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SIMPLE SKEW POLYNOMIAL RINGS

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Abstract. We treat two questions. First we give the general conditions for the existence of skew polynomial rings in finitely many variables over a given ring R (special cases of such rings are well, known, typifield by the n-th Weyl Algebras) and second we obtain the necesary and sufficient conditions for the simplicity of such rings.

Note that Amitsur [**1**] obtained conditions under which an Ore extension $R[x,d]$ over a simple ring R is simple, while more recently Jordan [**6**] obtained such conditions if R is d-simple.

1. Preliminaries. All the rings considered in this paper are with identities. We recal that a map $d: R \to R$ such that $d(a+b) = d(a) + d(b)$ and $d(ab) = d(a)b + ad(b)$, for all a, b in R is called a derivation of R. Given s in R it is easy to check that the map $d: R \to R$, defined by the relation $d(r) = sr - rs$ for all r in R, is a derivation of R called the inner derivation of R induced bu s. Any derivation of R which is not inner is called an outher derivation of R.

Let D be a family of derivations of R, then an ideal I of R is said to be a D-ideal if $d(I) \subseteq I$ for all d in D, and R is called a D-simple ring if it has no nonzero proper D-ideals. In the special case where $D = \{d\}$ we write d-ideal and d-simple ring respectively. It is clear that if R is d-simple for some d in D then R is D-simple.

Every D-simple ring contains the field $F_0 = C(R) \cap [\cap_{d \in D} \ker d]$, where $C(R)$ denotes the center of R, and therefore R is of characteristic either zero or of a prime number p.

Consider now the set S of all polynomials in one variable, say x, over R and define in S addition in the usual way and multiplication by the rule $xr - rx^{+} d(r)$ for all r in R, where d is a given derivation of R.

Then it is well known (e. g. [**2** p. 35]) that S becomes a ring denoted by $R[x,d]$, and it is called a skew polynomial ring, or an Ore extension over R.

Applying induction on n one finds that $x^n r = \sum_{i=0}^{n} \binom{n}{i} d^i(r) x^{n-i}$ for all r in R.

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2. Skew polynomial rings in finitely many varibales. We need first the following lemma:

2.1 LEMMA. *Let R and S be as before and let d'be an outer derivation of R. Then d'extends to a derivation of S by $d'(x) = 0$ if and if d'commutes with d.*

Proof. It is clear that d' extends to a derivation of S if $d'(x)$ can be defined in a way compaitible with the multiplication in S. Namely, if $d'(x) = h$, we should be able to write $d'(xr) = d'(rx) + d'(d(r))$, or $xd'(r) + hr = d'(r)x + rh + d'(d(r))$, for all r in R. But $xd'(r) = d'(r)x + d(d'(r))$ and therefore we get that $hr + d(d'(r)) = rh + d'(d(r))$, for all r in R. Thus if $h = 0$, d' comutes with d and, conversely, if that happens, one can extend d' to a derivation of S by putting $d'(x) = 0$.

Using the lemma above and applying induction on n we get the following result:

2.2 THEOREM. *Let R be a ring and $d_1, d_2, \ldots, d_n$ be derivations of R. Consider the set S_n of all polynomials in n variables, say $x_1, x_2, \ldots, x_n$ over R. Define addition in S_n in the usual way and define multiplication by the relations: $x_i r = rx_i + d_i(r)$ and $x_i x_j = x_j x_i$ for all r in R and all $i, j = 1, 2, \ldots, n$. Then S_i becomes an Ore extension of S_{i-1} for each $i = 1, 2, \ldots, n$ (where $S_0 = R$) if and only if d_i commutes with d_j, for all $i, j = 1, 2, \ldots, n$.*□

We call the ring construced in the previous theorem a skew polynomial ring in n variables over R (by derivations) and we denote it by

$$S_n = R[x_1, d_1] \cdots [x_n, d_n].$$

3. Simple skew polynomial rings. Throughout this section S_n is understood to be a skew polynomial ring in n variables over a ring R, defined with respect to a finite set $D = \{d_1, \ldots, d_n\}$ of commuting derivations of R.

We need the following two lemmas.

3.1 LEMMA. *Let $S = R[x, d]$ be an Ore extension over a ring R, where d is a derivation of R. Then:* (i) *If I is a d-ideal of R, IS is an ideal of S, and* (ii) *If I is an ideal of S, the set $A(I)$ of the leading coefficients of the elements of I of minimal degree together with zero is a d-ideal of R.*

Proof. See Lemma 1.3 in [**5**].□.

3.2 LEMMA. *Let $f(x_i) = \sum_{j=0}^{m} a_j x_i^j$ be a unit of S_n for some $i = 1, 2, \ldots n$, where a_j is in R for each j and a_m is a regular element of R. Then $m = 0$ and f is a unit of R.*

Proof. There exists a g in S_n such that $f \cdot g = 1$. We write $g = \sum_{k=0}^{q} g_k x_i^k$ with $g_k = g_k(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for each k. Assume that $q > 0$, then $1 = a_m x_i^m g_q x_i^q$+terms of lower degree with respect to x_i. We can write $g_q = \sum_{(t)} b(t) x^{(t)}$, with $x^{(t)} = \prod_{i \neq j=1}^{n} x_j^{t_j}$ and $b(t) = b(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots t_n)$ in R, where the $t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n$ are nonnegative integers and where the sum $t_1 + \cdots + t_{i-1} + t_{i+1} + \cdots + t_n$ is an integer from zero to a fixed positive integer. Then a straigtforward calculation shows that $x_i g_q - g_q x_i = \sum_{(t)} d_i(b(t))x^{(t)}$. Writing $x_i^m g_q = x_i^{m-1}(x_i g_q)$ and applying the previous relation m times we finally get that $1 = a_m g_q x_{m+q}^i +$ terms of lower degree with respect to x_i, therefore $a_m g_q = 0$ and $g_q = 0$. Thus we must have $q = 0$ and $f \cdot g_0 = 1$. Assuming that $m > 0$ we get that $a_m x_i^m g_0 + (\sum_{j=0}^{m-1} a_j x_i^j) g_0 = 1$, which gives that $a_m g_0 = 0$; hence $g_0 = 0$. Therefore $m = 0$ and $f = a_0$ is a unit of R.□

We are now ready to prove

3.3. THEOREM. *Assume that S_n is a simple ring. Then:* (i) *No element of D is an inner derivation of R induced by some $0 \neq r$ in $\cap_{d \in D}$* ker d, *and* (ii) *R is a D-simple ring.*

Proof. (i) Assume the contrary, and apply Lemma 3.2 to get a contradiction.

(ii) If I is a nozero D-ideal of R, then IS_1 is a nonzero d_2-ideal of S_1; therefore IS_2 is a nonzero ideal of S_2 etc. Finally we get that IS_n is a nonzero ideal of S_n and therefore $I = R$.□

3.4. THEOREM. *Let R be a D-simple ring of characteristic zero and let d_i be an outer derivation of S_{i-1}, for each $i = 1, \dots, n$ (where $S_0 = R$). Then S_n is a simple ring.*

Proof. Assume the contrary, and let I be a nonzero proper ideal of S_n. Write the elements of I as polynomials in x_n with coefficients in S_{n-1}; then the set $A(I) = I_{n-1}$ of Lemma 3.1 is a d_n-ideal of S_{n-1} while $A(I_{n-1}) = I_{n-2}$ is a d_{n-1}-ideal of S_{n-2}. Moreover, given $0 \neq f$ in I_{n-2} there exists $0 \neq g$ in I_{n-1} of minimal degree, say k, with respect to x_{n-1} and leading coefficient f. Then $d_n(g) = d_n(f)x_{n-1}^k +$ terms of lower degree (since $d_n(x_{n-1}) = 0$); therefore I_{n-2} is a $\{d_n, d_{n-1}\}$-ideal of S_{n-2}. In the same way $I_i = A(I_{i+1})$ is a nonzero $\{d_n, d_{n-1}, \dots, d_{i+1}\}$-ideal of S_i, for each $i = 1, \dots, n$ and therefore $I_0 = R$. Hence, if s is the minimal degree in I_1, there exists $f_1(x_1) = x_1^s + \sum_{i-0}^{s-1} a_i x_1^i$ in I_1, with a_i in R for each i.

Suppose first that $s > 0$, then $f_1 r = r x_1^s + (s d_1(r) + a_{s-1} r) x_1^{s-1} +$ terms of lower degree, for all r in R. Hence the polynomial $f_1(x_1)r - r f_1(x_1)$, which is also in I_1, has degree less than s and therefore $f_1(x_1)r = r f_1(x_1)$, for all r in R. On comparing the coefficients of x_1^{s-1} in the last equation we get that $s d_1(r) + a_{s-1} r = r a_{s-1}$, for all r in R.

But $0 \neq s$ and 1_R belongs to the field F_0; therefore $d_1(r) = (-s^{-1} a_{s-1} r - r(s^{-1} a_{s-1})$ for all r in R, a contradiction. Hence $s = 0$ and 1_R is in I_1. Therefore, we can find $f_2(x_2) = x_1^{s'} +$ terms of lower degree s' in I_2. If $s' > 0$, then repeating the previous argument we find that d_2 is an inner derivation of S_1; otherwise, we continue in the same way.

Finally, in the f_i's keep having degree zero, we find some f_{n-1} in I_{n-1} on the same form with deg$f_{n-1} > 0$ (otherwise $I = R$). Then d_n is an inner derivation of S_{n-1}, a contradiction. □.

An argument similar to the previous one gives the following result:

3.5. THEOREM. *Let R be a D-simple ring of prime characteristic, say p. Set $F_q = C(S_q) \cap (\bigcap_{j=q+1}^{n} \ker d_j)$, $0 \leq q \leq n-1$ (where $S_0 = R$), and suppose that no derivation of the form $\sum_{i=0}^{m} a_i d_k^{p^i}$, with m a non negative integer and a_i in F_{k-1} for each i, is an inner derivation of S_{k-1} induced by an element of $\bigcap_{j=k}^{n} \ker d_j$, for all $k = 1, \dots, n$. Then S_n is a simple ring.*

Conversely, if S_n is a simple ring, then no derivation of the form $\sum_{i=0}^{m} a_i d_k^{p^i}$, with a_i in F_0 for each i, can be an inner derivation of R induced by some nonzero element of $\bigcap_{d \in D} \ker d$, for all $k = 1, \dots, n$, and R is a simple ring.

The previous theorems give the following corollary for $n = 1$, due to Jordan [**6**].

3.6. COROLLARY. *Let R be a ring, and let $S = S[x, d]$ be an Ore extension over R. Then: (i) If R is of characteristic zero, S is simple if and only if, R is d-simple and d is an outher derivation of r, and (ii) if R is of prime characteristic, say p, S is simple if and only if, R id d-simple and no derivation of the form $\sum_{i=0}^{m} a_i d^{p_i}$ with a_i in $C(R) \cap \ker d$ for each i, is an inner derivation of R induced by some nonzero element of* $\ker d$.

4. Example. The following examples illustrate the previous results.

1) Let $R = T[y_1, \dots, y_n]$ be a polynomial ring over a given ring T. Then the skew polynomial ring over R defined with respect to the set $D = \{\partial/\partial y_1, \dots, \partial/\partial y_n\}$ of derivations of R is called the n-th Weyl Algebra over T, and it is denoted by $A_n(T)$. If T is a simple ring of characteristic zero, it is well know that $A_n(T)$ is simple (cf. [**3**, Prop. 7.30, p. 354] and apply induction on n). Alternatively, since R is D-simple, apply Theorem 3.4 to get the same result.

If T has nonzero characteristic, say p, since $(y_1^p, \dots, y_n^p)$ is a nonzero proper D-ideal of $R, A_n(T)$ iz not simple by Theorem 3.3 (otherwise show directly that $(x_1^p, \dots, x_n^p)$ is an ideal of $A_n(T)$).

2) Let k be a field of prime characteristic, say p and let $R' = k[y_1, \dots, y_n]$ be a polynomial ring over k. Denote by d_i the derivation of the ring $R = R'/(y_1^p, \dots, y_n^p)$ induced by $\partial/\partial y_i$ in the obvious way. Then $d_i^p = 0$, for each $i = 1, \dots, n$, and therefore $S_n = R[x_1, d_1] \cdots [xn, d_n]$ is not simple by Theorem 3.5, although it is easy to check that R is $\{d_1, \dots, d_n\}$-simple.

3) Let k be as before and let $R = k(y_1, \dots, y_n)$ be the field generated by the indeterminates $y_1, \dots, y_n$ over k. Then $d_i = y_i \partial/\partial y_i$ is a derivation of R while $d_i' = d_i^p - d_i$ is the zero derivation of R, for each $i = 1, \dots, n$. Hence the skew polynomial ring over R defined with respect to the d_i's is not simple, by Theorem 3.5.

4) Let k be a field of characteristic zero and let R be a commutative k-algebra with no zero divisors.

If R is a regular local ring of finitely generated type over k, then R is d-simple [**4**], therefore we can construct simple skew polynomial rings over R. On the other hand, if R is either a finitely generated k-algebra or a complete local ring, and it is also nonregular, then R is not d-simple (cf. [**7**] and [**8**]); therefore we cannote construct simple Ore extensions in one variable over R.

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