THE STRUCTURE ON A SUBSPACE OF A SPACE
WITH AN $f(3,-1)$-STRUCTURE

Jovanka Nikić

Abstract. Let $\mathcal{M}^n$ be a manifold with an $f(3,-1)$-structure of rank r and let $\mathcal{N}^{n-1}$ be a hypersurface in $\mathcal{M}^n$. The following theorem is proved: If the dimension of $T(\mathcal{V}^{n-1} \cap f(T\mathcal{N}^{n-1}))_p$ is constant, say s, for all $p \in \mathcal{N}^{n-1}$, then $\mathcal{N}^{n-1}$ possesses a natural $F(3,-1)$-structure of rank s. It is also proved that the naturally induced $F(3,-1)$-structure is integrable if the $f(3,-1)$-structure on $\mathcal{M}^n$ is integrable and if the transversal to $\mathcal{N}^{n-1}$ can be found to lie in the distribution M.

Introduction. Let $\mathcal{M}^n$ be an n-dimensional C^∞ manifold and f a tensor of type (1,1) such that $f^3 \pm f = 0$, and let the rank of f be constant, say $r < n$, on $\mathcal{M}^n$. We then say that $\mathcal{M}^n$ has an $f(3,\pm 1)$ structure of rank r.

Yano and Ishihara [5] have shown that if $\mathcal{M}^n$ is an almost complex manifold, then a submanifold of $\mathcal{M}^n$ satisfying a certain property, possesses a natural $f(3,+1)$-structure. In particular, Tashiro [4] has shown that if the submanifold is a hypersurface, then the induced $f(3,+1)$-structure has a maximal rank (i.e. is an almost contact structure).

It is known that a hypersurface of an almost contact manifold possesses a natural $f(3,+1)$-structure, which does not have a maximal rank. In [2] Miyzawa proved that if the submanifold in the manifold with an almost product structure is a hypersurface, then the induced $f(3,-1)$-structure has a maximal rank. (i.e. is an almost paracontact structure.)

The purpose in this paper is to show that if $\mathcal{M}^n$ has an $f(3,-1)$-structure, then a hypersurface of this manifold possesses a natural $F(3,-1)$-structure. In section 3 we shall study the integrability of the induced $f(3,-1)$-structure.

Prelimininaries. The (ψ, ξ, η)-structure is an almost paracontact structure if we have on $\mathcal{M}^n$ a tensor field ψ of type (1,1), a vector field ξ, and a 1-form η satisfying:

$$\psi^2 = I - \eta \otimes \xi, \ \ \eta\psi = 0, \ \ \psi\xi = 0, \ \ \eta(\xi) = 1.$$

In [**2**] the following theorem is proved:

THEOREM 1.1. *Let* $T(\mathcal{M}^n)_p$ *denote the tangent space to* $\mathcal{M}^n$ *at* p. *The almost product structure* Φ *induces the almost paracontact structure* ψ *on a hypersurface* $\mathcal{K}^n$ *in the following way:* $\Phi B = B\psi \oplus (\eta \oplus N)$, $\Phi N = B\xi$, *where* B *is the differential of the immersion* $i\,\mathcal{K}$ *into* $\mathcal{M}^n$, *and* $N \in (\mathcal{M}^n)_p$ $N \notin T(\mathcal{K})_p$, *for all* $p \in \mathcal{K}$.

THEOREM 1.2. *If is an* $f(3,-1)$-*structure on* , $\mathcal{M}^n$ *the operators* $l = f^2$, $m = I - f^2$, I *denoting the identity operator, applied to the tangent space at a point of the manifold, are complementary projection operators.*

There exist complementary distributions L and M corresponding to the projection operators l and m, respectively.

2. The structure on the hypersurface.

THEOREM 2.1 *Let* $\mathcal{M}^n$ *be a manifold with* $f(3,-1)$-*structure of rank* r *and let* $\mathcal{N}^m$ *be a hypersurface in* $\mathcal{M}^n$, $m = n - 1$. *It the dimnsion of* $T(\mathcal{N}^m)_p \cap f(T(\mathcal{N}^m))_p$ *is constant, say* s, *for all* $p \in \mathcal{N}^m$, *then* $\mathcal{N}^m$ *possesses a natural* $F(3,-1)$-*structure of rank* s.

Proof. Let C be a transversal defined on $\mathcal{N}^m$, i.e. $C \in T\ (\mathcal{M}^n)_p$ but $C \notin T(\mathcal{N}^m)_p$ for all $p \in \mathcal{N}^m$. Let B be a differential of the imbedding of $\mathcal{N}^m$ in $\mathcal{M}^n$. Then B is a map of $T(\mathcal{N}^m)$ into $T_R(\mathcal{M}^n)$, where $T_r(\mathcal{M}^n)$ denotes the restriction of $T(\mathcal{M}^n)$, the tangent bundle of $\mathcal{M}^n$ to $\mathcal{N}^m$. Then we can find a locally 1-form C^* defined on $\mathcal{N}^m$ such that:

$$B^{-1}B = I, \qquad\qquad BB^{-1} = I - C^* \otimes C,$$
$$C^*B = B^{-1}C = 0, \qquad\qquad C^*(C) = 1 \qquad\qquad .$$

Let F be defined locally on $T(\mathcal{N}^m)$ by $F = B^{-1}fB$. Then:

$$F^2 X = B^{-1}fBB^{-1}fBX = B^{-1}f(I_C^* \otimes C)f(BX) =$$
$$= B^{-1}f^2(BX) - C^*f(BX)B^{-1}fC.$$

If C is in distribution M, then $fC = 0$ we have that

$$(F^3 - F)X = B^{-1}fBB^{-1}f^2BX - fBX =$$
$$= B^{-1}f(I - C^* \otimes C)f^2BX - B^{-1}fBX = B^{-1}((f^3 - f)Bx) = 0$$

fur all X. On the other hand, suppose that C is in distribution L. Then:

$$(F^3 - F)X = (B^{-1}fB)B^{-1}f^2(BX) - (B^{-1}fB)C^*(fBX)B^{-1}fC -$$
$$- B^{-1}fBX = B^{-1}(f^3 - f)BX - C^*(f^2BX)B^{-1}fC -$$
$$- C^*(fBX)^{-1}f^2C + C^*(fBX)C^*(fC)B^{-1}fC = 0,$$

since $f^2C = C$ on L and $C^*B = B^{-1}C = 0$, and since we can choose C^* so that $C^*(fC) = 0$. Also $C^*(f^2BX) = C^*(BX + (f^2 - 1)BX) = 0$.

THEOREM 2.2. *If* (f, ξ, η) *is an almost paracontact structure on* $\mathcal{M}^n$ *then* $\mathcal{N}^{-1}$ *possesses a natural* $F(3,-1)$-*structure if* ξ *is tangent to* $\mathcal{N}^{n-1}$. *The hypersurface* $\mathcal{N}^{n-1}$ *possesses a natural almost product structure if* 5 *is not tangent to* $\mathcal{N}^{n-1}$.

Proof. When ξ is not tangent to $\mathcal{N}^{-1}$, ξ can be chosen for a pseudonormal. Then we have from Theorem 2.1. that $T(\mathcal{N}^{-1}) \cap f(T(\mathcal{N}^{-1})) = T(\mathcal{N}^{-1})$, and rank $F = \dim \mathcal{N}^{-1} = n - 1$. The almost paracontact structure F has a maximal rank, i.e. F is an almost product structure.

3. Integrability conditions. The structure f is integrable if $[f, f] = 0$; $[f, f]$ denotes the Nijenhuis tensor of f, i.e.

$$[f, f](X, Y) = [fX, fY] - [fX, Y] - f(X, fY) + f^2[X, Y]$$

for all vector fields X an Y on $\mathcal{M}^n$.

In this section we shall assume that f is an $f(3, -1)$-structure on $\mathcal{M}^n$ and that $F = B^{-1}fB$ is the naturally induced $F(3, -1)$-structure.

THEOREM 3.1. *Let $\mathcal{N}^{-1}$ be a hypersurface in $\mathcal{M}^n$, and suppose f is integrable. If, locally, the transversal to $\mathcal{N}^{-1}$ can be found to lie in tbe distribution M, then the induced $F(3, -1)$-structure on $\mathcal{N}^{-1}$ is integrable.*

Proof. We see that

$$[F, F](X, Y) = [FX, FY] - F[FX, Y] - F[X, FY] + F^2[X, Y] =$$
$$= B^{-1}[BB^{-1}fBX, BB^{-1}fBY] - B^{-1}f[BB^{-1}fBX, BY] -$$
$$- B^{-1}f[BX, BB^{-1}fBY] + B^{-1}fBB^{-1}f[BX, BY] =$$
$$= B^{-1}\{[f, f](BY, BY) - [C^*(fBX)C, fBY] - [fBX, C^*(fBY)C] +$$
$$+ [C^*(fBX)C, C^*(fBY)C] + f[C^*(fBX)C, BY] + f[BX, C^*(fBYC] -$$
$$- C^*(f[BX, BY])fC\},$$

where we have used the fact that $B[X, Y) = [BX, BY]$, for vector fields X, Y on $\mathcal{N}^{-1}$, and that $BB^{-1} = I - C^* \otimes C$. If the transversal C lies in the distribution M, then the form C^* can be chosen so that $C^*f = 0$. We see that: $[F, F](X, Y) = B^{-1}([f, f](BX, BY)) = 0$.

REFERENCES

[1] G. Luden, *Submanifolds of manifolds with an f-structure*, Kodai Math. Sem. Rep. **21** (1969), 160–166.

[2] T. Miyzawa, *Hypersurfaces immersed in an almost product Riemannien manifold*, Tensor, (N.S.) **33** (1979), 114–116.

[3] S. Sato, *On a structure similar to the almost contact structure*, Tensor (N.S.) **30** (1976), 219–224.

[4] Y. Tashiro, *On contact structure of hypersurfaces in complex manifolds I, II*, Tohoku Math. J. (2) **15** (1963), 62–78, 167–175.

[5] K. Yano and S. Ishihara, *The f structure induced on submanifolds of complex and almost complex spaces*, Kodai Math. Sem. Rep. **18** (1966), 120–160.

[6] K. Yano, *On a structure defined by a tensor field f of type (1,1) satisfying $f^2 + f = 0$*, Tensor (N.S.) **14** (1963), 99–109.

Fakultet tehničkih nauka
Institut za primenjene osnovne discipline
Veljka Vlahovića 3
21000 Novi Sad
Jugoslavija