

## ON SOME CLASSES OF LINEAR EQUATIONS, IV

*Jovan D. Kečkvić*

### 1. Introduction

Let  $V$  be a commutative algebra over  $\mathbf{R}$  (or  $\mathbf{C}$ ), and let  $L_1$  and  $L_2$  be linear operators on  $V$  satisfying the conditions

$$(1) \quad L_i(uv) = uL_iv \text{ iff } u \in \ker L_i \quad (i = 1, 2).$$

Consider the linear equation

$$(2) \quad P(L_1, L_2)u = 0$$

where  $P$  is a two variable polynomial over  $\mathbf{R}$  (or  $\mathbf{C}$ ) and  $u \in V$ .

Suppose that there exists an element  $v \in V$  which is a characteristic vector of  $L_1$  and the same time, belongs to the kernel of  $L_2$ ; and similarly, that there exists an element  $w \in V$  which is a characteristic vector of  $L_2$  and belongs to the kernel of  $L_1$ . In other words, we suppose the existence of elements  $v, w \in V$  such that

$$(3) \quad L_1v = \lambda v, \quad L_2v = 0,$$

$$(4) \quad L_1w = 0, \quad L_2w = \mu w$$

where  $\lambda, \mu \in \mathbf{R}$  (or  $\mathbf{C}$ ).

Put  $u = vw$ . Then, in virtue of (1), (3) and (4), we easily verify that

$$L_1^k u = \lambda^k u, \quad L_2^k u = \mu^k u, \quad L_1^p L_2^q u = \lambda^p \mu^q u \quad (k, p, q \in \mathbf{N}).$$

Hence, if  $v, w, \lambda, \mu$  are defined by (3) and (4), we see that  $u = vw$  is a solution of the equation (2) provided that

$$(5) \quad P(\lambda, \mu) = 0.$$

This simple observation reduces the problem of finding a *particular* solution of the equation (2) to the problem of solving the algebraic equation (5). However, the equation (5) may have an infinity of solutions, which means that it is, in certain cases, possible to obtain an infinity of solutions of (2). Some times those solutions can be combined to yield the *general* solution of the considered equation. Namely, suppose that  $v_\lambda$  and  $w_\mu$  ( $\lambda, \mu \in \mathbf{R}$ ) are such that

$$(6) \quad L_1 v_\lambda = \lambda v_\lambda, \quad L_1 w_\mu = 0, \quad L_2 v_\lambda = 0, \quad L_2 w_\mu = \mu w_\mu \quad (\text{for all } \lambda, \mu \in \mathbf{R}).$$

If the equation (5) implies  $\mu = f(\lambda)$ , then

$$u = v_\lambda w_{f(\lambda)}$$

is a solution of (2) for all  $\lambda \in \mathbf{R}$ , and hence we can combine those solutions into a formal series

$$(7) \quad u = \sum_{\lambda} C_\lambda v_\lambda w_{f(\lambda)}$$

which, can, in certain cases, be summed up, the sum of (7) being the actual general solution of the equation (2).

In this paper we shall give a number of examples of first order equations of the form (2).

## 2. Examples of first order equations

If  $P$  is a first degree polynomial, then the equation (2) becomes

$$(8) \quad (aL_1 + bL_2 + cI)u = 0 \quad (a, b, c \in \mathbf{R} \text{ or } \mathbf{C})$$

where  $I$  is the identity mapping.

The corresponding algebraic equation is

$$(9) \quad a\lambda + b\mu + c = 0,$$

and we, naturally, suppose that the coefficients  $a$  and  $b$  are not both zero.

EXAMPLE 1: *Partial differential equations.* Let  $V$  be the algebra of all real differentiable functions in two real variables. Let  $L_1 = \frac{\delta}{\delta x}$ ,  $L_2 = \frac{\delta}{\delta y}$ . Then  $v_\lambda = e^{\lambda x}$ ,  $w_\mu = e^{\mu y}$  satisfy the equations (6) for all  $\lambda, \mu \in \mathbf{R}$ . The equation (8) in this case becomes

$$(10) \quad au_x + bu_y + cu = 0.$$

Supposing that  $b \neq 0$ , we obtain the following solution of (10):

$$u = e^{\lambda x} e^{-(a\lambda + c)y/b}$$

where  $\lambda \in \mathbf{R}$  is arbitrary. Combining the obtained solutions into a formal series of the form (7), we find

$$u = \sum_{\lambda} C_{\lambda} e^{\lambda x} e^{-(a\lambda+c)y/b},$$

i.e.

$$(11) \quad u = e^{-\frac{c}{b}y} \sum_{\lambda} C_{\lambda} e^{\frac{\lambda}{b}(bx-ay)},$$

where  $C_{\lambda}$  are arbitrary constants. This suggests that the sum which appears in (11) is an arbitrary (differentiable) function of  $bx - ay$ , giving *general* solution of (10):

$$u(x, y) = e^{-\frac{c}{b}y} F(bx - ay),$$

where  $F$  is an arbitrary differentiable function.

EXAMPLE 2: *Partial difference equations.* Let  $V$  be the algebra of all real functions in two real variables  $x$  and  $y$ , and let  $L_1 = \Delta_x$ ,  $L_2 = \Delta_y$ , where

$$\Delta_x u(x, y) = u(x+1, y) - u(x, y), \quad \Delta_y u(x, y) = u(x, y+1) - u(x, y)$$

Then  $v_{\lambda} = (1 + \lambda)^x$ ,  $w_{\mu} = (1 + \mu)^y$  satisfy the equations (6) for all  $\lambda, \mu \in \mathbf{R}$ . The equation (8) in this case becomes

$$a\Delta_x u + b\Delta_y u + cu = 0,$$

i.e.

$$(12) \quad au(x+1, y) + bu(x, y+1) + (c - a - b)u(x, y) = 0.$$

Supposing that  $b \neq 0$ , we obtain the following solution of (12):

$$u = (1 + \lambda)^x \left(1 - \frac{1}{b}(a\lambda + c)\right)^y$$

with  $\lambda \in \mathbf{R}$  arbitrary. Hence

$$(13) \quad u(x, y) = \sum_{\lambda} C_{\lambda} (1 + \lambda)^x \left(1 - \frac{a}{b}\lambda - \frac{c}{b}\right)^y$$

is also a formal solution of (12) for arbitrary constants  $C_{\lambda}$ . We distinguish between

three cases:

(i)  $a + b \neq c$ ;  $a = 0$ ;  $b \neq 0$ . Then (13) becomes

$$\begin{aligned} u(x, y) &= \sum_{\lambda} C_{\lambda} (1 + \lambda)^x \left(1 - \frac{c}{b}\right)^y = \left(\frac{b-c}{b}\right)^y \sum_{\lambda} C_{\lambda} (1 + \lambda)^x \\ &= \left(\frac{b-c}{b}\right)^y f(x), \text{ where } f \text{ is arbitrary.} \end{aligned}$$

(ii)  $a + b = c$ ;  $ab \neq 0$ . Then (13) becomes

$$\begin{aligned} u(x, y) &= \sum_{\lambda} C_{\lambda} (1 + \lambda)^x \left(-\frac{a}{b} - \frac{a}{b}\lambda\right)^y = \left(-\frac{a}{b}\right)^y \sum_{\lambda} C_{\lambda} (1 + \lambda)^{x+y} \\ &= \left(-\frac{a}{b}\right)^y f(x + y), \text{ where } f \text{ is arbitrary.} \end{aligned}$$

(iii)  $a + b \neq c$ ;  $b \neq 0$ . Then (13) becomes

$$\begin{aligned} u(x, y) &= \left(-\frac{a}{b}\right)^y \sum_{\lambda} C_{\lambda} (1 + \lambda)^x \left(\frac{c-a-b}{a} + 1 + \lambda\right)^y \\ &= \left(-\frac{a}{b}\right)^y \sum_{\lambda} C_{\lambda} (1 + \lambda)^x \sum_{v=0}^y \binom{y}{v} \left(\frac{c-a-b}{a}\right)^{y-v} (1 + \lambda)^v \\ &= \left(\frac{a+b-c}{b}\right)^y \sum_{v=0}^y \binom{y}{v} \left(\frac{a}{c-a-b}\right)^v f(x + v), \end{aligned}$$

where  $f$  is arbitrary.

The obtained solutions are general.

**EXAMPLE 3:** *Differential-difference equations.* Let  $V = \{u_n(x) \mid n \in \mathbf{N}, x \in \mathbf{R}\}$  and define  $L_1$  and  $L_2$  by:

$$L_1 u_n(x) = u_{n+1}(x) - u_n(x), \quad L_2 u_n(x) = u_n'(x).$$

This leads to the equation

$$(14) \quad au_{n+1}(x) + bu_n'(x) + (c-a)u_n(x) = 0.$$

Since  $v_{\lambda} = (1 + \lambda)^n$ ,  $w_{\mu} = e^{\mu x}$  satisfy (6) for all  $\lambda, \mu \in \mathbf{R}$ , for  $b \neq 0$  we obtain the following solution of (14):

$$u_n(x) = (1 + \lambda)^n e^{-\frac{1}{b}(a\lambda+c)x},$$

for all  $\lambda \in \mathbf{R}$ . The corresponding formal series is given by

$$\begin{aligned} u_n(x) &= e^{-\frac{c}{b}x} \sum_{\lambda} C_{\lambda} (1 + \lambda)^n e^{-\frac{a}{b}\lambda x} = e^{-\frac{c}{b}x} \sum_v C_v \left(1 - \frac{b}{a}v\right)^n e^{vx} \\ &= e^{-\frac{c}{b}x} \sum_v C_v \left[1 - \binom{n}{1} \frac{b}{a}v + \binom{n}{2} \frac{b^2}{a^2}v^2 - \dots + (-1)^n \frac{b^n}{a^n}v^n\right] e^{vx}. \end{aligned}$$

Hence, putting

$$\sum_v C_v e^{vx} = f(x),$$

where  $f$  is an arbitrary  $n$  times differentiable function, we arrive at the following solution of (14):

$$u_n(x) = e^{-\frac{c}{b}x} \left[ f(x) - \binom{n}{1} \frac{b}{a} f'(x) + \binom{n}{2} \frac{b^2}{a^2} f''(x) - \dots + (-1)^n \frac{b^n}{a^n} f^{(n)}(x) \right].$$

EXAMPLE 4: *Functional-differential equations.* Let  $V$  be the algebra of all real differentiable functions in two real variables  $x$  and  $y$ . Define the operators  $L_1$  and  $L_2$  by

$$L_1 u(x, y) = u(\theta x, y) - u(x, y), \quad L_2 u(x, y) = u_y(x, y)$$

where  $\theta: \mathbf{R} \rightarrow \mathbf{R}$  is such that  $\theta^2 = I$ , i.e.  $\theta(\theta(x)) = x$ . The linear equation (8) becomes

$$(15) \quad au(\theta x, y) + bu_y(x, y) + (c - a)u(x, y) = 0.$$

As before, any function of the form  $u(x, y) = e^{\mu y}$  is a characteristic vector for  $L_2$  and also belongs to  $\ker L_1$ . In order to find the characteristic vectors of  $L_1$  we have to solve the equation

$$u(\theta x, y) = (1 + \lambda)u(x, y),$$

which is possible only if  $(1 + \lambda)^2 = 1$ , i. e.  $\lambda(\lambda + 2) = 0$ .

For  $\lambda = 0$  we find  $\mu = -\frac{c}{b}$ , while for  $\lambda = -2$  we get  $\mu = \frac{2a-c}{b}$ . The characteristic vectors which correspond to  $\lambda = 0$  and  $\lambda = -2$  respectively, are

$$u(x, y) = f(x) + f(\theta x) \quad \text{and} \quad u(x, y) = g(x) - g(\theta x),$$

where  $f$  and  $g$  are arbitrary. Hence, we obtain the following solution of the equation (15):

$$u(x, y) = [f(x) + f(\theta x)]e^{-\frac{c}{b}y} + [g(x) - g(\theta x)]e^{\frac{2a-c}{b}y}$$

where  $f$  and  $g$  are arbitrary functions.

EXAMPLE 5: *Functional-difference equations.* Let  $V$  be as in Example 2, and  $\theta$  as in Example 4. Define the operators  $L_1$  and  $L_2$  by

$$L_1 u(x, y) = u(\theta x, y) - u(x, y), \quad L_2 u(x, y) = u(x, y + 1) - u(x, y).$$

This leads to the equation

$$(16) \quad au(\theta x, y) + bu(x, y + 1) + (c - a - b)u(x, y) = 0.$$

Again we only have two characteristic values for  $L_1$ , namely  $\lambda = 0$  and  $\lambda = -2$ . Hence,  $\mu = -\frac{c}{b}$  and  $\mu = \frac{2a-c}{b}$  respectively, and we arrive at the following solution of (16):

$$u(x, y) = [f(x) + f(\theta x)] \left(1 - \frac{c}{b}\right)^y + [g(x) - g(\theta x)] \left(1 - \frac{2a-c}{b}\right)^y,$$

where  $f$  and  $g$  are arbitrary functions.

EXAMPLE 6: *Functional equations.* Let  $V$  be as in Example 2, and let  $\theta, \varphi: \mathbf{R} \rightarrow \mathbf{R}$  be such that  $\theta^2 = I, \varphi^3 = I$  ( $Ix = x$ ). Define the operators  $L_1$  and  $L_2$  by

$$L_1 u(x, y) = u(\theta x, y) - u(x, y), \quad L_2 u(x, y) = u(x, \varphi y) - u(x, y).$$

The linear equation (8) becomes

$$(17) \quad au(\theta x, y) + bu(x, \varphi y) + (c - a - b)u(x, y) = 0.$$

There are only two possible characteristic values  $\lambda$  for  $L_1:0$  and  $-2$ , and only one (real) characteristic value  $\mu$  for  $L_2:0$ . Since  $\lambda$  and  $\mu$  are tied by the equation (9), this equation has to be satisfied by the following two pairs:  $(0, 0)$  and  $(-2, 0)$ , which yields  $c = 0$  and  $c = 2a$ , respectively.

Hence, we can obtain solutions of the following equations

$$(18) \quad au(\theta x, y) + bu(x, \varphi y) - (a + b)u(x, y) = 0,$$

$$(19) \quad au(\theta x, y) + bu(x, \varphi y) + (a - b)u(x, y) = 0.$$

They are

$$(20) \quad u(x, y) = (f(x) + f(\theta x))(g(y) + g(\varphi y) + g(\varphi^2 y)),$$

$$(21) \quad u(x, y) = (f(x) - f(\theta x))(g(y) + g(\varphi y) + g(\varphi^2 y)),$$

respectively, where  $f$  and  $g$  are arbitrary functions.

REMARK 1. If we put  $f(x)g(y) = F(x, y)$ , then (20) and (21) can be written as

$$(22) \quad u(x, y) = F(x, y) + F(x, \varphi y) + F(x, \varphi^2 y) \\ + F(\theta x, y) + F(\theta x, \varphi y) + F(\theta x, \varphi^2 y)$$

$$(23) \quad u(x, y) = F(x, y) + F(x, \varphi y) + F(x, \varphi^2 y) \\ - F(\theta x, y) - F(\theta x, \varphi y) + F(\theta x, \varphi^2 y),$$

respectively.

It is easily verified that (22) and (23) satisfy the equations (18) and (19), when  $F$  is an arbitrary function. Moreover, it can be shown that the obtained solutions are *general*. Indeed, the equation (18) is equivalent to the equation

$$u(x, y) = \frac{1}{6} (u(x, y) + u(x, \varphi y) + u(x, \varphi^2 y) + u(\theta x, y) + u(\theta x, \varphi y) + u(\theta x, \varphi^2 y)),$$

which is a reproductive equation, i.e. has the form

$$u = Au, \quad \text{with} \quad A^2 = A.$$

Hence, see for example [1], its general solution is

$$u(x, y) = A\Pi(x, y),$$

where  $\Pi$  is arbitrary, which is precisely (22) with  $6F = \Pi$ .

Similar constutions hold for the solution (23) of the equation (19).

REMARK 2. It can also be shown that the equations (18) and (19) are the only equations of the form (17) which have nontrivial solutions.

### 3. A remark on first order equations

The exposed method gives only one or more *particular* solutions of the considered equation. However, in Examples 1, 2 and 6, using those particular solutions we arrived at the *general* solutions. Moreover, the solution of the equation (14) from Example 3 is also called the general solution in book [2], though no proof is given. Some other investigations, which are not the subject of this paper, lead us to conjecture that the solutions given in Examples 4 and 5 are also general.

### 4. A note on some second order equations

The method given here can clearly be applied to higher order equations, but the corresponding algebraic equation will be more complicated. In papers [3] and [4] we applied this method to second order partial differential and difference equations.

We shall briefly consider some examples of the second order equation

$$(24) \quad (L_1 L_2 + aL_1 + bL_2 + cI)u = 0 \quad (a, b, c \in \mathbf{R})$$

with the corresponding algebraic equation

$$\lambda\mu + a\lambda + b\mu + c = 0.$$

If  $V$ ,  $L_1$ ,  $L_2$  are defined as in Examples 3, 4, 5 and 6, respectively, we obtain the following equations

$$(25) \quad u_{n+1}'(x) + au_{n+1}(x) + (b-1)u_n'(x) + (c-a)u_n(x) = 0,$$

$$\begin{aligned}
(26) \quad & u_y(\theta x, y) + au(\theta x, y) + (b-1)u_y(x, y) + (c-a)u(x, y) = 0, \\
(27) \quad & u(\theta x, y+1) + (a-1)u(\theta x, y) + (b-1)u(x, y+1) + (c-a-b+1)u(x, y) = 0, \\
(28) \quad & u(\theta x, \varphi y) + (a-1)u(\theta x, y) + (b-1)u(x, \varphi y) + (c-a-b+1)u(x, y) = 0.
\end{aligned}$$

The equation (25) is formally satisfied by

$$(29) \quad u_n(x) = \sum_{\lambda} C_{\lambda} (1 + \lambda)^n e^{\frac{\alpha\lambda + c}{\lambda + b} x}$$

and this series can, in some cases, be summed. For example, if  $c = ab$  then (29) reduces to  $u_n(x) = f(n)e^{-ax}$ , where  $f$  is arbitrary.

For the equation (26) we obtain the solutions:

$$\begin{aligned}
u(x, y) &= (f(x) + f(\theta x)e^{-\frac{c}{b}y} + (g(x) - g(\theta x))e^{\frac{2a-c}{b-a}y}) \quad \text{if } b(b-2) \neq 0, \\
u(x, y) &= F(x, y) + F(\theta x, y) + (g(x) - g(\theta x))e^{-ay} \quad \text{if } b = c = 0, \\
u(x, y) &= (g(x) - g(\theta x))e^{\frac{c-2a}{2}y} \quad \text{if } b = 0, c \neq 0, \\
u(x, y) &= (f(x) + f(\theta x))e^{-ay} + G(x, y) - G(\theta x, y) \quad \text{if } b = 2, c = 2a, \\
u(x, y) &= (f(x) + f(\theta x))e^{-\frac{c}{2}y} \quad \text{if } b = 2, c \neq 2a,
\end{aligned}$$

where  $f, g, F, G$  are arbitrary functions.

Similar results hold for the equation (27).

Finally, regarding the equation (28) we conclude that we can obtain solutions only for the following two equations

$$\begin{aligned}
u(\theta x, \varphi y) + (a-1)u(\theta x, y) + (b-1)u(x, \varphi y) + (1-a-b)u(x, y) &= 0, \\
u(\theta x, \varphi y) + (a-1)u(\theta x, y) + (b-1)u(x, \varphi y) + (1+a-b)u(x, y) &= 0.
\end{aligned}$$

The solutions are again (22) and (23), respectively, where  $F$  is an arbitrary function.

#### REFERENCES

- [1] S. B. Prešić, *Une Classe d'équations matricielles et l'équation fonctionnelle  $f^2 = f$* . Publ. Inst. Math. (Beograd) **8** (22) (1968), 143–148.
- [2] M. R. Spiegel, *Finite differences and difference equations*. New York 1971.
- [3] J. D. Kečkić, *On integration of linear partial differential equations with constant coefficients*. Math. Balkanica **1** (1971), 134–139.
- [4] J. D. Kečkić, *Analogies between differential and difference equations II: First and non-parabolic second order partial equations with constant coefficients*. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. **634**–No. **677** (1979), 192–196.

Tikveška 2  
11000 Beograd