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DECOMPOSITION OF RECURRENT CURVATURE TENSOR FIELDS OF R-TH ORDER IN FINSLER MANIFOLDS

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Summary. We study the decomposition of Berwald curvature tensor fields in recurrent Finsler spaces of r-th order.

1. Introduction. The Berwald curvature tensor fields H^i_{kh} and H^i_{jkh} satisfy the identities

$$\text{(1.1)} \qquad \text{a)}\ H^i_{jhk} + H^i_{khj} + H^i_{hjk} = 0, \qquad \text{b)}\ H^i_{jkh} = -H^i_{jhk},$$

$$\text{(1.2)} \qquad H^i_{jk(h)} + H^i_{kh(j)} + H^i_{hj(k)} = 0,$$

$$\text{(1.3)} \qquad H^i_{jkh(l)} + H^i_{jhl(k)} + H^i_{jlk(h)} + H^m_{kh}G^i_{mjl} + H^m_{lk}G^i_{mjh}H^m_{hl}G^i_{mjk} = 0,$$

in an n-dimensional Finsler manifold.

The commutation formulae in the sense of Berwald are given by

$$\text{(1.4)} \qquad T^i_{j(h)(k)} - T^i_{j(k)(h)} = -\dot{\partial}_q T^i_j H^q_{hk} - T^i_q H^q_{jhk} + T^q_j H^i_{qhk},$$

$$\text{(1.5)} \qquad (\dot{\partial}_k T^i_j)_{(h)} - \dot{\partial}_k T^i_{j(h)} = T^i_q G^q_{jkh} - T^q_j G^i_{qkh}$$

The recurrent curvature tensor fields of first and second order are defined (Sinha, Singh [**5**]) as

$$\text{(1.6)} \qquad H^i_{jkh(m_1)} = V_{m_1} H^i_{jhk}$$

$$\text{(1.7)} \qquad H^i_{jkh(m_1)(m_2)} = V_{m_1 m_2} H^i_{jhk},$$

where V_{m_1} and $V_{m_1 m_2}$ are recurrence vector and tensor fields which are related by

$$\text{(1.8)} \qquad V_{m_1 m_2} = V_{m_1(m_2)} + V_{m_1} V_{m_2}.$$

The recurrent curvature tensor field of r-th order can be obtained as

$$\text{(1.9)} \qquad H_{jkh(m_1)(m_2)\dots(m_r)} = V_{m_1 m_2 \dots m_r} H^i_{jkh},$$

where

$$V_{m_1 m_2 \dots m_r} = V_{m_1 m_2 \dots m_{r-1}(m_r)} + V_{m_1 m_2 \dots m_{r-1}} V_{m_r}.$$

Transvecting (1.6), (1.7) and (1.9) by $\dot{x}^j$, we get

$$H^i_{kh(m_1)} = V_{m_1} H^i_{kh}, \quad H^i_{kh(m_1)(m_2)} = V_{m_1 m_2} H^i_{kh}$$
$$H_{kh(m_1)(m_2)\dots(m_r)} = V_{m_1 m_2 \dots m_r} H^i_{kh}.$$

2. Decomposition of Berwald curvature tensor field. Let us consider the decomposition

(2.1) $$H^i_{kh} = X^i \Phi_{jkh}.$$

where Φ_{kh} is a non-zero homogeneous tensor field of the first degree in $\dot{x}^i$ and X^i is a non-zero vector field independent of $\dot{x}^i$. Differentiating (2.1) with to $\dot{x}^j$, we get

(2.2) $$H^i_{jkh} = X^i \Phi_{jkh} \text{ where } \Phi_{jkh} = \dot{\partial} \Phi_{kh}.$$

The decomposition tensor field Φ_{kh} satisfies the relation

(2.3) $$\Phi_{jk} V_{hm_1 \dots m_{r-1}} + \Phi_{kh} V_{km_1 \dots m_{r-1}} + \Phi_{hj} V_{km_1 \dots m_{r-1}} = 0.$$

In an affinely connected space, if V_{m_1} is independent of $\dot{x}^i$ the decomposition tensor field satisfies the relations

(2.4) $$\Phi_{jkh} + \Phi_{khj} + \Phi_{hjk} = 0$$

(2.5) $$\Phi_{jkh} V_{lm_1 \dots m_{r-1}} + \Phi_{jlk} V_{hm_1 \dots m_{r-1} + \Phi_{jhl} V_k m_1 \dots m_{r-1}} = 0.$$

THEOREM 2.1. *If X^i in* (2.1) *is a covariant constant then the decomposition tensor fields Φ_{jkh} and Φ_{kh} behave like recurrent tensor fields of r-th order.*

Proof. Taking successive covariant derivatives of (2.2) with respect to $x^{m_1}, x^{m_2}, \dots, x^{m_r}$, we have

(2.6) $$H^i_{jkh(m_1)\dots(m_r)} = X^i \Phi_{jkh(m_1)\dots(m_r)}.$$

Using (1.9) and (2.2), (2.6) gives

(2.7) $$V_{m_1 m_2 \dots m_r} \Phi_{jkh} = \Phi_{jkh(m_1)(m_2)\dots(m_r)}.$$

Transvecting (2.7) by $\dot{x}^j$, we get

(2.8) $$V_{m_1 m_2 \dots m_r} \Phi_{kh} = \Phi_{kh(m_1)(m_2)\dots(m_r)},$$

which proves the statement.

THEOREM 2.2. *Under the decomposition* (2.1) *and* (2.2), *if V_{m_1} is independent of $\dot{x}^i$ in an affinely connected space, the following relation holds*

(2.9) $$\Phi_{kh} V_{m_1 \dots m_{r-2}}[m_{r-1} m_r p] = 0$$

Proof. Differentiating (2.8) covariantly with respect to x^p and commuting the indices m_r and p in the result, we get

$$(2.10)\quad \begin{aligned} &\{(V_{m_1\ldots m_{r-1}m_r(p)} - V_{m_1\ldots m_{r-1}p(m_r)}) + (V_{m_1\ldots m_r}V_p - V_{m_1\ldots m_{r-1}p}V_{mr})\}\Phi_{kh} \\ &\qquad = (\Phi_{kh(m_1)\ldots(m_{r-1})})_{(m_r)(p)} - (\Phi_{kh(m_1)\ldots(mr-1)})_{(p)(m_r)}. \end{aligned}$$

With the help of commutation formula (1.4), (2.10) yields

$$(2.11)\quad \begin{aligned} &\{(V_{m_1\ldots m_{r-1}m_r(p)} - V_{m_1\ldots m_{r-1}p(m_r)}) + (V_{m_1\ldots m_r}V_p - V_{m_1\ldots m_{r-1}p}V_{m_r})\}\Phi_{kh} \\ &\qquad = -\dot{\partial}_q\Phi_{kh(m_1)\ldots(m_{r-1})}H^q_{mrp} - \Phi_{qh(m_1)\ldots(m_{r-1})}H^q_{km_rp} - \\ &\qquad - \Phi_{qk(m_1)\ldots(m_{r-1})}H^q_{hm_rp} - \Phi_{kh(q)(m_2)\ldots(m_{r-1})}H^q_{m_1m_rp} \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad - \Phi_{kh(m_1)\ldots(m_{r-2})(q)}H^q_{m_{r-1}m_rp}. \end{aligned}$$

Using the decomposition (2.1) and (2.2) and the fact that Φ_{kh} is recurrent of $(r-1)^{-\text{th}}$ order and that in an affinely connected space V_{m_1} is independent of $\dot{x}^i$, we get

$$(2.12)\quad \begin{aligned} &\{(V_{m_1\ldots m_{r-1}m_r(p)} - V_{m_1\ldots m_{r-1}p(m_r)}) + (V_{m_1\ldots m_r}V_p - V_{m_1\ldots m_{r-1}p}V_{m_r})\}\Phi_{kh} \\ &\qquad = -V_{m_1\ldots m_{r-1}}\Phi_{qkh}\Phi_{m_rp}X^q - V_{m_1\ldots m_{r-1}}\Phi_{qh}\Phi_{km_rp}X^q \\ &\qquad - V_{m_1\ldots m_{r-1}}\Phi_{kq}\Phi_{hm_rp}X^q - V_{qm_2\ldots m_{r-1}}\Phi_{kh}\Phi_{m_1m_rp}X^q \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\qquad - V_{m_1\ldots m_{r-2}q}\Phi_{kh}\Phi_{m_{r-1}m_rp}X^q. \end{aligned}$$

Changing the indices m_{r-1}, m_r and p cyclically in (2.12) and adding up the results thus obtained, we get (2.9) by virtue of (2.3), (2.4) and (2.5).

THEOREM 2.3. *In an affinely connected recurrent space of order r, if V_{m_1} is independent of $\dot{x}^i$ the tensor field Φ_{kh} satisfies the relation*

$$(2.13)\quad \Phi_{kh(m_1)(m_2)\ldots(m_{r-2})[(m_{r-1})(m_r)(p)]} = 0.$$

Proof. Interchanging the indices m_{r-1}, m_r and p cyclically in (2.10) and adding up the expressions thus obtained, we get (2.13) by using (2.9).

3. Decomposition of Berwald curvature tensor field $H^i_{jkh}(x,\dot{x})$ in another form. Let us consider the decomposition of H^i_{jhk} in another form

$$(3.1)\quad H^i_{jkh} = A^i_j\psi_{kh},$$

where $\psi_{kh}(x,\dot{x})$ is a decomposition tensor field and $A^i_j(x,\dot{x})$ is a non zero tensor field.

Under the decomposition (3.1), the following identities are true: $S_j\psi_{kh} + S_k\psi_{hj} + S_h\psi_{jk} = 0$, and $\psi_{kh} + \psi_{hk} = 0$, where $S_j = V_iA^i_j$.

THEOREM 3.1. *Under the decomposition (3.1), if A^i_j is a covariant constant then the decomposition tensor field ψ_{kh} behaves like recurrent tensor field of r-th order.*

Proof. Differentiating (3.1) covariantly with respect to x^{m_1}, we have

$$H_{jkh(m_1)} = A^i_{j(ms)}\psi_{kh} + A^i_j\psi_{kh(m_1)}. \tag{3.2}$$

Since A^i_j is a covariant constant, that is $A^i_{(m_1)} = 0$, then (3.2) gives

$$H^i_{jkh(m_1)} = A^i_j\psi_{kh(m_1)}. \tag{3.3}$$

Differentiating (3.3) covariantly with respect to $x^{m_1}, x^{m_2}, \dots, x^{m_r}$ successively and using (1.9), (3.1) and $A^i_{j(m_1)} = 0$, we get

$$\psi_{kh(m_1)(m_2)\dots(m_r)} = V_{m_1m_2\dots m_r}\psi_{kh}.$$

REFERENCES

[1] P. N. Pandey, *On decomposability of curvature tensor of a Finisler manifold*, Acta Math. Acad. Sci. Hungar, 36 (1,2), 1981.

[2] D. H. Pandey, *The recurrent Finsler space of third order*, Indian J. Pure Appl. Math., to appear.

[3] H. Rund, *The differential geometry of Finisler spaces*, Springer-Verlag, Berlin,1959.

[4] A. K. Singh, R. D. Mishra, *Decomposition of Neo-pseudo projective curvature tensor field in a Q-current Finisler space of third order*, to appear.

[5] B. B. Sinha, S. P. Singh, *Recurrent Finsler space of second order*, Yokohama Math. J. **18** (1971).

[6] B. B. Sinha, *Decomposition of recurrent curvature tensor fields of second order*, Prog. Math. **6** (1972), 7–14.

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