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ON SEHGAL'S MAPS WITH A CONTRACTIVE ITERATE AT A POINT

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Abstract. Let (X, d) be a complete metric space and T a mapping of X into itself. Suppose that for each $x \in X$ there exists a positive integer $n = n(x)$ such that for all $y \in X$,

$$d(T^n x, T^n y) \leq \alpha \max\{d(x,y), d(x,Ty), d(x,T^2y), \dots, d(x,T^n y), d(x,T^n x)\},$$

holds tor some $\alpha < 1$. With these assumptions our main result states that T has a unique fixed point. This generalizes an earlier result of V. M. Sehgal and a recent result of the author.

1. We shall prove the following theorem, which is a generalization of Sehgal's Theorem [**3**].

THEOREM 1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping. If for each $x \in X$ there exists a positive integer $n = n(x)$ such that*

$$(1) \quad (T^n x, T^n y) \leq \alpha \cdot \max\{d(x,y), d(x,Ty), d(x,T^2y), d(x,T^3y), \dots, d(x,T^n y) d(x,T^n x)\}$$

holds for some $\alpha < 1$ and all $y \in X$, then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $\lim_{m\to\infty} T^m x = u$.

Proof. First we shall show that for every $x \in X$, the orbit $\{T^m x\}_{m=0}^{\infty}$ is bounded. To prove this assertion, we shall show that for any $x \in X$

$$(2) \qquad r(x) = \sup\{m > 0\{d(x,T^m x) \leq \max\{d(x,T^s x) : 0 < s \leq n(x)\}/(1-\alpha).$$

Let m be any, but fixed, positive integer and k $(k = k(x,m))$ a positive integer such that

$$(3) \qquad d(x,T^k x) = \max\{d(x,T^r x) : 0 < r \leq m\}.$$

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We may suppose that $m > n(x)$ and $k > n(x)$. Then by triangle inequality and by (1) we have

$$\begin{aligned} d(x, T^k) \leq d(x, T^n x) + d(T^n x, T^n T^{k-n} x) \leq\leq d(x, T^n x) + \\ + \alpha \cdot \max\{d(x, T^{k-n} x), d(x, T^{k-n+1} x), \dots, d(x, T^k x), d(x, T^n x)\} \leq \\ \leq d(x, T^n x) + \alpha \cdot \max\{d(x, T^r x) : 0 < r \leq m\}. \end{aligned}$$

Using (3) we obtain $d(x, T^k x) \leq d(x, T^n x) + \alpha \cdot d(x, T^k x)$ and hence

$$d(x, T^k x) \leq d(x, T^n x)/(1-\alpha), \text{ i.e.,}$$
$$\max\{d(x, T^r x) : 0 < r \leq m\} \leq d(x, T^n x)/(1-\alpha).$$

Since m was arbitrary, this implies

$$\sup_{m > n(x)} \{d(x, T^m x) \leq d(x, T^{n(x)} x)/(1-\alpha).$$

The relation (2) now follows immediately. Consequently, the orbit $\{T^m x\}_{m=0}^{\infty}$ is bounded.

Now, let $x_0 = x \in X$, $n_0 = n(x_0)$, $x_1 = T^{n_0} x_0$ and inductively

$$n_k = n(x_k), \quad x_{k+1} = T^{n_k} x_k \quad (k = 1, 2, \dots).$$

Evidently, $\{x_k\}$ is a subsequence of the orbit $\{T^m x_0\}_{m=0}^{\infty}$. Using this subsequence we shall show that $\{T^m x_0\}_{m=0}^{\infty}$ is a Cauchy sequence.

Let x_k be any fixed member of $\{x_k\}_{k=i}^{\infty}$ and let $x_p = T^p x_0$ and $x_q = T^q x_0$ be any two members of the orbit $\{T^m x_0\}_{m=0}^{\infty}$ which follow after x_k. Then $x_p = T^r x_k$ and $x_q = T^s x_k$ for some r and s, respectively. Now, using (1) we get

$$d(x_k, x_p) = d(x_k, T^r x_k) = d(T^{n_{k-1}} x_{k-1}, T^{n_{k-1}} x_{k-1}) \leq \alpha d(x_{k-1}, T^{r_1} x_{k-1})$$

where

$$\begin{aligned} d(x_{k-1}, T^{r_1} x_{k-1}) = \max\{d(x_{k-1}, T^r x_{k-1}), d(x_{k-1}, T^{r+1} x_{k-1}), \dots, \\ d(x_{k-1}, T^{r+n_{k-1}} x_{k-1}), d(x_{k-1}, T^{n_{k-1}} x_{k-1})\}. \end{aligned}$$

Similarly, $d(x_{k-1}, T^{r_1} x_{k-1}) \leq \alpha d(x_{k-2}, T^{r_2} x_{k-2})$, where

$$d(x_{k-2}, T^{r_1} x_{k-2}) = \max\{d x_{k-2}, T^{r_1} x_{k-2}), \dots, d(x_{k-1}, T^{n_{k-2}} x_{k-1}\}.$$

Repeating this argument k-times we get

$$d(x_k, x_p) \leq \alpha d(x_{k-1}, T^{r_1} x_{k-1}) \leq \alpha^2 d(x_{k-2}, T^{r_2} x_{k-2}) \leq \dots \leq \alpha^k d(x_0, T^{r_k} x_0).$$

Hence $d(x_k, x_p) \leq \alpha^k r(x)$. Similarly, $d(x_k, x_q) = d(x_k, T^s x_k) \leq \alpha^k r(x)$.

Therefore,

$$d(x_p, x_q) \leq d(x_k, x_p) + d(x_k, x_q) \leq \alpha^k \cdot 2r(x). \tag{4}$$

Since $\alpha < 1$, (4) implies that the orbit $\{T^m x_0\}_{m=0}^{\infty}$ is a Cauchy sequence.

By the completeness of X there is $u \in X$ such that $u = \lim_{m\to 0} T^m x_0$. We shall show that $T^{n(0)}u = u$. For $m \geq n = n(u)$, we now have

$$d(T^n u, T^n T^m x_0) \leq \alpha \cdot \max\{d(u, T^m x_0), d(u, T^{m+1}x_0), \ldots, \\ d(u, T^{m+n}x_0), d(u, T^n u)\}$$

and on letting m tend to infinity it follows that

$$d(T^n u, u) \leq \alpha d(u, T^n u).$$

Since $\alpha < 1$, we see that u is a fixed point of $T^{n(u)}$.

To show that u is a fixed point of T, let us assume that $Tu \neq u$ and let $d(u, T^k u) = \max\{d(u, T^r u) : 0 < r \leq n = n(u)\}$. Then

$$\begin{aligned} d(u, T^k u) =& d(T^n u, T^k T^n u) = d(T^n u, T^n T^k u) \leq \\ & \alpha \cdot \max\{d(u, T^k u), d(u, T^{k+1}u), \ldots, d(u, T^{k+n}u), d(u, T^n u)\} \leq \\ & \alpha d(u, T^k u). \end{aligned}$$

Since $\alpha \leq 1$, it follows that $d(u, T^k u) = 0$, which implies that u is a fixed point of T. The uniqueness of a fixed point of T follows immediately from (1). This completes the proof of the Theorem.

2. If we suppose that T is continuous, then we may prove the following theorem.

THEOREM 2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a continuous mapping which satisfies the following condition: for each $x \in X$ there is a positive integer $n = n(x)$ such that for all $y \in X$,*

$$d(T^n x, T^n y) \leq \alpha \cdot \max\{d(x, y), d(x, Ty), d(x, T^2 y), \ldots, d(x, T^n y), \\ d(x, Tx), d(x, T^2 x), \ldots, d(x, T^n x)\},$$

where $0 \leq \alpha < 1$. Then T has a unique fixed point $u \in X$. Moreover, for every $x \in X$, $\lim_{k\to\infty} T^k x = u$.

Proof. Let x be an arbitrary point in X. Then, as in the proof of Theorem 1, the orbit $\{T^m x\}_{m=0}$ is bounded and is a Cauchy sequence in the complete metric space X and so it has a limit u in X. Since by the hypothesis T is continuous, it follows that $T^{n(u)}$ is continuous, which implies that

$$T^{n(u)}u = T^{n(u)} \lim_{m\to\infty} T^m x = \lim_{m\to\infty} T^{m+n(u)}x = u.$$

Therefore, u is a fixed point of $T^{n(u)}$. By the same arguments as in the proof of Theorem 1, it follows that u is a unique fixed point of T. This completes the proof of the Theorem.

Remark. The condition that T be continuous in Theorem 2 may be relaxed by the following: $T^{n(x)}$ is continuous at a point $x \in X$.

We now note that the condition that $T^{n(u)}$ be continuous at u is necessary in Theorem 2. This is easily seen by letting X be the closed interval $[0,1]$ with the usual metric. X is then complete. Define a discontinuous mapping T on X by $T(0) = 1$ and $Tx = x/2$, if $x \neq 0$. We then have

$$d(T^2x, T^2y) \leq \max\{d(x,Ty), d(x,Tx)\}/2$$

for all x and y in X and so T satisfies (5) with $\alpha = 1/2$. T however has no fixed point, because T^n is not continuous at 0 for any $n = 2, 3, \ldots$.

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