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A fixed point theorem for random asymptotically nonexpansive mappings

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Abstract. In this paper, we first establish the following fixed point theorem for a random asymptotically nonexpansive mapping, which can be regarded as a random generalization of the classical Goebel-Kirk fixed point theorem: let $(E, \|\cdot\|)$ be a complete random uniformly convex random normed module and G be an almost surely bounded closed L^0-convex subset of E, then every random asymptotically nonexpansive mapping f from G to G has a fixed point in G. Second, we prove that the set Y of fixed points of f is closed and L^0-convex. Finally, we show that every eventually random asymptotically nonexpansive mapping f also has a fixed point. Since the classical method used to prove the Goebel-Kirk fixed point theorem for an asymptotically nonexpansive mapping does not work directly for the current random setting, we are forced to make use of the connection between the random uniform convexity of the complete random normed module $(E, \|\cdot\|)$ and the uniform convexity of the abstract $L^p(E)$-space generated by E, where p is a given positive number with $1 < p < +\infty$. Specifically, we decompose a random asymptotically nonexpansive operator on G into a sequence of smaller operators on a bounded closed convex subset of $L^p(E)$ such that each smaller operator is a classical eventually asymptotically nonexpansive mapping on the corresponding bounded closed convex subset. Consequently, by using the σ-stability of f and G, we can establish a precise relation between the fixed point set of f and the fixed point sets of these smaller operators, which makes us finally complete the proofs of the above mentioned main results.

Contents

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1. Introduction and the main results

In 1965, Browder and Kirk established a remarkable fixed point theorem in [1, 18] for a nonexpansive mapping, which states that, for a weakly compact convex subset G with normal structure of a Banach space, every nonexpansive mapping f from G to G has a fixed point. Browder-Kirk fixed point theorem shows the strong intersection of fixed point theory with geometry of Banach spaces.

In 1972, Goebel and Kirk generalized Browder's work [1] from nonexpansive mappings to asymptotically nonexpansive mappings. Precisely, they gave the following famous fixed point theorem in [4]: let $(B, \|\cdot\|)$ be a uniformly convex Banach space, C a nonempty bounded closed convex subset of B and $f : C \to C$ an asymptotically nonexpansive mapping, namely, there exists a sequence $\{k_m, m \in \mathbb{N}\}$ of nonnegative real numbers such that $\lim_{m\to\infty} k_m = 1$ and $\|f^m x - f^m y\| \leq k_m\|x - y\|$ for any $x, y \in C$ and $m \in \mathbb{N}$, then f has a fixed point in C, where f^m denotes the m-th iteration of f. Since then, the work in [4] has attracted the attention of many scholars in the field of nonlinear analysis, see, for example, [17, 19, 23, 24, 27]. The purpose of this paper is to extend the Goebel-Kirk fixed point theorem from asymptotically nonexpansive mappings in a uniformly convex Banach space to random asymptotically nonexpansive mappings in a complete random uniformly convex random normed module.

Random normed modules(briefly, RN modules) are a random generalization of ordinary normed spaces, which were independently introduced by Guo in [6, 7] and Gigli in [3] and have been one of basic frameworks in random functional analysis. Just based on such an idea of randomizing the traditional space theory, random functional analysis has been deeply and widely developed [7, 10], in particular random functional analysis has also been successfully applied to dynamic risk measures [8, 9] and nonsmooth differential geometry [3, 20, 21].

With the deep development of random functional analysis in financial applications, the current central task of random functional analysis is to extend fixed point theory from Banach spaces or locally convex spaces to complete random normed modules and random locally convex modules. In 2020, the notion of L^0-convex compactness for an L^0-convex set was introduced by Guo [15], where a characterization theorem for a closed L^0-convex set in a complete RN module to have L^0-convex compactness was established, which can be regarded as a generalization of the famous James characterization theorem for a closed convex subset of a Banach space to be weakly compact. Furthermore, the notion of random normal structure was introduced in [16], the authors of [14] successfully generalized Kirk's fixed point theorem [18] from a Banach space to a complete RN module, which states that, for an L^0-convexly compact closed L^0-convex subset G with random normal structure of a complete RN module $(E, \|\cdot\|)$ over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, every nonexpansive mapping f from G to G has a fixed point. Recently, Mu, et.al further established several common fixed point theorems for a commutative family of nonexpansive mappings in complete random normed modules in [22], which is an important advance in the fixed point theory of nonexpansive mappings in complete random normed modules. In 2024, the two notions of random sequentially compact sets and random sequentially continuous mappings were introduced by Guo, et.al. in [10], where they successfully generalized the Schauder fixed point theorem [5] to *RN* modules. In 2024, by utilizing the theory of random sequential compactness in random normed modules, Wang, et.al. established a noncompact Dotson fixed point theorem in [26]. Shortly afterwards, based on the notion of stable compactness, Tu, et.al. established the random Markov-Kakutani fixed point theorem together with its connection with the random Hahn-Banach theorem in [25].

The work of this paper depends on geometry of *RN* modules, which began with Guo and Zeng's work in [12], where the notions of random strict and uniform convexities were introduced and the equivalence between random uniform convexity of an *RN* module E and uniform convexity of the abstract $L^p(E)$-space generated by E was established when $1 < p < +\infty$. Besides the important advance on random uniform convexity, this paper is also considerably motivated by work in [10]. The authors in [10] provided a good idea when they studied the fixed point problem for a σ-stable mapping f, namely, first decomposing f to countably many smaller operators, then considering the fixed point problem for each smaller operator and eventually obtaining a fixed point of f by countably concatenating the fixed points of these smaller operators. Compared with the theory of classical uniformly convex spaces, the theory of random uniformly convex spaces has not developed so fully that we can adopt a similar method used to prove the Goebel-Kirk fixed point theorem for an asymptotically nonexpansive mapping, for example, we do not know if random convexity modulus is continuous, which forces us to adopt a new method different from that used in [4]. Just motivated by the idea of [10], in this paper we first decompose a random asymptotically nonexpansive operator on an almost surely (briefly, a.s.) bounded closed L^0-convex subset of a complete random uniformly convex random normed module E into a sequence of smaller operators on a bounded closed convex subset of $L^p(E)$ such that each smaller operator is a classical eventually asymptotically nonexpansive mapping on the corresponding bounded closed convex subset. Consequently, by using the σ-stability of f and G, we can establish a precise relation between the fixed point set of f and the fixed point sets of these smaller operators, which makes us finally complete the proofs of the main results of this paper.

To present the main results of this paper, we need to introduce some necessary preliminaries in random functional analysis.

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a given probability space, $\mathbb{N}$ the set of positive integers, $\mathbb{K}$ the scalar field $\mathbb{R}$ of real numbers or $\mathbb{C}$ of complex numbers, $L^0(\mathcal{F}, \mathbb{K})$ the algebra of equivalence classes of $\mathbb{K}$-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$ (as usual, two random variables equal a.s. are said to be equivalent). Specially, $L^0(\mathcal{F}) := L^0(\mathcal{F}, \mathbb{R})$ and $\bar{L}^0$ the set of equivalence classes of extended real valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

Proposition 1.1 can be regarded as a random version of the classical supremum and infimum principle. The partial order $\leq$ on $\bar{L}^0(\mathcal{F})$ is defined by $\xi \leq \eta$ iff $\xi^0(\omega) \leq \eta^0(\omega)$ for P-almost surely all $\omega \in \Omega$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η respectively.

Proposition 1.1 ([2]). *$(\bar{L}^0(\mathcal{F}), \leq)$ is a complete lattice, for any nonempty subset H of $\bar{L}^0(\mathcal{F})$, $\vee H$ and $\wedge H$ denote the supremum and infimum of H, respectively. Also, the following statements hold:*

(1) There are two sequences $\{a_n, n \in \mathbb{N}\}$ and $\{b_n, n \in \mathbb{N}\}$ in H such that $\vee_{n\geq 1} a_n = \vee H$ and $\wedge_{n\geq 1} b_n = \wedge H$.
(2) If H in (1) is directed upwards (resp., downwards), i.e., there exists $h_3 \in H$ for any two elements h_1 and h_2 in H such that $h_1 \vee h_2 \leq h_3$ ($h_3 \leq h_1 \wedge h_2$), then $\{a_n, n \in \mathbb{N}\}$ (resp., $\{b_n, n \in \mathbb{N}\}$) can be chosen as nondecreasing (resp., nonincreasing).
(3) $(L^0(\mathcal{F}), \leq)$ is a Dedekind complete lattice, i.e., every nonempty subset with an upper bound has a supremum.

As usual, $\xi < \eta$ means $\xi \leq \eta$ and $\xi \neq \eta$ for any ξ and η in $\bar{L}^0(\mathcal{F})$, whereas, for any $A \in \mathcal{F}$, $\xi < \eta$ on A ($\xi \leq \eta$ on A) means $\xi^0(\omega) < \eta^0(\omega)$ ($\xi^0(\omega) \leq \eta^0(\omega)$) for almost all ω in A, where ξ^0 and η^0 are respectively arbitrarily chosen representatives of ξ and η.

In this paper, the following notation are always employed:
$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \geq 0\}$;
$L^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$;
$\bar{L}^0_+(\mathcal{F}) = \{\xi \in \bar{L}^0(\mathcal{F}) \mid \xi \geq 0\}$.

For any $A \in \mathcal{F}$, $\tilde{I}_A$ stands for the equivalence class of I_A, where I_A stands for the characteristic function of A, namely $I_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise.

Definition 1.2 ([6]). *An ordered pair $(E, \|\cdot\|)$ is called a random normed module(briefly, an RN module) over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ if E is a left module over the algebra $L^0(\mathcal{F}, \mathbb{K})$ (briefly, an $L^0(\mathcal{F}, \mathbb{K})$-module) and $\|\cdot\|$ is a mapping from E to L^0_+ such that the following are satisfied:*

(RNM-1) $\|x\| = 0$ implies $x = \theta$ (the null in E);
(RNM-2) $\|\xi \cdot x\| = |\xi| \cdot \|x\|$ for any $(\xi, x) \in L^0(\mathcal{F}, \mathbb{K}) \times E$;
(RNM-3) $\|x + y\| \leq \|x\| + \|y\|$ for all x and $y \in E$.

As usual, $\|\cdot\|$ is called the L^0-norm on E.

$(L^0(\mathcal{F}, \mathbb{K}), |\cdot|)$ is a simplest *RN* module, where $|\cdot|$ is the usual absolute value mapping on $L^0(\mathcal{F}, \mathbb{K})$. In this paper, an *RN* module is always endowed with the (ε, λ)-topology as follows. It is easy to see that the (ε, λ)-topology for $(L^0(\mathcal{F}, \mathbb{K}), |\cdot|)$ is exactly the topology of convergence in probability P.

Proposition 1.3 ([7]). *Let $(E, \|\cdot\|)$ be an RN module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon > 0$ and $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in E | P\{\omega \in \Omega : \|x\|(\omega) <$

$\varepsilon\} > 1-\lambda\}$, called the (ε,λ)-neighborhood of θ. Then $\mathcal{U}_\theta = \{N_\theta(\varepsilon,\lambda), \varepsilon > 0, 0 < \lambda < 1\}$ forms a local base for some metrizable linear topology on E, called the (ε,λ)-topology, denoted by $\mathcal{T}_{\varepsilon,\lambda}$. Furthermore, the following statements hold:

(1) $L^0(\mathcal{F},\mathbb{K})$ is a topological algebra over $\mathbb{K}$;
(2) E is a topological module over the topological algebra $L^0(\mathcal{F},\mathbb{K})$;
(3) A sequence $\{x_n, n\geq 1\}$ in E converges to $x\in E$ in the (ε,λ)-topology iff $\{\|x_n - x\|, n\in\mathbb{N}\}$ converges to 0 in probability P.

Let $(E,\|\cdot\|)$ be an *RN* module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$. A nonempty subset G of E is said to be L^0-convex if $\xi x+\eta y\in G$ for any x and $y\in G$ and any ξ and $\eta\in L^0_+(\mathcal{F})$ such that $\xi+\eta=1$, and **a**.**s**. bounded or L^0-bounded if $\bigvee\{\|x\| : x\in G\}\in L^0_+(\mathcal{F})$. The set $supp(E)=\{\omega\in E \mid \xi^0(\omega)=+\infty\}$ is called the support of E ($supp(E)$ is unique **a**.**s**.), where $\xi=\bigvee\{\|x\| : x\in E\}$ and ξ^0 is an arbitrarily chosen representative of ξ. If $P(supp(E))=1$, then E is said to have full support. In the remainder of this paper, it is always assumed that all *RN* modules mentioned have full support.

Further, we employ the following notation for a brief introduction to random uniformly convex *RN* modules:
$\varepsilon_{\mathcal{F}}[0,2]=\{\varepsilon\in L^0_{++}(\mathcal{F}) :$ there exists a positive number λ such that $\lambda\leq\varepsilon\leq 2\}$.
$\delta_{\mathcal{F}}[0,1]=\{\delta\in L^0_{++}(\mathcal{F}) :$ there exists a positive number η such that $\eta\leq\delta\leq 1\}$.

For any x, y in E, denote the equivalence class of $\mathcal{F}$-measurable set $\{\omega\in\Omega : \|x\|^0(\omega)\neq 0\}$ by A_x, called the support of x, where $\|x\|^0$ is an arbitrarily chosen representative of $\|x\|$; and we briefly write $A_{x,y}=A_x\cap A_y$ and $B_{x,y}=A_x\cap A_y\cap A_{x-y}$.

Definition 1.4 ([12]). *Let $(E,\|\cdot\|)$ be a complete RN module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$. E is said to be random uniformly convex if for each $\varepsilon\in\varepsilon_{\mathcal{F}}[0,2]$ there exists $\delta\in\delta_{\mathcal{F}}[0,1]$ such that $\|x-y\|\geq\varepsilon$ on D always implies $\|x+y\|\leq 2(1-\delta)$ on D for any x and $y\in U(1)$ and any $D\in\mathcal{F}$ such that $D\subset B_{x,y}$ and $P(D)>0$, where $U(1)=\{z\in E \mid \|z\|\leq 1\}$, called the random closed unit ball of E.*

Definition 1.5. *Let $(E,\|\cdot\|)$ be an RN module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ and G be a nonempty subset of E. A mapping $f : G\to G$ is said to be random asymptotically nonexpansive if there exists a sequence $\{\xi_m, m\in\mathbb{N}\}$ in $L^0_+(\mathcal{F})$ with $\{\xi_m, m\in\mathbb{N}\}$ convergent **a.s.** to 1, such that*

$$\|f^m x - f^m y\|\leq \xi_m\|x-y\|, \forall x,y\in G \text{ and } m\in\mathbb{N}.$$

*A mapping $f : G\to G$ is called an eventually random asymptotically nonexpansive mapping if there exist some $l\in\mathbb{N}$ and a sequence $\{\xi_m, m\geq l\}$ in $L^0_+(\mathcal{F})$ with $\{\xi_m, m\geq l\}$ convergent **a.s.** to 1, such that*

$$\|f^m(x)-f^m(y)\|\leq\xi_m\|x-y\|, \forall x,y\in G \text{ and } m\geq l.$$

Theorems 1.6, 1.7 and 1.8 below are the main results of this paper, as random generalizations of Theorems 1, 2 and 3 in [4], respectively.

Theorem 1.6. *Let $(E,\|\cdot\|)$ be a complete random uniformly convex RN module over $\mathbb{K}$ with base $(\Omega,\mathcal{F},P)$ and G be an* **a**.**s**. *bounded closed L^0-convex subset of*

E, then every random asymptotically nonexpansive mapping $f : G \to G$ has a fixed point.

Theorem 1.7. *Under the same assumptions as in Theorem* 1.6*, then the set Y of fixed points of f is closed and L^0-convex.*

Theorem 1.8 below shows that f in Theorem 1.6 only needs to be eventually random asymptotically nonexpansive.

Theorem 1.8. *Let $(E, \|\cdot\|)$ be a complete random uniformly convex RN module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ and G be an* **a.s.** *bounded closed L^0-convex subset of E. Further suppose that $f : G \to G$ is an eventually random asymptotically nonexpansive mapping, then f has a fixed point.*

2. Proofs of the main results

Proposition 2.1 below is Theorem 3 of [4], which will be used in the proof of Theorem 1.6.

Proposition 2.1. *Suppose C is a nonempty, closed, bounded and convex subset of a uniformly convex Banach space $(B, \|\cdot\|)$ and $F : C \to C$ is an arbitrary (even noncontinuous) transformation such that for some positive integer n,*

$$\|F^i x - F^i y\| \leq k_i \|x - y\|, i \geq n,$$

where $\{k_i, i \geq n\}$ is a sequence of nonnegative real numbers with $\lim_{i\to\infty} k_i = 1$. *Then F has a fixed point.*

Remark 2.2. *It is easy to see from the proof of Theorem 3 of* [4] *that the set of fixed points of F in Proposition* 2.1 *is also closed and convex.*

Lemma 2.3 below is crucial for the proofs of the main results of this paper, as it establishes a connection between *RN* modules and normed spaces (see [7] for details).

Lemma 2.3. *Let $(E, \|\cdot\|)$ be an RN module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$, and $1 \leq p \leq +\infty$. Let $L^p(E) = \{x \in E | \|x\|_p < +\infty\}$, where $\|\cdot\|_p : E \to [0, +\infty]$ is defined by:*

$$\|x\|_p = \begin{cases} (\int_\Omega (\|x\|^p) dP)^{\frac{1}{p}}, & \text{when } 1 \leq p < +\infty; \\ inf\{M \in [0, +\infty] \mid \|x\| \leq M\}, & \text{when } p = +\infty \end{cases}$$

for all $x \in E$. Then $(L^p(E), \|\cdot\|_p)$ is a normed space and $L^p(E)$ is $\mathcal{T}_{\varepsilon,\lambda}$-dense in E.

Remark 2.4. *For $1 \leq p \leq +\infty$, if $(E, \|\cdot\|)$ is a complete RN module, then $(L^p(E), \|\cdot\|_p)$ is a Banach space (see* [11]*). For $1 < p < +\infty$, if $(E, \|\cdot\|)$ is a complete RN module, then $(L^p(E), \|\cdot\|_p)$ is uniformly convex iff E is random uniformly convex (see* [12, 13]*).*

Definitions 2.5 and 2.6 below provide the notions of stability of sets and mappings, and they have played an important role in random functional analysis (see [7, 10, 14, 15, 16, 22, 25, 26]), particularly in the proofs of the main results of this paper.

Definition 2.5. *Let E be an $L^0(\mathcal{F},\mathbb{K})$-module and G be a nonempty subset of E. G is said to be stable if $\tilde{I}_A x + \tilde{I}_{A^c} y \in G$ for any $x, y \in G$ and any $A \in \mathcal{F}$, where $A^c = \Omega \setminus A$. G is said to be σ-stable (or, to have the countable concatenation property in the original terminology of [7]) if for any sequence $\{x_n, n \in \mathbb{N}\}$ in G and any countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ (namely, each $A_n \in \mathcal{F}$, $A_i \bigcap A_j = \emptyset$ for any $i \neq j$, and $\bigcup_{n=1}^{\infty} A_n = \Omega$) there exists $x \in G$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} x_n$ for each $n \in \mathbb{N}$.*

It is known from [7] that when $(E, \|\cdot\|)$ is an RN module or a more general regular L^0-module (see [10] for the notion of a regular L^0-module), x in Definition 2.5 is unique and can be written as $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$. Also, any closed L^0-convex subset of a complete RN module is always σ-stable.

Definition 2.6. *Let E be a regular $L^0(\mathcal{F},K)$-module and G be a nonempty subset of E. The mapping $f : G \to G$ is said to be*

(1) σ-stable, if G is σ-stable and

$$f(\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n) = \sum_{n=1}^{\infty} \tilde{I}_{A_n} f(x_n)$$

for every sequence $\{x_n, n \in \mathbb{N}\}$ in G and every countable partition $\{A_n, n \in \mathbb{N}\}$ of Ω to $\mathcal{F}$;

(2) have the local property if G is stable, $\theta \in G$ and

$$\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$$

for any $A \in \mathcal{F}$ and any $x \in G$ (Here, we should like to remind the reader of the fact that $\tilde{I}_A x \in G$ for any $A \in \mathcal{F}$ and any $x \in G$ when G is stable and $\theta \in G$).

By Lemma 2.11 of [14], any L^0-Lipschitzian mapping defined on a σ-stable set of an RN module is σ-stable. So, if f is a random asymptotically nonexpansive mapping defined on a σ-stable set, then f is σ-stable. According to Remark 3.2 of [15], if G is a σ-stable set with $\theta \in G$, then f is σ-stable iff f has the local property.

With the above preparations, we are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Without loss of generality, we can assume that $\theta \in G$(otherwise, take an arbitrary $p_0 \in G$, and replace G and f with $G' = G - p_0$ and $f' : G' \to G'$ defined by $f'(p) = f(p + p_0) - p_0$, $\forall p \in G'$).

Since $f : G \to G$ is a random asymptotically nonexpansive mapping, there exists a sequence $\{\xi_m, m \in \mathbb{N}\}$ in $L^0_+(\mathcal{F})$ with $\{\xi_m, m \in \mathbb{N}\}$ convergent **a.s.** to 1, such that $\|f^m x - f^m y \| \leq \xi_m \|x - y\|$, $\forall x, y \in G$, $m \in \mathbb{N}$. By Egoroff's theorem, there exists $E_k \in \mathcal{F}$ for each $k \in \mathbb{N}$ such that $P(\Omega \setminus E_k) < \frac{1}{k}$ and $\{\xi_m, m \in \mathbb{N}\}$ converges uniformly to 1 on E_k. Since $P(\bigcup_{k=1}^{\infty} E_k) = 1$, without loss of generality, we can assume that $\Omega = \cup_{k=1}^{\infty} E_k$. Furthermore, let $\Omega_1 = E_1$ and $\Omega_k = E_k \setminus \cup_{i=1}^{k-1} E_{i-1}$ for any $k \in \mathbb{N}$ with $k \geq 2$, then $\{\Omega_k, k \in \mathbb{N}\}$ is a countable partition of Ω to $\mathcal{F}$ and $\{\xi_m, m \in \mathbb{N}\}$ converges uniformly to 1 on Ω_k for any $k \in \mathbb{N}$.

Since G is an **a.s** bounded subset of E, there exists $\eta \in L^0_+(\mathcal{F})$ such that $\|g\| \leq \eta$ for all $g \in G$. Let

$$A_n = \{\omega \in \Omega | n - 1 \leq \eta^0(\omega) < n\}$$

for any $n \in \mathbb{N}$, where η^0 is an arbitrarily chosen representative of η. Then $\{A_n, n \in \mathbb{N}\}$ is a countable partition of Ω to $\mathcal{F}$.

It is clear that $\{A_n \cap \Omega_k, n, k \in \mathbb{N}\}$ is a countable partition of Ω to $\mathcal{F}$. For any $k \in \mathbb{N}$ and $n \in \mathbb{N}$, define $f_{n,k} : \tilde{I}_{A_n \cap \Omega_k} G \to \tilde{I}_{A_n \cap \Omega_k} G$ by

$$f_{n,k}(x) = \tilde{I}_{A_n \cap \Omega_k} f(x), \ \forall x \in \tilde{I}_{A_n \cap \Omega_k} G.$$

One can easily check that $f_{n,k}(\tilde{I}_{A_n \cap \Omega_k} g) = \tilde{I}_{A_n \cap \Omega_k} f(g)$ for any $g \in G$ by the local property of f.

The remainder of the proof is divided into two steps.

Step 1. We prove that each $f_{n,k}$ has a fixed point in $\tilde{I}_{A_n \cap \Omega_k} G$.

By Proposition 2.1, it suffices to prove $\tilde{I}_{A_n \cap \Omega_k} G$ is a bounded $\| \cdot \|_p$-closed convex subset of $(L^p(E), \| \cdot \|_p)$, where p is any fixed number such that $1 < p < +\infty$, and $f_{n,k}$ is an eventually asymptotically nonexpansive mapping.

First, for any $h \in \tilde{I}_{A_n \cap \Omega_k} G$, since

$$\|h\|_p = (\int_{\Omega} \|\tilde{I}_{A_n \cap \Omega_k} h\|^p dP)^{\frac{1}{p}} \leq n,$$

then $\tilde{I}_{A_n \cap \Omega_k} G$ is a bounded subset of $(L^p(E), \| \cdot \|_p)$.

Second, since $\tilde{I}_{A_n \cap \Omega_k} G$ is a $\mathcal{T}_{\varepsilon,\lambda}$-closed subset of E, by the Lebesgue dominance convergence theorem it is easy to see that $\tilde{I}_{A_n \cap \Omega_k} G$ is a $\| \cdot \|_p$-closed subset of $(L^p(E), \| \cdot \|_p)$.

Third, since G is L^0-convex, it is naturally convex, it is obvious that $\tilde{I}_{A_n \cap \Omega_k} G$ is also convex.

Finally, for any $g \in \tilde{I}_{A_n \cap \Omega_k} G$, we have

$$\begin{aligned}
f^m_{n,k}(g) &= f_{n,k}(f^{m-1}_{n,k}(g)) \\
&= \tilde{I}_{A_n \cap \Omega_k} f(f^{m-1}_{n,k}(g)) \\
&= \tilde{I}_{A_n \cap \Omega_k} f(\tilde{I}_{A_n \cap \Omega_k} f(f^{m-2}_{n,k}(g))) \\
&\cdots \\
&= \tilde{I}_{A_n \cap \Omega_k} f(\tilde{I}_{A_n \cap \Omega_k} f(\cdots \tilde{I}_{A_n \cap \Omega_k} f(\tilde{I}_{A_n \cap \Omega_k} f(g))) \\
&= \tilde{I}_{A_n \cap \Omega_k} f^m(g) \quad (since\ f\ has\ the\ local\ property)
\end{aligned}$$

for any $m \in \mathbb{N}$, where $f^m_{n,k}$ denotes the m-th iteration of $f_{n,k}$.

For $m \in \mathbb{N}$, let $\gamma_m = \|\tilde{I}_{A_n \cap \Omega_k} \xi_m\|_\infty$, then $\lim_{m\to\infty} \gamma_m = 1$ since $\{\xi_m, m \in \mathbb{N}\}$ converges uniformly to 1 on Ω_k for any $k \in \mathbb{N}$. It follows that

$$\begin{aligned}
\|f^m_{n,k}(x) - f^m_{n,k}(y)\|_p &= (\int_\Omega \|f^m_{n,k}(x) - f^m_{n,k}(y)\|^p dP)^{\frac{1}{p}} \\
&= (\int_{A_n \cap \Omega_k} \|f^m(x) - f^m(y)\|^p dP)^{\frac{1}{p}} \\
&\le (\int_{A_n \cap \Omega_k} |\xi_m|^p \|x - y\|^p dP)^{\frac{1}{p}} \\
&\le \gamma_m \|x - y\|_p
\end{aligned}$$

for any $x, y \in \tilde{I}_{A_n \cap \Omega_k} G$ and any $m \in \mathbb{N}$, which implies that $f_{n,k}$ is an eventually asymptotically nonexpansive mapping (Here, we can only assert $f_{n,k}$ is an eventually asymptotically nonexpansive mapping since it is possible that there exists some $m_0 \in \mathbb{N}$ such that $\gamma_m = +\infty$ for some $m \le m_0$).

Hence, there exists $x_{n,k} \in \tilde{I}_{A_n \cap \Omega_k} G$ such that $f_{n,k}(x_{n,k}) = x_{n,k}$.

Step 2. We prove that f has a fixed point.

Let $x = \sum_{n=1}^\infty \sum_{k=1}^\infty \tilde{I}_{A_n \cap \Omega_k} x_{n,k}$, then $x \in G$. Since f is σ-stable, then f has the local property, we have

$$\begin{aligned}
f(x) &= \sum_{n=1}^\infty \sum_{k=1}^\infty \tilde{I}_{A_n \cap \Omega_k} f(x) \\
&= \sum_{n=1}^\infty \sum_{k=1}^\infty \tilde{I}_{A_n \cap \Omega_k} f(\tilde{I}_{A_n \cap \Omega_k} x) \\
&= \sum_{n=1}^\infty \sum_{k=1}^\infty \tilde{I}_{A_n \cap \Omega_k} f_{n,k}(x_{n,k}) \\
&= \sum_{n=1}^\infty \sum_{k=1}^\infty \tilde{I}_{A_n \cap \Omega_k} x_{n,k} \\
&= x.
\end{aligned}$$

Thus, f has a fixed point x in G. □

Remark 2.7. *When $(\Omega, \mathcal{F}, P)$ is trivial, namely $\mathcal{F} = \{\Omega, \emptyset\}$, the complete random uniformly convex RN module $(E, \|\cdot\|)$ reduces to a uniformly convex Banach space, G to a bounded closed convex subset of E and f to an asymptotically nonexpansive mapping, and then the classical Goebel-Kirk fixed point theorem for an asymptotically nonexpansive mapping, namely, Theorem 1 of* [4] *is a special case of Theorem* 1.6.

$L^0(\mathcal{F}, [0,1])$ denotes the set of equivalence classes of random variables from $(\Omega, \mathcal{F}, P)$ to $[0,1]$. Before we give the proof of Theorem 1.7, we first prove Lemma 2.8 below.

Lemma 2.8. *Let $(E, \|\cdot\|)$ be a complete RN module over $\mathbb{K}$ with base $(\Omega, \mathcal{F}, P)$ and G be a stable closed convex subset of E. Then G is L^0-convex.*

Proof. Since the set of simple elements in $L^0(\mathcal{F}, [0,1])$ is $\mathcal{T}_{\varepsilon,\lambda}$-dense in $L^0(\mathcal{F}, [0,1])$ and G is closed, it suffices to show that

$$\xi x + (1-\xi)y \in G$$

for any $x, y \in G$ and any simple element ξ in $L^0(\mathcal{F}, [0,1])$.

Let $\xi = \sum_{i=1}^{n} \tilde{I}_{A_i}\alpha_i$, where α_i is a nonnegative number with $0 \leq \alpha_i \leq 1$ for any $i = 1 \sim n$ and $\{A_i, i = 1 \sim n\}$ is a finite partition of Ω to $\mathcal{F}$, since G is stable and convex, we have

$$\begin{aligned} \xi x + (1-\xi)y &= \sum_{i=1}^{n} \tilde{I}_{A_i}\alpha_i x + \sum_{i=1}^{n} \tilde{I}_{A_i}(1-\alpha_i)y \\ &= \sum_{i=1}^{n} \tilde{I}_{A_i}(\alpha_i x + (1-\alpha_i)y) \\ &\in \sum_{i=1}^{n} \tilde{I}_{A_i} G \\ &\subset G \end{aligned}$$

for any $x, y \in G$. Hence G is L^0-convex. □

Let us first recall that, for any countable partition $\{B_m, m \in \mathbb{N}\}$ of Ω to $\mathcal{F}$ and any sequence $\{G_m, m \in \mathbb{N}\}$ of nonempty subsets of E,

$$\sum_{m=1}^{\infty} \tilde{I}_{B_m} G_m := \{\sum_{m=1}^{\infty} \tilde{I}_{B_m} g_m,\ g_m \in G_m,\ \forall m \in \mathbb{N}\}$$

is called the countable concatenation of $\{G_m, m \in \mathbb{N}\}$ along $\{B_m, m \in \mathbb{N}\}$.

Now we can give the proof of Theorem 1.7.

Proof of Theorem 1.7. First, since G is σ-stable and closed, and since f is σ-stable and continuous, then Y is σ-stable and closed.

Second, we prove that $Y = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{I}_{A_n\cap\Omega_k} Y_{n,k}$, where $Y_{n,k}$ is the fixed point set of $f_{n,k}$ for any $n, k \in \mathbb{N}$. Indeed, by the step 2 of the proof of Theorem 1.6, it is clear that $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{I}_{A_n\cap\Omega_k} Y_{n,k} \subset Y$. On the other hand, for any $x \in Y$ and any $n, k \in \mathbb{N}$, since

$$\begin{aligned} f_{n,k}(\tilde{I}_{A_n\cap\Omega_k} x) &= \tilde{I}_{A_n\cap\Omega_k} f(\tilde{I}_{A_n\cap\Omega_k} x) \\ &= \tilde{I}_{A_n\cap\Omega_k} f(x) \\ &= \tilde{I}_{A_n\cap\Omega_k} x, \end{aligned}$$

then $\tilde{I}_{A_n\cap\Omega_k} x \in Y_{n,k}$, and this implies that $x \in \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{I}_{A_n\cap\Omega_k} Y_{n,k}$. Hence $Y = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \tilde{I}_{A_n\cap\Omega_k} Y_{n,k}$.

Finally, each $Y_{n,k}$ is convex by Remark 2.2, then $Y = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \tilde{I}_{A_n \cap \Omega_k} Y_{n,k}$ is convex.

To sum up, by Lemma 2.8, Y is L^0-convex. □

The following proof of Theorem 1.8 is motivated by Proposition 2.1.

Proof of Theorem 1.8. Let $f : G \to G$ be an eventually random asymptotically nonexpansive mapping as defined in Definition 1.5, then the mapping $g = f^l$ is a random asymptotically nonexpansive mapping, which implies that its set Y of fixed points is closed and L^0-convex. For any $x \in Y$, we have

$$f(x) = f(g(x)) = f^{l+1}(x) = g(f(x)),$$

thus f maps Y into Y. Moreover, $f = f^{pl+1}$ on Y for any $p \in \mathbb{N}$. As a result,

$$\|f(x) - f(y)\| = \|f^{pl+1}(x) - f^{pl+1}(y)\| \leq \xi_{pl+1}\|x - y\|, \ \forall x, y \in Y,$$

which implies that $\|f(x) - f(y)\| \leq \|x - y\|$ for any $x, y \in Y$. Therefore, by Corollary 3.9 in [14], f has a fixed point in Y. □

Remark 2.9. *Let Y be the set of fixed points of g as defined in the proof of Theorem* 1.8*, then it is easy to see that the set Fix(f) of fixed points of f is exactly $\{y \in Y | f(y) = y\}$, and thus Fix(f) is also a closed L^0-convex subset of G by Theorem* 1.7.

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