PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 42 (56), 1987, pp. 3–12

FREE POWER OR WIDTH OF SOME KINDS OF MATHEMATICAL STRUCTURES

Đuro R. Kurepa

Abstract. The present work consists of 3 sections. In section 1 we have Theorem 1:1 which gives an sufficient condition to exhibit a kind of antichains in pseudotrees. In section 2 the problem of attainability of p_sE is examined: since simple examples show that even in well-founded sets W the number p_sW might be unattained one examines the case of p_sT for trees; we prove the main Theorem 2:4 and formulate ATH (Antichain Tree Hypothesis) in 2:7 and prove that ATH is implied by the RH (Ramification Hypothesis) (v. 2:8 Theorem). We stress the fact how limit regular cardinals occur in considerations in section 2. Section 3 examines p_sT^n for squares, cubes and hypercubes of trees it is proved that for any index set I of cardinality > 1 the cardinal ordering of the hypercube T^I is such that the number p_sT^I is attained. One has the beautiful result 3:5.

Introduction

0: Width is a current word in everyday practice (width of a solid physical or geometrical body or figure) and could be used everywhere where a length (measure) is occurring. With many mathematical structures S a width, wid S, could be associated.

0:1. For ordered sets $(E, \leq)$ a width was introduced in Kurepa [1937] and was denoted by $p_s(E, \leq)$ in order to indicate that one deals with a power (or cardinality) of some free sets (in all slavic languages the word free starts with s (svoboda or sloboda). It was $p_s(E, \leq) := \sup_A |A|$, A running through the system of all free subsets or antichains in $(E, \leq)$. In particular, for any set M in which no order or structure is introduced p_sM becomes the power pM or $|M|$ of M.

0:2. For metrical sets M, lying in a metric space (E, d) where d is the distance function or metrics, one could define a width of M as $\sup d(x, y)$ $(x, y \in M)$, called the diameter of M. In this case, wid M is a member of $R[0, \infty]$. A similar definition is possible for topological spaces (M, d), defined by a distance function $d(x, y)$ taking values in a given ordered set $(E, \leq)$ (for such spaces see Kurepa [1956], [1976], ...)

AMS Subject Classification (1980): Primary 04A10; Secondary 05C38

0:3. For any mathematical structure of the form (M, R) where M is a set and R a binary relation on M, i.e. $R \subset M \times M$, let us define a width as follows: $\mathrm{wid}(M, R) := \sup_A |A|$, where A runs through the system of all subsets A of M such that if $x, y \in A$ and $x \neq y$ then neither xRy nor yRx, i.e. $(A^2 \backslash D) \cap R = \nu$ (empty) .

0:4. For a ternary relation R on a set M one defines free subsets as subsets A of M such that $(A^3 \backslash D) \cap R = \nu$, where D is the diagonal of M, i.e. the set of all 3-uns (m, m, m) $(m \in M)$.

0:5. In a general way, we have the following. Given an (index) set I and any I-un (1) $f : x \in I \to fx$ of sets fx; any subset R of the products (2) $M := \prod_{x \in I} fx$ of all sets fx is called an I-ary relation in the given I-un f.

0:6. *Definition.* A subset A of (2) is said to be free or an antichain in the structure (M, R) if for time restriction $f|A$ the corresponding product $\prod fx$ is disjoint with R.

0:7. In particular, given an index set I and a set M one has the I-cube of M, i.e. the set M^I of all mapping $f : I \to M$; the diagonal D of M^I is the set of all constant mappings $c : I \to M$. Any $R \subset M^I$ is called an I-ary relation in M.

0:8. A first question arises whether wid is attained, i.e. given (M, R), is there a subset A of M such that $|A| = \mathrm{wid}(M, R)$ and such that if $x, y \in A$, then neither xRy nor yRx. Such subsets of M are called free relatively to R, or relatively to (M, R); they are also called *antichains.*

0:9. In this paper, we restrict ourselves mainly to ordered sets. One of the main results is the attainability of width for every tree T such that $p_s T$ is no limit regular cardinal and such that the question whether the width is attained for *every* tree has probably a *postulational* character.

0:10. The question of supremum and maximum was one of the main points in my doctoral dissertation (Paris 1935:2). There trees T and some cardinal functions one trees were introduced; in particular for any tree T of decreasing sets a cardinal $b'T$ was defined as the supremum of $|D|$, D running through the system of all disjoint subsystems of $T^d := \{X, X \in T \text{ or } X = Y \backslash Z, \text{ where } Y, Y \in T \text{ and } Y \supset Z\}$. Then I proved the following:

0:11. THEOREM. [These p. 110, Théoreme 3] *Unless the tree T is of inaccessible height (rank)*, the supremum $b'T$ is attained.

This theorem implies:

0:12. THEOREM. *For every ordered chain $(L, \leq)$, unless $p_2(L, \leq)$ is a regular limit, the celullarity $p_2(L, \leq) := \sup |D|$, D running through disjoint system of open intervals, is attained.*

0:13. The fact is transferable to topological spaces, as was published, without quotation of any result in Erdos-Tarski [1943] (the Thèse was not quoted but my definition of tree or ramification (p. 3274_{4-1}) and some results of Aronszajn, and Kurepa (p. 328_{12-11}) are mentioned without quoting my name).

1. Some General Statements

1:0. GRAPH LEMMA. *Let (G, R) be a reflexive symmetric graph and t a point of G such that $p_s G(t) > 1$; for every maximal complete subgraph L of $G(t)$, there is a 2-un (ft, gt) of incomparable points ft, gt such that $gt \in L$, $G(t) := \{g : g \in G, gRt\}$.*

Proof. By assumption, t is a point of G such that $p_s G(t) > 1$; i.e. there is an antichain $\{x, y\} \subset G(t)$. Since, by assumption, L is a complete subgraph of $G(t)$, one has not $x, y \in L$, thus there is an element in $\{x, y\} \backslash L$; let us denote it by $f(t)$. Now, there is at least one member l of L such that ft, l are incomparable. As a matter of fact, if for every $l \in L$ the points ft, l were R- comparable, this would mean that $L \cup \{ft\}$ is a complete subgraph of $G(t)$ more extensive than the maximal subgraph L, which is an absurdity. Consequently, there is a point in $L \backslash G(ft)$, and it suffices to denote it by gt in order to see that the statement of L is true. Q.E.D.

1:0:1. COROLLARY. *If $(E, <)$ is ordered and if $e \in E$ is such that $p_s E(e) > 1$, then for every maximal chain L of $E(e)$ there is a 2-un (fe, ge) of free points in $E(e)$ such that $gt \in L$; one has either fe, $ge < e$ or fe, $ge > e$.*

Proof. It is sufficient to put: $G = E$, $R = \leq \cup \geq$, $t = e$ and to apply the Graph lemma: one gets wording of Corollary 1:0:1.

1:0:2. COROLLARY. *Let $(R, \leq)$ be a pseudotree and $t \in R$ be such that $p_s R(t, \cdot) > 1$. If L is a branch ($\equiv$ maximal chain) in $R(t)$, then there is a 2-un (ft, gt) of free points ft, gt in $R(t)$, such that $gt \in L$.*

1:0:3. COROLLARY. *If $(T, \leq)$ is any tree and t a point of T having at least 2 followers, then for every maximal chain $L \subset T(t)$ there is a 2-un $(f(t), g(t))$ of incomparable members in $T(t, \cdot)$ such that $f(t)$, $g(t)$ belong to the first row of $T(t, \cdot)$ containing at least 2 points; again $g(t) \in L$.*

Proof. Since L is a branch in $T(t, \cdot)$, T intersects every row of $T(t, \cdot)$; so also the first one, R_t, which is not a singleton; therefore $\{g(t)\} = L \cap R_t$ and ft could denote any point of $R_t \backslash L$.

As an application of the Graph Antichain Lemma, we have the following

1:1. THEOREM. *Let* (1) *$(R, \leq)$ be any nonempty pseudotree and L a maximal sub-chain ($\equiv$ branch) of* (1) *such that every $l \in L$ satisfies $p_s(l, \cdot)_R > 1$; then $(R, \leq)$ contains an antichain of* (1) *of cardinality $\geq cf L$; i.e. $p_s(R, \leq) \equiv cf L$.*

Proof. By an induction argument we are going to exhibit a biunique sequence

$$a_j \quad (j < cf\, L) \tag{2}$$

of free points $a_j \in R$. The thing is obvious if L has a least point i.e. if $cf\, L = 1$. Therefore let us consider the case that $cf\ L$ is a regular initial ω_n. Let then

$$w := \{l_0, \ldots, l_j, \ldots\} \quad (j < cf\, L) \tag{3}$$

be a well-ordered subset W of type ω_n of L which is cofinal to L. To start let us consider the point $c_0 := l_0$ in (3) and apply the Lemma for $t = c_0$; we get the points $a_0 := fc_0$, $b_0 := gc_0$; assume that $0 < j < cf\,L$, and that a strictly increasing j-sequence c_i $(i < j)$ of points of (3) is formed such that in connection with L one has a j-sequence $a_i := fc_i$ $(i < j)$ of free points and a strictly increasing j-sequence $b_i := gc_i$ $(i < j)$ such that

$$b_i, c_i \in W \setminus \bigcup_{r<i} (\cdot, a_r]. \tag{4}$$

Let us define c_j, a_j, b_j. If $j - 1 < j$, we put $c_j := gb_{j_i}$, $a_j := fc_j$, $b_j := gc_j$. If j is a limit ordinal $< \omega_n$, let c_j denote any point such that

$$c_j \in W \setminus \bigcup (\cdot, a_i] \quad (i < j). \tag{5}$$

Such a c_j exists because the set $(5)_2$ is nonempty — a fact implied by the regularity of the order type $\omega_n > j$ of W. Then we apply the Lemma for $t = c_j$ and get

$$a_j := fc_j, \quad b_j := gc_j. \tag{6}$$

In virtue of (5), (6) if $i < j$, then one does not have $c_j = a_i$, still less $a_j = a_i$, becaause $a_j := fc_j > c_j$. Neither does one have $a_i < a_j$. Assume on the contrary that for some $i < j$ one has $a_i < a_j$, i.e. $fc_i < fc_j$, and $fc_i || gc_i < c_j < fc_j$: the point $fc_j := a_j$ would be preceded by incomparable points $fc_i = a_i$ and c_j, contrarily to the fact that each left cone in $(R, \leq)$ is a chain. So the induction step for each $j < cf\,L$ is performable, one gets a requested antichain a_i $(i < cf\,L)$ of power ef L. Q.E.D.

1:1:1. *Remark.* The statement of the Theorem might be false if the involved subchain L is not maximal. Example: If $(E, \leq)$ is any totally ordered set of cofinality $> Al_0$ and if $(N, \leq)$ is the tree of all finite sequences of natural numbers where $\leq$, denotes the relation "is an initial segment of", then the ordinal sum $(E, \leq) + (N, \leq)$ is a pseudotreee of width Al_0, and contains no free sunbset or cardinality $cf\,L$.

1:1:2. *Remark.* The statement of the Theorem might be false for well-founded sets $(E, \leq)$.

As a maatter of fact, let ω_n be any initial ordinal number and let f_i $(i < \omega_n)$ be any ω_n-sequence of disjoint sets each having just 2 points; let then the sum $S := \Sigma f_i$ $(i < \omega_n)$ be ordered in such a way that the members of f_i be incomparable for each $i < \omega_n$ and that $f_i < f_j$ for $i < j < \omega_n$; then each maximal chain L in S is of power Al_n, while $p_s E := R^2 < cf\,L$.

2. The question of attainability of width.

2:0. ln this section we shall present some interesting results on the question whether the width is attained in a given structure. Since already in well-founded sets the width might be unattained (v. 2:1 Lemma), we pass to trees and establish the main theorem 1:2. We also announce the proposition ATP (Antichain Tree Proposition) stating that for every tree T the member p_sT is attained. This proposition is examined with other tree propositions, especially with our Ramification Hypothesis (RH), and our Tree Axiom. We denote by T, R any tree any pseudo-tree respectively.

2:1. LEMMA. *For every limit cardinall there exists an ordered set* $(E, \leq)$ *such that* $p_s(E, \leq) = l$ *and* l *is not attained.*

Proof. Let us consider any strictly increating $cf\ l$-sequence a_n of cardinals such that sup $a_n = l := \omega_\alpha$ and an l-sequence E_n $(n < l)$ of pairwise disjoint sets such that

(1) $|E_n| = r_n + 1$; where r_n is such that $n = k\omega + r_n$;

let $<$ in (2) $E := \cup E_n$ mean that

(3) $x < y$ holds if and only if $x \in E_m$, $y \in E_n$, $m < n$ and that $x||y$ means $\{x, y\} \subset E_n$ for some $n < l$. Then obviously, $p_sE = Al_0$ and every free subset is finite. Analogously, if instead of (1) one requires $|E_n| = |n|$ $(n < \omega_\alpha)$, then the ordering (3) yields the structure (2) for which $p_s = l$ and in which every free subset is of a cardinality $< p_r$.

2:2. What about p_sT for trees?

2:2:1. In our Thesis we defined $mT := \sup_{\alpha<\gamma T}|R_\alpha T|$; of course $mT \leq p_sT$; the difference betwen mT, and p_sT could be great; e.g. if T consist of a well-ordered set W and of points W'_n such that for each $n < \gamma W$, W_n, W'_n are incomparable points as a row R_nT, then $mT = 2$, $W' := \{W'_n\}_{n<\gamma T}$ is an antichain of cardinality $|W|$.

2:2:2. The number p_sT need not be attained for T as the power of a row of T. As a matter of fact, for every ordinal n there is a tree T_n such that $\gamma T_n = \omega_n$, $mT_n = p_sT_n = Al_n$ and mT_n is not attained as the power of a row of T_n.

Proof. Let T_n consist of the 2-uns $(n, n')_{n' \leq n}$ for every $n < \gamma W$ in which $(a, b) < (c, d) \leftrightarrow a = b < c = d$. Then the diagonal $L := \{(i, i)\}_{i<\omega_n}$ is a maximal chain; its complement is free, it is of power $Al_n = mT$ and it is not attained as the power of a row.

2:2:3. Well-founded set U_n. For any ordinal number n let us consider the upper part U_n of the square of the set ω_n of ordinals $< \omega_\alpha$, i.e. $U_n := \{(x, y) : x < y < \omega_n\}$ ordered in such a way that for $(a, b), (c, d) \in U_n$ the relation $(a, b) < (c, d)$ means $a \leq c \wedge b < d$. One proves easily that $\leq$ is an order relation in U_n and that $(U_n, \leq)$ is well-founded. Also one verifies that $(a, b)||(c, d)$ means $((a \neq c) \wedge (b = d)) \vee ((a < c) \wedge (b \geq d)) \vee ((a > c) \wedge (b \leq d))$.

LEMMA. *In the graph* $(U_n, ||)$ *every complete subgraph* A *is* $< Al_n$*, i.e. every antichain* A *in* $(U_n, \leq)$ *is* $< Al_n$*; if* $n = 0$ *or if* n *is a limit, then* $p_s(U_n, \leq) = Al_n$ *and the number* p_s *is not attained.*

Proof. Let $(x_0, y_0) := gA$ be the element of A having minimal first coordinate x_0; gA is uniquely determined; let $(x_1, y_1) :== g(A\backslash\{gA\})$; thus $x_0 < x_1$ and $y_0 \geq y_1 > x_1$: the procedure is performed as far as possible. In the sequence $(y_i)_i$ there is only a finite number of distinct terms because they form a decreasing sequence of ordinal numbers. On the other hand, the number of consecutive signs $=$ starting with y_i is $\leq -x_0 + y_i$, thus $< Al_n$. Consequently, $|A| < Al_n$. Obviously, if $n = 0$ or if n is limit, then $p_s(U, \leq) = Al_n$ because for any $y < \omega_n$ one has the antichain $\{(x, y)$ where $x < y\}$ of power $|y| - 1$. The set $(U_n, \leq)$ served me in 1952 as an example of an ordered set in which the relation $||$ is not trivial, and in which the p_s-number is not attained; the case $n = 0$ was considered also by Rado [1954, § 2].

2:3. LEMMA. *Every tree T such that $p_sT = Al_0$ contains a free subset of cardinality p_sT.*

Proof. The statement is trivial if p_sT is finite or if T contains an infinite row. Therefore let us consider any tree γT such that $p_sT = Al_0$ and every level of T is $< Al_0$. Let $T_0 := \{t : t \in T, p_sT < Al_0\}$. If T_0 is infinite, then necessarily R_0T_0 is infinite because $T_0 = \cup T_0[x, \cdot)$ $(x \in R_0T_0)$ and each summand is finite. If T_0 is finite, one could assume that T_0 is empty: it is sufficient to denote by T all points t of T such that $\gamma t > \gamma t_0$ for every $t_0 \in R_0T_0$; γt is defined by $t \in R_{\gamma t}T$. Consequently, we are in the position that $p_sT(t) > 1$ for every $t \in T$.

Since $p_sT = \sum p_s(t)$ $(t \in R_0T)$ and $|R_0T| < Al_0$, one concludes that there exists a point $t_0 \in R_0T$ such that $p_sT(t) = Al_0$. Let n_0 be the first ordinal such that the row $R_{n_0}T$ contains 2 points $a_0 \neq b_0$ such that $p_sT(b_0) = Al_0$; the existence of $n_0 = f(t)$, $a_0 = g(T)$, $b_0 = h(T)$ being obvious, let n_1, a_1, b_1 be determined as $f[b_0, \cdot)_T$, $g[b_0, \cdot)_T$, $h[b_0, \cdot)_T$ respectively; one proceeds by an induction argument: if $k > 0$ is any ordinal $< \omega_0$ such that: the ordinals $n_0 < n_1 < \cdots < n_i$ $(i < k)$, the free points $a_i \in R_{n_i}T)$ $(i < k)$, the linked points $b_i \in R_{n_i}T)$ are determined so that $a_{n_i}||b_{n_i}$, then we determine also n_k, a_k, b_k putting $n_k = f[b_{k-1}, \cdot)_T$, $a_k = g[b_{k-1}, \cdot)_T$, $b_k = f[b_{k-1}, \cdot)_T$. The set $\{a_0, a_1, \dots\}$ is free and has p_sT points.

2:4. MAIN THEOREM. *If p_sT is not a regular limit uncountable cardinal, then p_sT is attained.*

The case when p_sT is finite or of the form Al_{n+1} being obvious, we assume that p_sT is infinite and not of a form Al_{n+1}. Let us consider the set $\{t : t \in T, p_sT(t) < p_sT\} := T_0$. Of course $p_sT_0 \leq p_sT$.

1. *First subcase:* $p_sT_0 = p_sT$; then the first row R_0T_0 is of a power $= p_sT$ because $p_sT = p_sT_0$ and $T_0 = \sum p_sT(t_0);)_{T_0}$ $(t_0 \in R_0T_0)$. If incidentally R_0T_0 is of the power p_sT, all is proved; this occurs in particular if p_s is regular.

1:1. If $|R_0T_0| < p_sT$, p_sT singular, and there is a s'-sequence $(s' := cf\, p_0T)$ a_j $(j < s')$ of points of R_0T_0 such that $\sup [a_j, \cdot]_{T_0} = p_sT$. Let then Al_{k_m} $(m < cf\, s)$ be a strictly increasing $cf\ s$-sequence of alephs of the first kind tending to p_sT. Let f_m be a one-to-one mapping $m < cf\ s \to f_m \in R_0T_0$ such that $p_s[f_m, \cdot)_{T_0} = Al_{k_m}$; the existence of f_m is obvious: by an induction argument one defines $f_0 =$ the first

i such that $p_s[a_i, \cdot) \geq Al_{k_0}$ and for any $0 < i < cf\,s$ let f_i be the least m such that $p_s[a_m, \cdot)_{T_0} > f_i$ $(i < m)$ and $p_s[a_m, \cdot)_{T_0} = Al_{k_m}$. Choosing for every $m < cf\,s$ a free set $A_m \subset ([f_m, \cdot)_{T_0}$ such that $pA_m > Al_{k+1}$, the union $A := \cup_m A_m$ is a requested free subset of T_0 of cardinality $p_s T$.

2. *Second case:* $sT_0 < p_s T$; let $T_1 := T \backslash T_0$; then $T_1 \neq \nu$ and for every $t \in T_1$ one has $p_s T_1(t, \cdot) = p_s T$ — we say that T_1 is width-homogeneous. So the Theorem is reduced to the following:

2:5. THEOREM. *In every width-homogeneous tree T, disregarding the case when $p_s T$ is limit regular not countable, the width is attained.*

Proof. First, T contains a free subset A of cardinality $s' := cf\, p_s T$. This is obvious if s' is isolated; if s' is not isolated, then, by assumption, $s' < p_s T$ and by the definition of p_s as supremum of $|A| < p_s$, a requested A exists. Again, if k_i $(i < s')$ is any strictly increasing s'-sequence of cardinals tending to p_s and a_i any normal well-order of A, then for every $i < s'$ the cone $T(a_i)$ contains a free subset f_i of cardinality $> k_i$ (because T is width-homogeneous); then the union of all f_i $(i < s')$ is a requested free subset of cardinality p_s. Q.E.D.

2:6. COROLLARY. *In every tree of a singular width the width is attained.*

Since the question of attainability of $p_s T$ is reducible to width-homogeneous trees T and since the question is settled for all T having a subchain L or an antichain A of cardinality $\geq s'$ and for all T such that $s' < p_s$, the open remaining case is the following: Every row is $< l$, where l is any limit regular $> Al_0$, and at the same time: every chain is $< l$. For such trees, the question of attainability is open. Do such trees exist? According to our tree axiom, TA, such trees do exist for every l. Consequently, the negation of TA implies the attainability of width in every tree.

Therefore we formulate the following.

2:7. Antichain Tree Hypothesis (ATH). In every tree T the width $p_s T$ is attained.

By 2:3, 2:4, ATH is provable for every T such that $p_s T$ is singular or of countable cofinality, or of the form Al_{n+1}.

In my Doctorial dissertation the following hypothesis was introduced — Ramificatiofn Hypothesis (RH): For every tree T the cardinal bT is attained, where for any ordered set $(E, \leq)$ one defines $b(E, \leq) := \sup |D|$, D running through the system of all d-subsets D of $(E, \leq)$ i.e. such that the corresponding cones $D(a)$ $(a \in D)$ are pairwise disjoint chains.

2:8. THEOREM. $RH \Rightarrow ATH$. *The Ramification Hypothesis implies the Antichain Tree Hypothesis.*

Proof. Since RH is equivalent to the Reduction Princip RH[1)] that every infinite tree is equinumerous to one of his d-subtrees and since ATH was proved for all cases except when T is such that the height $\gamma T = l$, $|R_\alpha| < l$ for every $\alpha < l$ and if each subchain is $< l$, then a d-set D of T such that $|D| = |T|$ is necessarily such that its first row $R_0 D$ is of cardinality $|T|$ — this is implied by the disjoint partition $D = \cup D[d, \cdot)$ $(d \in D)$ and the fact that $D[d, \cdot)$ is a subchain of $(T, \leq)$.

2:9. Disjoint systems of open intervals in ordered chains. I had the opportunity to stress several times that antichains in trees $(T, \leq)$ are closely connected with disjoint systems of intervals of natural, total extensions of order $(T, \leq)$; e.g. any complete bipartition of any ordered chain $(L, \leq)$ yields a tree T of intervals of $(L, \leq)$.

Therefore the previous proof of the main Theorem 2:4 implies my result in the Thèse quoted above as Theorem 0:12. And vice versa: Theorem 0:12 implies Theorem 2:4.

The Antichain Tree Hypothesis is equivalent to

2:10. Disjointnes Chain Hypothesis: For every ordered chain $(L, \leq)$ there exists a disjoint system J of open intervals such that $pJ \geq pX$ for any disjoint system X of open intervals of $(L, \leq)$.

3. Width and Cartesian multiplication

3:0. It is very important and very interesting to see how the p_s-operator behaves with respect to combinatorial (cartesian) multiplication. Since obviously, the cartesian product of any antichains (in a given structure) is again an antichain in the corresponding product and since, in particular, for any antichain A and any index set I one has (1) $p_s A^I = A^{pI}$, one infers that unless $p_s A = 1$, the number (1) might be arbitrarily high. Already the simplest case $p_s(E, \leq) = 1$ of ordered chains $(E, \leq)$ shows situations of maximal change of during the transition from the structure to the square. E.g. for the real line Re one has $p_s\mathrm{Re} = 1$ and $p_s\mathrm{Re}^2 = c =$ power of continuum, because the second diagonal $(x, -x)$ $(x \in \mathrm{Re})$ is an antichain in the square Re^2.

3:1. LEMMA. (i) *For any infinite ordered chain L one has $p_s L^2 \geq Al_0$;* (ii) *If L is order-dense and infinite then L^2 contains also infinite antichains.* (iii) *For any infinite well-ordered (or inversely well-ordered) set W, $p_s W^2 = Al_0$, but every antichain in W^2 is finite.*

Proof. (i) In fact, since L is infinite, one has, for any given integer n a strictly increasing n-sequence a_i $(i < n)$ and a strictly decreasing n-sequence b_i $(i < n)$ of points in L; then the set of all points (a_i, b_i) $(i < n)$ is an antichain in L^2 of power n. (ii) If L is order-dense, then the preceding sequences a_i, b_i could be taken to be infinite and they yield an infinite antichain in L^2. (iii) If W is well-ordered and A is any given antichain in W^2, let $x_0 := \inf pr_1 A$ and let y_0, be the unique point of W such that $a_0 := (x_0, y_0) \in A$. Let us write fulnctionally $a_0 = fA$; if $\{a_0\} \neq A$, let us define $a_1 := f(A \backslash \{a_0\})$, and inductively $a_i := f(A \backslash \{a_0, a_1, \ldots, a_{i-1}\})$ as long as the f-and is $\neq \nu$. But the procedure stops at most after y_0 steps because the second projections $y_0 > y_1 > \ldots$ are strictly decreasing. In fact, if we assume e.g.

[1] In Kurepa 1935 b, c p. 130–133 there were listed 12 mutually equivalent tree propositions $P_1, P_2, \ldots, P_{12}$, of which $P_1 = RH$, $P_2 = RP$; one more equivalent statement P_0 was formulated on p. 93: If T satisfies height $T(a) =$ height T $(a \in T)$ and $p_s T(a) > 1$ $(a \in T)$, then T contains an antichain of cardinality $|\text{height}\, T|$.

$y_1 \le y_2$, then this relation jointly with $x_1 < x_2$ would imply $a_1 < a_2$, contradicting the fact that a_1, a_2 are two members of the antichain A.

3:2. COROLLARY. *An ordered chain L is well-ordered or inversely well-ordered if and only if L^2 contains no infinite antichain.*

2:2. An analogous statement holds for L^n, where $n = 2, 3, 4, \ldots$; whereas the hypercube L^I for any infinite index set I contains an infinite antichain of power 2^I, provided that L contains at least a chain $\{0, 1\}$ of two points 0, 1.

3:4. Left (right) nodes. Given $(E, \le)$; a left node of $(E, \le)$ is any *maximal* subset M such that $x, y \in M$ implies $E(\cdot, x) = E(\cdot, y)$ where $E(\cdot, x) := \{z \mid z \in E \text{ and } z < x\}$. Dually, one defines right nodes of $(E, \le)$. Of course, each node is an antichain.

3:4:1. Let $l(E, \le)$ be the system of all left nodes of $(E, \le)$. Then we have a well-defined system $l(E, \le)$ of antichains in $(E, \le)$ such that $\cup\, l(E, \le) = E$, as well as, for any index set I, a well-determined system of antichains $X^I \backslash D$ $(X \in l(E, \le)$ in the hypercube without the diagonal: $(E, \le)^J \backslash D$, where D denotes the diagonal of the hypercube, i.e. the set of all constant functions from I into E. It is extremely interesting that for trees one has the following:

3:5. THEOREM. *Let* (1) $(T, \le)$ *be a tree and* I *an index set; then the set*

$$AT^I := \bigcup X^I \backslash D, \quad (X \in l(T, \le)) \tag{2}$$

is a maximal antichain (= antibranch) in the cardinal ordering of the hypercube without the diagonal

$$(T, \le)^I \backslash D, \tag{3}$$

where D is the diagonal of the hypercube.

(ii) *The antibranch (2) has a power $\ge pA$ for every antichain A in (3), provided every node of T has at least two points and $pT \ge Al_0$ and $1 < pI < Al_0$ or $pI \ge pT$.*

Proof. First, (2) is an antichain. As a matter of fact, let $f, g \in (2)$, thus $f \in F$, $g \in G$ for well-determined left nodes F, G of (1). Assume that $f \le g$ in (3); then $f_i \in G$ and $f_i \le g_i$ $(i \in I)$; but G is a left node, and $f_i \le G$. Consequently, fI is a singleton of the node F because fI belongs to the G-ideal, of the tree T, thus $f|I \in D$, in contradiction to $f \in (2)$. Let us prove that each member f of (3) is comparable to some member f' of (2). For this, let $L := \cap (T, \le)(\cdot, f_i)$ $(i \in I)$; L is a chain $< f_i$ for every $i \in I$; thus the set $T(L, \cdot)$ of proper majorants of L in T contains f_i for each $i \in I$; let then $N := R_0(L, \cdot)$ be the first row of $(L, \cdot)$, for each $i \in I$ there is a unique $f'_i \in N$ such that $f'_i \le f_i$. The set $f' \le I$ of all f'_i $(i \in I)$ has at least two points; in the opposite case, if $f'I$ were a singleton $\{h\}$, one would have (4) $h \le f_i$ $(i \in I)$, thus $h = f_i$ $(i \in I)$ because if in (4) we had $<$ instead of $\le$, the point h would be in L in contradiction to $N \cap I = \nu$. Since, $pf'I > 1$, f' is a member of $N^I \backslash D$ and all is proved.

3:6. A deleting operation in trees. Let T be a given tree; let p be any most extensive path of T which is the union of monopunctual nodes of T; we replace p by its last member gp if p has a such one; in the opposite case, we replace p by its first member gp. The result of such a substitution in T will be denoted by T_n. For example, if T is well-ordered, then T_n is a singleton consisting of the last point of T if this point exists; otherwise $T_n = R_0T$. If every knot of T has as least two points, then $T = T_n$. Anyway, we have a mapping g which associates with p a point $gp \in p$; we also have a self-mapping g of $\cup_p p$.

3:6:1. LEMMA. *The mapping g preserves the incomparability relation: if a, b are $||$ in $\cup p$, so $ga||gb$ and one has $p_sT = p_sT_n$.*

Proof. Let a, b be incomparable in $M := \cup p$; then $ga||gb$. As a matter of fact let p, q be summands of M such that $a \in p$, $b \in q$. Since $a||b$, we infer that $p||q$; in the opposite case, there would be comparable points $c \in p$, $d \in q$; one does not have $c = d$ because p, q are disjoint. Assume $c < d$.

First case: a is last in p, b is last in q; thus $c \le a$, $d \le b$; since all members of p are monoknots, we infer the rows of $T(c,\cdot)$ are monoknots for at least the rank γa; thus in particular $a \le d$ and $a \le b$, contrarily to $a||b$.

Second case: a is last in p, b is first in q. Again one infers that $c < d$ would imply $a \le d$, and this with $b < d$ would mean that the point d would be preceded by incomparable points a, b — absurdity.

Third case: a is first in p, b is last in q. This case is not possible because we would have $a \le c < d \le b$ i.e. $a < b$, in contradiction to $a||b$.

Forth case: a is first in p, b is first in q — not possible, because otherwise d would be preceded by free points a, b.

The equality in 3:6:1 is implied by the fact that g carries every antichain A of T in an isomorphic antichain gA in T_n. Therefore, let us examine T_n. Let Z be the set of all terminal points of T_n. If $pZ = p_sT_n$, all is settled, because Z is an antichain. If $pZ < p_sT_n$, then the complement $U = T_n \backslash Z$ satisties $p_sU = p_sT_n$ and every node of U has at least two points. Now, the identical partition $U = \cup X$, $(X \in lU)$ implies $pU = \sum pX$ $(X \in lU)$ the last number is $\le \sum (pX)^2 p \le (U \times U) = pU$; thus in this case the antichain (2) has pU $(= pT)$ points and is not only maximal but also maximum. The same holds for every finite I of power > 1. Again if $pI \ge pT$, then the identity $T^I = (\cup X)^I$ implies for cardinals $(pT)^{pJ} = 2^{pJ} = (\sum_x pX)^{pJ} \ge \sum_x 2^{pJ}$ (X running through lU); consequently, in this case also the cardinality of (2) is again $(pT)^{pJ}$ and the antichain (2) is again maximum.

Probably the statement (ii) holds for every I of power > 1.

REFERENCES

[1943] P. Erdos and A. Tarski, *Families of mutually exclusive sets*, Ann. of Math 44 (1943), 315–323.

[1961] P. Erdos and A. Tarski, *On some problems involving inaccessible cardinals*, in: *Essays on the Foundations of Mathematics*, Jerusalem, 1961, pp. 50–82.

[1934c] Đ. Kurepa, *Tableaux ramifies d'ensembles. Espaces-pseudo-distancies*, C. R. Acad. Sci. Paris **198** (1934), 1563–1565.

[1934d] Đ. Kurepa, *Tableaux ramifies d'ensembles*, C. R. Acad. Sci. Paris **199** (1934), 112–114.

[1935b,c] Đ. Kurepa, *Ensembles ordonné et rifiés*, Thèse de doctorat souterue à la Facltè des Science a Paris le 1935;12;19; pp. Vl+138+11; resp. Publ. Math. Univ. (Beograd) **4** (1935), 1–138.

[1936a] Đ. Kurepa, *L'hypothèse de ramification*, C. R. Acad. Sci. Paris **202** (1936), 185–187.

[1937e] Đ. Kurepa, *L'hypothèse du continu et les ensembles partiellement ordonnés*, C. R. Acad. Sci. Paris **205** (1937), 1196–1198.

[1939b] Đ. Kurepa, *Sur la puissance des esembles partiellement ordonnés*, C. R. Soc. Varsovie, Classe Math. **32.** (1939) 61–69; reprinted in Glasnik Mat. Fiz. Astr. Zagreb (2) **14** (1959), 205–211.

[1985a] Đ. Kurepa, *A tree axiom*, Publ. Inst. Math. (Beograd) **38** (52), (1985), 7–11.

[1954] R. Rado, *Partial well-ordering of sets of vectors*, Mathematika, (London) **1** (1954), 89–95.

Laze Simića 9
11000 Beograd
Jugoslavija

(Received 27 05 1987)