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# ON A FUNCTIONAL WHHICH IS QUADRATIC ON A-ORTHOGONAL VECTORS

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**Abstract**. Let $X$ be a complex Hilbert space, $\dim X \geq 3$ and $A$ be a bounded selfadjoint operator defined on $X$. We give a representation of a continuous functional $H$ defined on $X$ under the condition that $H$ is quadratic on $A$-orthogonal vectors.

In [3] a continuous functional $F : X \to \Phi$ is studied which is additive on $A$-orthogonal vectors. Let us note that the square of functional which is additive on $A$-orthogonal vectors does not have to be quadratic. The purpose of this paper is to give a representation of the functional $H : X \to \Phi$ under the condition that is quadratic on $A$-orthogonal vectors. In [2] a representation is given in the case when $A = I$ ($I$ denotes the identical operator).

The following theorem will be proved:

THEOREM 1. *Let $H$ be a continuous functional defined on a (real or complex) Hilbert space $X$ with $\dim X \geq 3$. Suppose that if $(x, Ay) = 0$ $(x, y \in X)$ then*

$$H(x+y) + H(x-y) = 2H(x) + 2H(y), \tag{$*$}$$

*where $A : X \to X$ is a continuous selfadjoint operator with $\dim A(X) \neq 1, 2, 3$. Then there is a continuous linear operator $B$ and quasi-linear continuous operator $C$ and $D$ such that*

$$H(x) = (Bx, x) + (Cx, x) + (x, Dx) \tag{$**$}$$

*for all $x \in X$.*

We will use the same technique as in [2] and the proof of the theorem will be based upon the following lemmas.

LEMMA 1. *Under the hypotheses of Theorem 1 there exist functionals $B(x)$, $C(x)$ and $D(x)$ (defined on $X$) satisfying $(*)$ such that for all complex numbers $\lambda$ and for all $x$ in $X$:*

$$B(\lambda x) = |\lambda|^2 B(x), \quad C(\lambda x) = \lambda^2 C(x), \quad D(\lambda x) = \bar{\lambda}^2 D(x)$$

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*Moreover,* $H(x) = B(x) + C(x) + D(x)$.

*Proof.* We first show that for the functional $H(x)$ we have $H(rx) = r^2 H(x)$ for all $x \in X$, where $r$ is a real number. It is obvious that $H(0) = 0$.

1° Let $(x, Ax) = 0$ for some $x \in X$ $(x \neq 0)$; then, applying relatio $(*)$, we obtain $H(2x) = 2^2 H(x)$. Thus

$$H(3x) + H(x) = 2H(2x) + 2H(x), \qquad H(3x) = 2H(2x) + H(x)$$
$$= 2 \cdot 2^2 H(x) + H(x), \quad H(3x) = 3^2 H(x).$$

Similarly we obtain $H(4x) = 4^2 H(x)$, $H(5x) = 5^2 H(x), \ldots$. Suppose that $H(nx) = n^2 H(x)$ holds for a natural number $n$. We shall prove that $H[(n+1)x] = (n+1)^2 H(x)$. For this we have:

$$H[(n+1)x] + H[(n-1)x] = 2H(nx) + 2H(x)$$
$$H[(n+1)x] = 2H(nx) + 2H(x) - H(n-1)x$$
$$= 2n^2 H(x) + 2H(x) - (n-1)^2 H(x).$$
$$H[(n+1)x] = [2n^2 + 2 - (n-1)^2]H(x), \quad H[(n+1)x] = (n+1)^2 H(x).$$

Thus, $H(nx) = n^2 H(x)$ holds for all natural $n$.

Similarly we obtain $H(nx) = n^2 H(x)$, if $n = -1, -2, -3, \ldots$. It also follows easily (because of the continuity of $H$) that $H(rx) = r^2 H(x)$ for all real $r$.

2° Let $(Ax, x) \neq 0$. Then there exist a $y \in X$ $(y \neq 0)$ such that $(x, Ay) = 0$ and $(Ay, y) = \pm(Ax, x)$.

(a) If $(Ay, y) = (Ax, x)$, then the vectors $nx+y$ and $x-ny$ are pairwise $A$-orthgonal. According to $(*)$ we can write

$$H[(nx+y) + (x-ny)] + H[(nx+y) - (x-ny)] = 2H(nx+y) + 2H(x-ny), \quad (1)$$
$$H[(n+1)x - (n-1)y] + H[(n-1)x + (n+1)y] = 2H(nx+y) + 2H(x-ny), \text{ or}$$
$$H[(n+1)y - (n-1)x] + H[(n-1)y + (n+1)x] = 2H(ny+x) + 2H(y-nx) \quad (2)$$

If we add (1) and (2) and take into consideration $(*)$, we get

$$2H[(n+1)x] + 2H[(n-1)y] + 2H[(n-1)x] + 2H[(n+1)y]$$
$$= 4H(nx) + 4H(y) + 4H(x) + 4H(ny)$$

or

$$H[(n+1)x] + H[(n-1)y] + H[(n-1)x] + H[(n+1)y]$$
$$= 2H(nx) + 2H(y) + 2H(x) + 2H(ny).$$

Let

$$H(kx) + H(ky) = k^2[H(x) + H(y)] \quad (3)$$

hold for all $k = 0, 2, 3, \dots, n$. It is easy to prove that (3) is true for $n = k+1$. In [1] it has been proved that there exists a $z \in X$ such that $(x, Az) = (y, Az) = 0$ and $(Ax, x) = (Ay, y) = (Az, z)$, and on the basis of (3) we can write

$$H(nx) + H(ny) = n^2[H(x) + H(y)] \tag{3'}$$

$$H(nx) + H(nz) = n^2[H(x) + H(z)] \tag{3''}$$

$$H(ny) + H(nz) = n^2[H(y) + H(z)]. \tag{3'''}$$

Subtracting (3‴) from (3″), we obtain $H(nx) - H(ny) = n^2[H(x) - H(y)]$, which together with (3′) gives $H(nx) = n^2 H(x)$. Due to the continuity of the functional $H$, $H(rx) = r^2 H(x)$ holds for all real numbers $r$.

(b) Let $(Ay, y) = -(Ax, x)$. It follows that $(A(x \pm y, x \pm y) = 0$ and according to 1° we get $H[n(x+y)] = n^2 H(x+y)$, $H[n(x-y)] = n^2 H(x-y)$. Besides that we have $H[n(x+y)] + Hn(x-y)] = 2H(nx) + 2H(ny)$ or

$$n^2 H(x+y) + n^2 H(x-y) = 2H(nx) + 2H(ny). \tag{4}$$

In [1] it has been shown that there exists a $z \in X$ such that $(Ax, z) = 0$ and $(Az, z) = -(Ax, x), (Ay, z) = 0$, $(Ay, y) = (Az, z)$. On the basis of (a) we can write

$$n^2 H(y+z) + n^2 H(y-z) = 2H(ny) + 2H(nz) = 2n^2 H(y) + 2n^2 H(z) \tag{5}$$

or

$$n^2 H(x+z) + n^2 H(x-z) = 2H(nx) + 2H(nz). \tag{6}$$

If we subtract (5) from (6), we get

$$n^2 H(x+z) + n^2 H(x-z) - n^2 H(y+z) - n^2 H(y-z) = 2H(nx) - 2H(ny).$$

If we add this last relation to (4) we obtain

$$\begin{aligned} n^2 H(x+z) + n^2 H(x-z) - n^2 H(y+z) - n^2 H(y-z) + n^2 H(x+y) \\ + n^2 H(x-y) = 4H(nx) \end{aligned}$$

or

$$2n^2 H(x) + 2n^2 H(z) - 2n^2 H(y) - 2n^2 H(z) + 2n^2 H(x) + 2n^2 H(y) = 4H(nx)$$

or $H(nx) = n^2 H(x)$. Since the functional $H$ is continuous then $H(rx) = r^2 H(x)$ holds for all real numbers $r$. Therefore $H(rx) = r^2 H(x)$ holds for all real numbers $r$ nad for each $x \in X$.

Let $2B(x) = H(ix) + H(x)$. It is easy to see that $B(x)$ is a continuous and quadratic functional on $A$-orthogonal vectors, as well as it satisfies $B(rx) = r^2 B(x)$, that $B(ix) = B(x)$.

1° Let $(Ax, x) = 0$ for some $x \in X$. Then $(A\alpha x, i\beta x) = 0$ ($\alpha$, $\beta$ real numbers). For $\lambda = \alpha + i\beta$ we have

$$\begin{aligned} B(\lambda x) + B(\bar{\lambda} x) &= B((\alpha + i\beta)x) + B((\alpha - i\beta)x) = B(\alpha x + i\beta x) + B(\alpha x - i\beta x) \\ &= 2B(\alpha x) + 2B(i\beta x) = 2\alpha^2 B(x) + 2\beta^2 B(ix) \\ &= 2\alpha^2 B(x) + 2\beta^2 B(x) = 2(\alpha^2 + \beta^2) B(x) = 2|\lambda|^2 B(x). \end{aligned}$$

Hence $B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x)$.

2° Let $(Ax, x) \neq 0$. Then there exists a $y \in X$ such that $(x, Ay) = 0$ and $(Ay, y) = \pm(Ax, x)$. Let us consider the case when (a) $(Ay, y) = (Ax, x)$. Then if $\lambda = \alpha + i\beta$ ($\alpha$, $\beta$ real) and $e_1 = (x+y)/2$, $e_2 = (x-y) \mid 2i$, it follows that

$$B(\lambda x) + B(\lambda y) + B(\bar{\lambda} x) + B(\bar{\lambda} y) = 2|\lambda|^2 [B(x) + B(y)].$$

We can select a $z \in X$ such that $(x, Az) = 0$, $(y, Az) = 0$ and $(x, Ax) = (y, Ay) = \pm(z, Az)$. Let us consider the case when the sign is $\pm$. By analogy with the equation above, we can write the following.

$$B(\lambda x) + B(\lambda z) + B(\bar{\lambda} x) + B(\bar{\lambda} z) = 2|\lambda|^2 [B(x) + B(z)]$$
$$B(\lambda y) + B(\lambda z) + B(\bar{\lambda} y) + B(\bar{\lambda} z) = 2|\lambda|^2 [B(y) + B(z)].$$

From the last three equalities we have $B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x)$.

Let us consider the case when (b) $(Ay, y) = -(Ax, x)$. Then $(A(x \pm y), x \pm y) = 0$. On the basis of 1° we have

$$B(\lambda(x+y)) + B(\bar{\lambda}(x+y)) = 2|\lambda|^2 B(x+y)$$
$$B(\lambda(x-y)) + B(\bar{\lambda}(x-y)) = 2|\lambda|^2 B(x-y).$$

Summing these two equations we obtain

$$B(\lambda(x+y)) + B(\lambda(x-y)) + B(\bar{\lambda}(x+y)) + B(\bar{\lambda}(x-y)) = 2|\lambda|^2 (B(x+y) + B(x-y))$$

or

$$2B(\lambda x) + 2B(\lambda y) + 2B(\bar{\lambda} x) + 2B(\bar{\lambda} y) = 4|\lambda|^2 B(x) + 4|\lambda|^2 B(y)$$

or

$$B(\lambda x) + B(\lambda y) + B(\bar{\lambda} x) + B(\bar{\lambda} y) = 2|\lambda|^2 (B(x) + B(y)). \tag{7}$$

As before, there exists a $z \in X$ such that $(Ax, z) = (Ay, z) = 0$, $(Ay, y) = (Az, z)$ and $(Az, z) = -(Ax, x)$. We have

$$B(\lambda x) + B(\lambda z) + B(\bar{\lambda} x) + B(\bar{\lambda} z) = 2|\lambda|^2 (B(x) + B(y)) \tag{8}$$
$$B(\lambda y) + B(\lambda z) + B(\bar{\lambda} y) + B(\bar{\lambda} z) = 2|\lambda|^2 (B(y) + B(z)) \tag{9}$$
$$B(\lambda x) - B(\lambda y) + B(\bar{\lambda} x) - B(\bar{\lambda} y) = 2|\lambda|^2 B(x) - 2|\lambda|^2 B(y). \tag{10}$$

From (7) and (10) it follows that $B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x)$. Thus from these considerations we can conclude that for each $x \in X$ and each complex $\lambda$ we have

$$B(\lambda x) + B(\bar{\lambda} x) = 2|\lambda|^2 B(x) \tag{11}$$

If in (11) we replace $\lambda$ by $e^{i\varphi}$ ($\varphi$ real) and $ix$ by $e^{i\varphi} x$, we obtain $B(e^{2i\varphi} x) + B(x) = 2B(e^{i\varphi} x)$. Similarly we get $B(e^{-2i\varphi} x) + B(x) = 2B(e^{-i\varphi} x)$. Thus we have

$$B(e^{2i\varphi} x) - B(e^{-2i\varphi} x) = 2[B(e^{i\varphi} x) - B(e^{-i\varphi} x)]. \tag{12}$$

For fixed $x \in X$ let us set

$$I(\alpha) = B(\alpha x) - B(\alpha^{-1}x) \quad (\alpha = e^{i\varphi}). \tag{13}$$

It is easy to show that $I(\alpha) = 0$ for all $\alpha = e^{i\varphi}$ ($\varphi$ real). From this fact it follows that $B(\bar{\lambda}x) = B(\lambda x)$, and from that (due to (11)) we have

$$B(\lambda x) = |\lambda|^2 B(x) \quad (x \in X \text{ and } \lambda\text{-complex}).$$

Let us put

$$2S(x) = H(ix) - H(x). \tag{15}$$

The functional $S(x)$ is continuous, quadratic on $A$-orthogonal vectors and quadratic homogenous, i.e. $S(rx) = r^2 S(x)$, and besides that

$$S(ix) = -S(x), \quad (x \in X). \tag{16}$$

In the same way as with the functinal $B(x)$, we obtain

$$S(\lambda x) + S(\bar{\lambda}x) = (\lambda^2 + \bar{\lambda}^2)S(x) \tag{17}$$

for each $x$ in $X$ and for each $\lambda$. If in (17) we put $\lambda = \alpha$ ($|\alpha| = 1$, $\alpha^{4n} \neq 1$, $n = 1, 2, \dots$) and $\alpha x$ instead of $x$, we obtain

$$S(\alpha^2 x) + S(x) = (\alpha^2 + \bar{\alpha}^2)S(x) \tag{17'}$$

or

$$\alpha^4/(\alpha^8 - 1) \cdot [S(\alpha^2 x) - \bar{\alpha}^4 S(x)] = \alpha^2/(\alpha^4 - 1) \cdot [S(\alpha x) - \alpha^2 S(x)].$$

By induction we can prove

$$\alpha^{2n}/(\alpha^{4n} - 1) \cdot [S(\alpha^n x) - \alpha^{2n} S(x)] = \alpha^2/(\alpha^4 - 1) \cdot [S(\alpha x) - \bar{\alpha}^2 S(x)].$$

If $\beta = \alpha^n$, then

$$\beta^2/(\beta^4 - 1) \cdot [S(\beta x) - \bar{\beta}^2 S(x)] = \alpha^2/(\alpha^4 - 1) \cdot [S(\alpha x) - \bar{\alpha}^2 S(x)]$$

or

$$1/(\beta^2 - \bar{\beta}^2) \cdot [S(\beta x) - \bar{\beta}^2 S(x)] = 1/(\alpha^2 - \bar{\alpha}^2) \cdot [S(\alpha x) - \bar{\alpha}^2 S(x)]. \tag{17''}$$

In the last relation $\alpha$ and $\beta$ are arbitrary numbers such that $|\alpha| = |\beta| = 1$, $\alpha^4 \neq 1$ and $\beta^4 \neq 1$ and (17) holds for each $x$ in $X$. Since $S(rx) = r^2 S(x)$, from (17″) it follows immediately that

$$[S(\lambda x) - \bar{\lambda}^2(x)]/(\bar{\lambda}^2 - \lambda^2) = [S(\lambda_1 x) - \bar{\lambda}_1^2 S(x)]/(\lambda_1^2 - \bar{\lambda}_1^2) \tag{17'''}$$

($\lambda^2 \neq \bar{\lambda}^2$, $\lambda_1^2 = \bar{\lambda}_1^2$) for all $x$ in $X$.

The right-hand side of relation (17‴) is constant, for any $\lambda$ ($\lambda^2 = \bar{\lambda}^2$) and if for fixed $\lambda_1$ we put

$$C(x) = [\bar{\lambda}_1^2 S(x) - S(\lambda_1 x)]/(\lambda_1^2 - \bar{\lambda}_1^2)$$

we obtain $C(\lambda x) = [\bar{\lambda}_1^2 S(\lambda x) - S(\lambda_1 \lambda x)]/(\lambda_1^2 - \bar{\lambda}_1^2)$. According to (17‴) we conclude that $C(x)$ and $C(\lambda x)$ do not depend on $\lambda_1$, and if we put $\lambda_1 = \lambda$ (in the relation for $C(x)$), $\lambda_1 = \bar{\lambda}$ (in the relation for $C(\lambda x)$), we obtain $C(\lambda x) = \lambda^2 C(x)$, for each complex $\lambda$ and $x$ in $S$. Let us put $D(x) = -S(x) - C(x) = [S(\lambda_1 x) - \lambda_1^2 S(x)]/(\lambda_1^2 - \bar{\lambda}_1^2)$. Then it follows that $D(\lambda x) = \bar{\lambda}^2 D(x)$ ($x$ in $X$, $\lambda$ a complex number). Since $H(x) = B(x) - S(x)$ and $-S(x) = C(x) + D(x)$ it follows that $H(x) = B(x) + C(x) + D(x)$ Q.E.D.

LEMMA 2. *Suppose that the functional $H$ satisfies the conditions of THeorem 1 and that*

$$H(\lambda x) = |\lambda|^2 H(x) \tag{18}$$

*for all in $X$ and for every complex number $\lambda$. Then there exists a unique continuous linear operator $B$ such that for all $x$ in $X$*

$$H(x) = (Bx, x). \tag{19}$$

*Proof.* Let us put

$$F(x,y) = H(x+y) - H(x-y) \quad (x, y \text{ in } X) \tag{20}$$

Let further

$$X_y = \{x \mid x \in X,\ (Ax, y) = 0\}. \tag{21}$$

For a fixed $y$ and for $x$ in $X$, $F(x,y)$ is a continuous functional (on $X$) and moreover from $(x, Az) = 0$, $x$, $z$ in $X$ it follows that $F(x+z, y) = F(x,y) + F(z,y)$. On the basis of [3] there exist unique vectors $a_y$ and $b_y$ in $X_y$ and a unique complex number $\alpha_y$ such that

$$F(x,y) = 2(a_y, x) + 2(x, b_y) + 2\alpha_y (Ax, x) \tag{22}$$

for all $x$ in $X_y$. Since the functional $H$ is quadratic on $A$-orthogonal vectors we have

$$H(x+y) = H(x) + H(y) + (a_y, x) + (x, b_y) + \alpha_y (Ax, x), \quad ((Ax, y) = 0). \tag{23}$$

1° Let $x \in X$ be such that $(Ax, x) = 0$. Then the relation (23) has a form

$$H(x+y) = H(x) + H(y) + (a_y, x) + (x, b_y), \quad ((Ax, y) = 0).$$

2° Let $x \in X$ be such that $(Ax, x) \neq 0$. Then due to the continuity of the functional $F$ we conclude that $\alpha_y = 0$, and relation (23) becomes

$$H(x+y) = H(x) + H(y) + (a_y, x) + (x, b_y), \quad ((Ax, y) = 0). \tag{23'}$$

We can write the space $X$ as the direct sum of orthogonal and $A$-orthogonal invariant subspaces $X^0$, $X^-$, $X^+$ of the operator $A$, where $X^0 = \{(\in X \mid Ax = 0\}$. In $X^-$ it holds that $(Ax, x) < 0$ for $x \neq 0$, and in $X^+$ it holds that $(Ax, x) > 0$ for $x \neq 0$. In each of these subspaces we can select a maximal $A$-orthonormal system. Let $\{e_i\}$ be a maximal $A$-orthonormal system in the space $X$, which is equal to the union of these maximal $A$-orthonormal systems. Let us take an arbitrary $x$ in $X$; then $x = \sum_{i=1}^{\infty} \alpha_i e_i$. Let us put $x_n = \sum_{i=1}^{n} \alpha_i e_i$. Applying relation (23′) we obtain $H(x_n) = H_1(x_n) + H_2(x_n)$ where $H_1(x_n) = \sum_{i=1}^{n} |\alpha_i|^2 H(e_i)$; $H_2(x_n) = \sum_{k=1}^{n-1}[(\alpha_k a_k,\, \bar{x}_{k+1}) + (\bar{x}_{k+1}, \alpha_k b_k)]$, $a_k = a_{e_k}$, $b_k = b_{e_k}$, $\bar{x}_k = \sum_{i=k}^{n} \alpha_i e_i$ $(1, 2, \ldots, n-1)$. We claim that $H_1(x_n)$ and $H_2(x_n)$ are quadratic on vectors of the form $x_n = \sum_{i=1}^{n} \alpha_i e_i$. Let $x_n = \sum_{i=1}^{n} \alpha_i e_i$, $y_m = \sum_{i=1}^{m} \beta_i e_i$ (Set $n = \max\{n, m\}$). Then[1]

$$\begin{aligned}
&H_1(x_n + y_m) + H_1(x_n - y_m) = H_1(\textstyle\sum(\alpha_i + \beta_i)e_i) + H_1(\sum(\alpha_i - \beta_i)e_i) \\
&= \textstyle\sum(|\alpha_i + \beta_i|^2 H(e_i) + \sum|\alpha_i - \beta_i|^2 H(e_i) = \sum[|\alpha_i + \beta_i|^2 + |\alpha_i - \beta_i|^2]H(e_i) \\
&= \textstyle\sum(2|\alpha_i|^2 + 2|\beta_i|^2)H(e_i) = 2\sum|\alpha_i|^2 H(e_i) + 2\sum|\beta_i|^2 H(e_i) = 2H_1(x_n) + 2H_1(y_m).
\end{aligned}$$

Thus, $H_1(x_n + y_m) + H_1(x_n - y_m) = 2H_1(x_n) + 2H_1(y_m)$. Similarly it can be proved that

$$H_2(x_n + y_m) + H_2(x_n - y_m) = 2H_2(x_n) + 2H_2(y_m).$$

Therefore for all vectors $x_n = \sum\alpha_i e_i$, $y_m = \sum\beta_i e_i$,

$$H(x_n + y_m) + H(x_n - y_m) = 2H(x_n) + 2H(y_m).$$

Thus the functional $H$ is quadratic on the set $S = \{x_n \mid x_n = \sum\alpha_i e_i,\ e_i - A\text{-orthonormal vectors}\}$. Taking into consideration that the set $S$ is everywhere $X$-dense and that $H$ is a continuous functional, the equation $H(x + y) + H(x - y) = 2H(x) + 2H(y)$ holds for $x$, $y$ in $X$. Hence Lemma 2 follows from (18) and the continuity of $H$. Q.E.D.

LEMMA 3. *If the functional $H$ satisfies the conditions of Theorem 1 and moreover*

$$H(\lambda x) = \lambda^2 H(x) \quad (\text{or } H(\lambda x) = \bar{\lambda}^2 H(x))$$

*holds for every complex number $\lambda$ and all $x \in X$, then*

$$H(x + y) + H(x - y) = 2H(x) + 2H(y) \text{ holds for all } x, y \in X.$$

*Proof.* 1° Let $(Ax, y) = 0$ for some $x$, $y$ in $X$. Then due to the hypothesis the statement holds.

2° Let $(Ax, y) \neq 0$ for some $x$, $y$ in $X$[2]. We can suppose that $(Ax, x) \neq 0$. Then there exists a $z \in X$ such that $(Az, z) \neq 0$ and $(Ax, z) = 0$, $(Ay, z) = 0$. We can write $H(x + z) + H(x - z) = 2H(x) + 2H(z)$, $H(x + iz) + H(x - iz) = 2H(x) - 2H(z)$.

Thus

$$4H(x) = H(x + z) + H(x - z) + H(x + iz) + H(x - iz). \tag{24}$$

[1] $\sum$ means $\sum_{i=1}^{n}$.

[2] We can suppose that $(Ax, x) \neq 0$

Let us select the number $\alpha$ such that $(Ax+y), x-y) + \bar{\alpha}(Az, z) = 0$. Taking this condition into consideration we obtain

$$(A(x+y+z), x-y+\alpha z) = 0, \quad (A(x+y-z), x-y-\alpha z) = 0$$
$$(A(x+y+iz), x-y+\alpha iz) = 0, \quad (A(x+y-iz), x-y-\alpha iz) = 0.$$

Applying relation (24) we get

$$4H(x+y) = H(x+y+z) + H(x+y-z) + H(x+y+iz) + H(x+y-iz)$$
$$4H(x-y) = H(x-y+z) + H(x-y-z) + H(x-y+iz) + H(x-y-iz).$$

Now making use of the fact that the functional $H$ is quadratic on $A$-orthogonal vectors we obtain

$$H(x+y) + H(x-y) = 2H(x) + 2H(y).$$

This holds when $(A(x+y), x-y) \neq 0$. If $(A(x+y), x-y) = 0$, the statement obviously holds. From 1° and 2° we conclude that $H(x+y)+H(x-y) = 2H(x) + 2H(y)$ holds for $x, y \in X$. Now, let us consider the functional $H$ with the property $H(\lambda x) = \lambda^2 H(x)$. As with the proof of Lemma 2 it is also easy to show that

$$F(x,y) = 2(a_y, x) + 2(x, b_y) \tag{25}$$

and that $F(x,y) = H(x+y) - H(x-y)$. Relation (25) holds for all $x, y \in X$. Besides that

$$F(x,y) = H(x+y) - H(x-y) = H(y+x) - H(y-x) = F(y,x)$$

and

$$\begin{aligned} F(x_1+x_2, y) &= 2(a_y, x_1+x_2) + 2(x_1+x_2, b_v) \\ &= 2(a_y, x_1) + 2(x_1, b_y) + 2(a_y, x_2) + 2(x_2, b_y) \\ &= F(x_1, y) + F(x_2, y). \end{aligned}$$

Thus

$$F(x_1+x_2, y) = F(x_1, y) + F(x_2, y), \quad F(x, y_1+y_2) = F(x, y_1) + F(x, y_2).$$

It is now easy to obtain

$$a_{y_1+y_2} = a_{y_1} + a_{y_2} \quad b_{y_1+y_2} = b_{y_1} + b_{y_2} \tag{26}$$
$$a_{\lambda y} = \bar{\lambda}^2/\lambda \cdot a_y \qquad (\lambda \text{ – complex number} \neq 0)$$
$$b_{\lambda y} = \bar{\lambda} b \tag{27}$$

since $a_y = 0$ for $y$ in $X$. Thus $H(x+y) - H(x-y) = (x, b_y)$ holds for all $x, y \in X$. For $x = y$, $H(2x) = (x, b_x)$ or $H(x) = (x, Dx)$ where $Dx = b_x/4$ and $D$ is a quasi-linear operator. For $H(x)$ instead of $H(\lambda x) = \lambda^2 H(x)$, the condition $H(\lambda x) = \bar{\lambda}^2 H(x)$, should be added and it is easy to show in this way that $H(x) = (Cx, x)$, and that $C$ is a quasi-linear operator. Continuity of $D$ and $C$ is clear. So, we obtain

LEMMA 4. *If the functional $H$ satisfies the conditions of Lemma 2, there exists a unique continuous quasi-linear operator $D(C)$ such that*

$$H(x) = (x, Dx), \quad (H(x) = (Cx, x)).$$

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