members of U' have nonempty intersection.

THEOREM 2.3. [17] (Fredholm alternative). *Let $A : X \to Y$ be a continuous linear Fredholm map of index zero and* $\operatorname{codim} R(A) = m > 0$ *and $N : X \to Y$ be continuous and such that $|N| < c/\delta$, $R(N) \subset R(A)$ and $T = A + N$ is A-proper w.r.t. $\Gamma_0 = \{X_n, P_n; Y_n, Q_n\}$ with $X_0 \subset X_n$ and $Y_0 \subset Y_n$. Then, for each $f \in R(A)(= N(A^*)^{\perp})$, and only such ones, there is a connected closed subset K of $T^{-1}(f)$ whose dimension at each point is at least m and the projection P maps K onto Y_0.*

Proof. Let $V_n = Y_n \cap \tilde{Y}$, $X_n = X_0 \oplus U_n$ with $\dim U_n = \dim V_n$ and $\tilde{Q}_n = Q_n|\tilde{Y}|$. Then $T = A + N : X \to \tilde{Y}$ is A-proper w.r.t. $\Gamma_m = \{X_n, P_n; V_n, \tilde{Q}_n\}$ with $\dim X_n - \dim V_n = m$, $n \geq 1$. For a given $f \in R(A)$, let $Bx = Nx - f$. Let $\varepsilon > 0$ be such that $|N| + \varepsilon < c/\delta$ and $R = R(E) > 0$ such that

$$||NX|| \leq (|N| + \varepsilon)||x|| \quad \text{for all} \quad ||x|| \geq R.$$

We need to show that $A+B : X_0 \oplus \tilde{X} \to \tilde{Y}$ is complemented by P. To that end it suffices to show (see [2]) that $\deg(\tilde{Q}_n(A+B)|_{U_n}, U_n, 0) \neq 0$ for all large n. Define the homotopy $H_n : [0,1] \times U_n \to V_n$ by $H_n(t, x_1) = \tilde{Q}_n A x_1 + \tilde{Q}_n B(x_1)$ We claim that there are $n_0 \geq 1$ and $r \geq R$ such that if, $H_n(t, x_1) = 0$ for some $x_1 \in U_n$ with $n \geq n_0$ and $t \in [0,1]$ then $||x_1|| < r$. If not, then there would exist $x_{1n_k} \in U_{n_k}$ with $||x_{1n_k}|| \to \infty$ and $t_k \in [0,1]$ such that $H_{n_k}(t_k, x_{1n_k}) = 0$ for each k. Hence,

$$c||x_{1n_k}|| \leq ||\tilde{Q}_{n_k} A x_{1n_k}|| \leq \delta(|N| + \varepsilon)||x_{1n_k}|| + \delta||f||$$

and, dividing by x_{1n_k} and passing to the limit, we arrive at a contradiction to $|N| + \varepsilon < c/\delta$. Thus, the claim is valid and for each $n \geq n_0$, and $\deg(\tilde{Q}_n(A+B)|_{U_n}, U_n, 0) = \deg(\tilde{Q}_n A|_{U_n}, U_n, 0) \neq 0$.

Next, we need to show that $P : X_0 \oplus \tilde{X} \to X_0$ is proper on $(A+B)^{-1}(0)$. To see this, it suffices to show that if $\{x_n\} \subset X$ is such that $Ax_n + Bx_n \to 0$ and $\{Px_n\}$ is bounded, then $\{x_n\}$ is bounded since the A-proper map $A+B$ is proper restricted to bounded sets ([21]). We have that $x_n = x_{0n} + x_{1n}$ with $x_{0n} \in X_0$ and $x_{1n} \in \tilde{X}$, and $c||x_{1n}|| \leq ||Ax_{1n}|| \leq (||N|| + \varepsilon)||x_{1n}|| + ||f||$ for some $\varepsilon > 0$ with $|N| + \varepsilon < c$ if $||x_{1n}|| \geq R$. This implies that $\{x_{1n}\}$ is bounded as before. Since $\{x_{0n}\} = \{Px_n\}$ is bounded, it follows that $\{x_n\}$ is also bounded. Hence, the conclusions of the theorem follow from Theorem 1.2 in Fitzpatrick-Massabó-Pejsachowicz [2]. □

Analogously, a dimension assertion on the solution set of the corresponding "adjoint" equation treated in Theorem 2.3 in [23] can be proven when the involved maps are A-proper.

Remark 2.2. Theorem 2.2 extends a result of Petryshyn [23] dealing with weakly A-proper maps. Moreover, Theorem 2.3 includes the weaker form of the Fredholm alternative (not dealing with the dimension of the solution set) of Kachurovsky [5, 6] for compact maps and of Nečas [18, 19] and Hess [3] for maps of type (S), (S_+) and monotone ones, respectively.

Remark 2.3. Using similar arguments, it can be shown that Theorem 2.3 holds for nonlinearities N of superlinear growth, i.e. if $N = N_1 + N_2$ with N_1, A-proper, odd, α-homogeneous for some $\alpha > 1$ and $N_1 x = 0$ implies $x = 0$, and $||N_2 x|| \leq a + b||x||^k$ for some $a, b, k < \alpha$ and all $x \in X$.

3. Applications. We begin by looking at some applications of the abstract results in Section 2 to semilinear equations of the form (1.2) with $\dim \ker A \leq \infty$ when there is no resonance at infinity. By this we mean that there is some linear map $C : V \subseteq X \to Y$ such that $0 \notin \sigma(A-C)$, the spectrum of $A - C$, and $N - C$ stays away from $\sigma(A-C)$ at infinity (e.g., (3.1) holds).

Let H denote a real Hilbert space and X and Y be Banach spaces. In the self-adjoint case we have

THEOREM 3.1. *Let $A : D(A) \subset H \to H$ be self-adjoint, $V = (D(A), ||\cdot||_0)$ be a Banach space densily and continuously embedded in H, $C : D(C) \subset H \to H$ be bounded and symmetric with $V \subset D(C)$ and $0 \notin \sigma(A-C)$. Suppose that $N : V \to H$ is nonlinear and such that*

(3.1) *There are positive constants a, b, c, r and $k \in (0,1)$ such that*

$$||Nx - Cx|| \leq a||x|| + b||x||_0^k + c \quad for ||x||_0 \geq r$$

(3.2) $0 < a < \min\{|\lambda| \mid \lambda \in \sigma(A - C)\}$.

Then, if $A - N : V \to H$ is pseudo A-proper w.r.t. $\Gamma_0 = \{X_n, P_n; Y_n, Q_n\}$ for (V, H) with $Q_n(A - C)x = (A - C)x$, $x \in X_n$, $n \geq 1$, it is surjective.

Proof. Note first that $B = (A - C)^{-1} : H \to V$ is continuous. Indeed, by the closed graph theorem, it suffices to show that it is closed. Let $x_n \to x$ in H and $Bx_n \to v$ in V. Then $Bx_n \to v$ in H and $Bx = v$ by the closedness of B in H. Hence, for each $x \in V$

$$||(A - C)x|| \geq ||B||^{-1}||x||_0.$$

Next, since C is bounded and symmetric, $A-C$ is self-adjoint (see Kato [7, Thm. V. 4.3.]) and therefore $\min\{||\lambda|| \mid \lambda \in \sigma(A-C)\} = ||(A-C)^{-1}||$ and $a||(A-C)^{-1}|| < 1$ by (3.2). Moreover, for each $||x_0|| \geq r$, we have $x = (A - C)^{-1}y$ for some $y \in H$ and

$$\begin{aligned} ||Nx - Cx|| &\leq a||(A - C)^{-1}y|| + b||(A - C)^{-1}y||_0^k + c \\ &\leq a||(A - C)^{-1}|| \, ||y|| + b||B||^k||y||^k + c, \end{aligned}$$

or

$$\frac{||Nx - Cx||}{||(A - C)||} \leq a||(A - C)^{-1}|| + b||B||^k \, ||(A - C)x||^{k-1} + c||(A - C)x||^{-1}.$$

Hence,

$$|N - C| = \limsup_{||x_0||_0 \to \infty} \frac{||Nx - Cx||}{||(A - C)x||} \leq a||(A - C)^{-1}|| < 1$$

and the conclusion follows from Corollary 2.2. □

Remark 3.1. If there are real numbers $\alpha < \beta$ such that a $\sigma(A) \cap (\alpha, \beta)$ consists of at most finitely many eigenvalues, then we can take $C = \lambda I$, $\lambda = (\lambda_k + \lambda_{k+1})/2$, in Theorem 3.1 for some consecutive eigenvalues $\lambda_k < \lambda_{k+1}$ in (α, β). Then (3.2) holds if $a < \gamma = (\lambda_{k+1} - \lambda_k)/2$. Indeed, the spectral gap for $A - \lambda I$ induced by the gap $(\lambda_k, \lambda_{k+1})$ is $(-\gamma, \gamma)$ and therefore $(A - \lambda I)^{-1} : H \to H$ is a bounded self adjoint map whose spectrum lies in $(-1/\gamma, 1/\gamma)$. Hence, $||(A - \lambda I)^{-1}|| = 1/\gamma$. Moreover, the scheme $\Gamma_0 = \{(A - \lambda I)^{-1}(Y_n), P_n; Y_n, Q_n\}$ for (V, H) has the required property in Theorem 3.1.

Analyzing the proof of Theorem 3.1, we see that the following more general result holds when A is not selfadjoint.

THEOREM 3.2. *Let $(V, ||\cdot||_0)$ be densily and continuously embedded in X, $A : V \to Y$ and $C : X \to Y$ be closed linear maps with $A - C : V \to Y$ bijective. Suppose that $N : V \to Y$ is nonlinear and*

(3.3) *There are positive constants a, b and r, with a sufficiently small such that*

$$||Nx - Cx|| \leq a||x||_0 + b \quad for \quad ||x|| \geq r.$$

Then, if $A - N : V \to Y$ *is pseudo A-proper w.r.t.* Γ *for* (V, Y) *with* $Q_n(A-C)x = (A-C)x$, $x \in X_n$, $n \geq 1$, *it is surjective.*

Next, we shall look at Eq. (1.2) with nonlinearities of the form $Nx = B(x)x - Mx$, where $B(x) : X \to X$ is a continuous linear map for each $x \in V$ such that for some $\lambda \notin \sigma(A)$, $A_\lambda = A - \lambda I$ and $B_\lambda(x) = B(x) - \lambda I$ satisfy

$$m = \limsup_{||x||_0 \to \infty} ||B_\lambda(x)|| < \frac{1}{||A_\lambda^{-1}||}. \tag{3.4}$$

THEOREM 3.3. *Let* $A : D(A) \subset X \to X$ *be a closed linear map,* $V = (D(A), ||\cdot||_0)$ *be a Banach space densily continuously embedded in* X *and* (3.4) *hold. Suppose that* $M : V \to X$ *is nonlinear and* $T : V \to X$, $Tx = A(x) - B(x)x - Mx$, *is pseudo A-proper w.r.t.* $\Gamma = \{X_n, P_n; Y_n, Q_n\}$. *Then*

(a) *If* $Q_n A_\lambda x = A_\lambda x$, $x \in X$, $n \geq 1$, *and there are positive constants* a, b, c, r *and* $k \in (0,1)$ *such that* $\delta(a+m) \cdot ||A_\lambda^{-1}|| < 1$ *and*

$$||Mx|| \leq a||x|| + b||x||_0^k + c \quad \text{for} \quad ||x||_0 \geq r,$$

then T *is surjective*

(b) *If* $T_1 x = Ax - B(x)x$ *is A-proper w.r.t.* Γ_0 *and*

$$|M| = \limsup_{||x||_0 \to \infty} \frac{||Mx||}{||x||_0} < \infty$$

is sufficiently small, then T *is surjective.*

Proof. (a) As in Theorem 3.1, we obtain that

$$||A_\lambda x|| \geq ||A_\lambda^{-1}||_{(X\to V)}^{-1} ||x||_0, \quad x \in X.$$

Moreover, for $\varepsilon > 0$ small with $(m + a + \varepsilon)||A_\lambda^{-1}|| < 1$ there is an $R > 0$ such that for $||x||_0 \geq R$

$$||B_\lambda(x)x + Mx|| \leq (m + a + \varepsilon)||x|| + b||x_0||^k + c.$$

Then, setting $Nx = B(x)x + Mx$ and $C = \lambda I$, the conclusion follows from Corollary 2.2 as in Theorem 3.1.

(b) By (3.4), there is an $R > 0$ such that $||B_\lambda(x)|| < 1/||A_\lambda^{-1}||$ for all $||x||_0 \geq R$. Hence, for such x's, the map $B_\lambda(x)A_\lambda^{-1} : X \to X$ satisfies

$$||B_\lambda(x)A_\lambda^{-1}|| \leq ||B_\lambda(x)||\,||A_\lambda^{-1}|| < \theta < 1$$

for some θ independent of x. Consenquently, $I - B_\lambda(x)A_\lambda^{-1} : X \to X$ is invertible and

$$||(I - B_\lambda(x)A_\lambda^{-1})^{-1}|| < 1/(1-\theta) \quad \text{for} \quad ||x||_0 \geq R.$$

As before, $A_\lambda^{-1} : X \to V$ is continuous and therefore $c||x||_0 \leq ||A_\lambda x||$ for $x \in V$ and some $c > 0$. Moreover, for $||x||_0 \geq R$

$$c_1||x||_0 \leq ||[I - B_\lambda(x)A_\lambda^{-1}]^{-1}[I - B_\lambda(x)A_\lambda^{-1}]A_\lambda x|| \leq ||A_\lambda(x) - B_\lambda(x)||/(1-\theta).$$

or

$$c_1||x||_0 \le ||A_\lambda x - B_\lambda(x)x|| \quad \text{for} \quad ||x||_0 \ge R, c_1 = (1-\theta)c. \tag{3.5}$$

Since $T_1 x = A_\lambda x - B_\lambda(x)x = Ax - B(x)x$ is A-proper, arguing by contradiction and using (3.5), we obtain an $n_0 \ge 1$ and $c_0 \ge 0$ such that

$$c_0||x||_0 \le ||Q_n(A - B(x))x|| \quad \text{for all} \quad x \in X_n \backslash \overline{B}(0,R), \quad n \ge n_0. \tag{3.6}$$

Since $|M|$ is sufficiently small, the conslusion follows from Corollary 2.1, where one needs only to assume (2.1) on $X_n\backslash\overline{B}(0,R)$. □

To give some conditions for the A-properness of T_1 and T, we recall that a *ball-measure of noncompactness* of a set $D \subset X$ is defined by $\chi(D) = \inf\{r > 0 \,|\, D = \cup_{i=1}^n B(x_i, r), x_i \in X \text{ and some } n\}$. A map $T : D \to Y$ is *k-ball-contractive* if $\chi(T(Q)) \le k\chi(Q)$ for each $Q \subset D$. We have

PROPOSITION 3.1. *Let $U(x,y) = B(x)y$ for $(x,y) \in V \times V$ and*

(3.7) *For each $x \in V$, $U(x,\cdot) : V \to X$ is k_1-ball-contractive;*

(3.8) *For each $y \in V$, $U(\cdot,y) : V \to X$ is completely continuous.*

Suppose that $A : V \to X$ is Fredholm of index zero and $M : V \to X$ is k_2-ball-contractive with $k = k_1 + k_2$ sufficiently small. Then $T_1, T : V \to X$ are A-proper w.r.t. Γ_0 for (V,X) with $Q_nAx = Ax$ on X_n.

Proof. It is known that the map $B_1 : V \to X$, $B_1(x) = U(x,x)$ is k_1-ball-contractive by (3.7)-(3.8). Since $B_1 + M : V \to X$ is k-ball-contractive, T_1 and T are A-proper w.r.t. Γ_0 (cf. [15]). □

Remark 3.2. Condition (3.7) is implied by the compactness of the embedding of V into X or by $||B(x)||_{(V\to X)} \le k_1$ for all $x \in V$. In applications various natural conditions imply (3.7)-(3.8).

So far we have studied Eq. (1.2) with nonlinearities N asymptotically close to linear maps (i.e. when condition of type (3.1) holds). It turns out that when $A = I$, we can allow more general nonlinearities studied first by Perov [20] and Krasnoselskii-Zabreiko [8]. To introduce this class, we consider a pair of self adjoint maps $B_1, B_2 : H \to H$ such that $B_1 \le B_2$, i.e. $(B_1x, x) < (B_2x, x)$ for $x \in H$, and 1 is not in their spectrum $\sigma(B_1) \cup \sigma(B_2)$. Let $\sigma(B_1) \cap (1,\infty) = \{\lambda_1, \dots, \lambda_k\}$ and $\sigma(B_2) \cap (1,\infty) = \{\mu_1, \dots, \mu_m\}$, where the λ_i's and μ_j's are eigenvalues of B_1 and B_2, respectively, of finite multiplicities and assume that the sum of the multiplicities of the λ_i's is equal to the sum of the μ_j's. Then we say that B_1 and B_2 form a regular pair.

Recall that ([8]) a (nonlinear) map $K : H \to H$ is said to be $\{B_1, B_2\}$-quasilinear on a set $S \subset H$ if for each $x \in S$ there exists a linear selfadjoint map $B : H \to H$ such that $B_1 \le B \le B_2$ and $Bx = Kx$. A map $N : H \to H$ is said to be *asymptotically* $\{B_1, B_2\}$-quasilinear if there is a $\{B_1, B_2\}$-quasilinear outside some ball map K such that

$$|N - K| = \limsup_{||x||\to\infty} \frac{||Nx - Kx||}{||x||} < \infty. \tag{3.9}$$

It has been shown in [8] that if B_1 and B_2 form a regular pair, then there is a constant $c > 0$ such that for each self-adjoint map B with $B_1 \leq B \leq B_2$ we have that

$$||x - Bx|| \geq c||x|| \quad \text{for each} \quad x \in H. \tag{3.10}$$

For example, if $N : H \to H$ is such that $N'(x)$ is self-adjoint for each x in H and satisfies

$$B_1 \leq N'(x) < B_2 \quad \text{for} \quad x \in H, \tag{3.11}$$

then N is asymptotically $\{B_1, B_2\}$-quasilinear since we can represent $Nx = B(x)x + N(0)$, where $B(x) = \int_0^1 N'(tx)\,dt$. Moreover, if $Nx = B(x)x + Mx$ for some nonlinear M with $|M| < \infty$ and $B(X) : H \to H$ is self-adjoint and $B_1 \leq B(x) \leq B_2$ for each x in H, then N is asymptotically $\{B_1, B_2\}$-quasilinear (cf. [20] for some other criteria). For equations with such nonlinearities we have

THEOREM 3.4. [17]. *Let $\{B_1, B_2\}$ form regular pair, $M, N : H \to H$ be bounded and N be asymptotically $\{B_1, B_2\}$-quasilinear with $|M + N - K| < c$. Let $B_0 : H \to H$ be self-adjoint with $B_1 \leq B_0 \leq B_2$ and $H_t = I - t(M+N) - (1-t)B_0$, $0 \leq t \leq 1$. Then*

(a) *If H_t, is A-proper w.r.t. $\Gamma_0 = \{H_n, P_n\}$ for each $t \in [0,1]$, then the equation $x - Mx - Nx = f$ is f.a. solvable for each $f \in H$.*

(b) *If H_t, is A-proper w.r.t. Γ_0 for each $t < 1$ and H_1 is either pseudo A-proper w.r.t. Γ_0 or satisfies condition (*), then $(I - M - N)(H) = H$.*

(c) *Let $G : H \to H$ be such that $||Gx|| < a||x||$ on H for some a, and for each large r, $\deg(P_nB_0 + \mu P_nG, B(0,r) \cap X_n, 0) \neq 0$ for each large n and $\mu > 0$ small. Suppose that $H_t + \mu G$ is A-proper w.r.t. Γ_0 for each $t \in [0,1]$ and $\mu > 0$ small and H_1 satisfies condition (*). Then $(I - M - N)(H) = H$.*

Proof. Since $N_f x = Nx - f$ has the same properties as N for any t in H, it suffices to study the equation $x - Mx - Nx = 0$. Let $\mu_0 > 0$ and $\varepsilon > 0$ be such that $|M + N - K| + \varepsilon + a\mu_0 < c$.Then there is an $r > 0$ such that $||Mx + Nx - Kx|| \leq (|M + N - K| + \varepsilon)||x||$ for each $||x|| \geq r$. Moreover, $H(t,x) + \mu Gx \neq 0$ for $||x|| = r$, $t \in [0,1]$ and $\mu \in [0, \mu_0)$. If not, then there are $t \in [0,1]$, $||x|| = r$ and $\mu \in [0, \mu_0)$ such that $H(t,x) + \mu Gx = 0$. Hence,

$$||x - tKx - (1-t)B_0x|| \leq t||Mx + Nx - Kx|| + \mu||Gx|| < c.$$

Since K is $\{B_1, B_2\}$-quasilinear, there is a self-adjoint map $B_* : H \to H$ such that $Kx = B_*x$ and therefore

$$||x - tB_*x - (1-t)B_0x|| < c||x|| \tag{3.12}$$

But $B_1 \leq B \leq B_2$ for $B = tB_* + (1-t)B_0$ and consequently (3.10) holds. This contradicts (3.12) and our claim is valid. Hence, the conclusions of (a), (b) and (c) follow from Theorems .1 and 3.1 [16], respectively. □

Remark 3.3. Theorem 3.4 is applicable if B_0 is compact and $M+N$ is the sum of a k-ball-contraction and a monotone map, $k<1$, or N is compact and $(Mx-My, x-y) \geq -||x-y||^2$, etc. When B_0 and N are compact, $M=0$ and $|N-K|=0$, the solvability of $x-Nx=f$ in part (a) has been proven by Krasnoselskii-Zabreiko [8] and in a less general form by Perov [20], using completely different arguments.

Finally, we shall consider Eq. (1.2) when $D(A)$ is not a linear subset of X and $A: D(A) \subset X \to Y$ is such that

(3.13)
$$(A+C)^{-1}: Y \to D(A) \subset X \quad \textit{is surjective and} \quad ||(A+C)^{-1}y|| \leq K(||y||+1)$$

for some bounded map $C: X \to Y$, each $y \in Y$ and some constant $K>0$. Condition (3.13) is satisfied if, e.g., $Y=X$ and $C=\lambda I$, $\lambda>0$, and A is m-accretive (cf. [1]). In applications considered in part II (3.13) holds with $Y \neq X$.

THEOREM 3.5. [17]. *Let* (3.13) *hold and* $N: D(A) \subset X \to Y$ *be such that for some constants* $a>0$, $b>0$ *with* $\delta Ka<1$, $\delta = \max ||P_n||$,

$$||Nx - Cx|| \leq a||x|| + b \quad \textit{for} \quad x \in D(A). \tag{3.14}$$

Suppose that $T = I + (N-C)(A+C)^{-1} + \mu C(A+C)^{-1}$ *is A-proper w.r.t.* $\Gamma_0 = \{X_n, P_n\}$ *for* Y *and* $\mu \in [0,1)$ *and* T_0 *satisfies condition* (*). *Then* $(A+N)(D(A)) = Y$.

Proof. It is easy to see that Eq. (1.2) is solvable if and only if so is the equation $T_0 y = f$ in Y. In view of Corollary 2.1, with $A=I$ and $G=-C(A+C)^{-1}$, it suffices to show that $|(N-C)(A+C)^{-1}| < 1/\delta$. But, this follows easily from (3.13)–(3.14) since

$$\limsup_{||y|| \to \infty} \frac{||(N-C)(A+C)^{-1}y||}{||y||} \leq \limsup_{||y|| \to \infty} \frac{b + a||(A+C)^{-1}y||}{||y||} \leq aK < 1/\delta. \quad \square$$

Next, we shall give an extension of Theorem 3.5 when (3.13) does not hold. We need

Definition 3.1. A homotopy $H: [0,1] \times D \to Y$, $D \subset X$, is said to satisfy condition (+) if $\{x_n\}$ is bounded in X whenever $H(t_n, x_n) \to f$, $t_n \in [0,1]$.

THEOREM 3.6. [17]. *Let* $A, N: D(A) \subset X \to Y$ *and* $C: X \to Y$ *be nonlinear maps,* C *and* N *be bounded and* $(A+C)^{-1}: Y \to D(A)$ *be bounded and surjective. Suppose that* $H(t,x) = Ax + tNx + (1-t)Cx$ *satisfies condition* (+), $F_t = I + t(N-C)(A+C)^{-1}$ *is A-proper w.r.t.* $\Gamma_0 = \{Y_n, P_n\}$ *for each* $t \in [0,1)$ *and* F_1 *satisfies condition* (*). *Then* $(A+N)(D(A)) = Y$.

Proof. Let $f \in Y$ be fixed. Condition (+) implies that the set $U = \{x \in D(A) \,|\, H(t,x) = tf$ for some $t \in [0,1]\} \subset B(0,R_1)$ for some $R_1>0$. Then $x = (A+C)^{-1}y \in U$ whenever $F(t,y) = tf$ and, since C and N are bounded, there is an $R>0$ such that

$$||y|| \leq ||(N+C)(A+C)^{-1}y|| \leq R.$$

Hence, $F(t,y) \neq tf$ for $(t,y) \in [0,1] \times \partial B(0,R)$. Next, let $\varepsilon_k \in (0,1)$ and $\varepsilon_k \to 1$. By the A-properness of F_t for $t \in [0, \varepsilon_k]$, there is an $n_k = n(\varepsilon_k) \geq 1$ such that

$$P_n F(t,y) \neq t P_n f \quad \text{for} \quad t \in [0,\varepsilon_k],\ y \in Y_n \cap \partial B(0,R), n \geq n_k$$

and $n_{k_1} \geq n_{k_2}$ if $k_1 \geq k_2$. Hence, for each k fixed and each $n \geq n_k$

$$\deg(P_n H(\varepsilon_k \cdot), B(0,R) \cap Y_n, P_n f) = \deg(I, B(0,R) \cap Y_n, 0) \neq 0$$

and therefore $P_n F(\varepsilon_k, y_n) = \varepsilon_k P_n f$ for some $y_n \in B(0,R) \cap Y_n$ and each $n \geq n_k$. Since F_{ε_k} is A-proper, there is an $y_k \in \overline{B}(0,R)$ such that $F(\varepsilon_k, y_k) = \varepsilon_k f$. Then $y_k + (N-C)(A+C)^{-1} y_k = \varepsilon_k f + (1-\varepsilon_k)(N-C)(A+C)^{-1} y_k \to f$ as $k \to \infty$. Thus by condition (*) for F_1, there is an $y \in Y$ such that $F(1,y) = f$ and so $x = (A+C)^{-1} y$ is a solution of $Ax + Nx = f$. □

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