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Iterations of the functor of naive $\mathbb{A}^1$-connected components of varieties

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ABSTRACT. For any sheaf of sets $\mathcal{F}$ on Sm/k, it is well known that the universal $\mathbb{A}^1$-invariant quotient of $\mathcal{F}$ is given as the colimit of sheaves $\mathcal{S}^n(\mathcal{F})$ where $\mathcal{S}(F)$ is the sheaf of naive $\mathbb{A}^1$-connected components of $\mathcal{F}$. We show that these infinite iterations of naive $\mathbb{A}^1$-connected components in the construction of universal $\mathbb{A}^1$-invariant quotient for a scheme are certainly required. For every n, we construct an $\mathbb{A}^1$-connected variety X_n such that $\mathcal{S}^n(X_n) \neq \mathcal{S}^{n+1}(X_n)$ and $\mathcal{S}^{n+2}(X_n) = *$.

CONTENTS

1. Introduction

Let k be a field and X be any smooth, finite-type scheme over k. In the unstable $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ [8], there are two notions of $\mathbb{A}^1$-connectedness for X. The genuine notion is the sheaf of $\mathbb{A}^1$-connected components $\pi_0^{\mathbb{A}^1}(X)$, which is given by the Nisnevich sheafication of the presheaf that associates to any smooth scheme U the set of morphisms from U to X in $\mathcal{H}(k)$. The naive notion is given by the sheaf of $\mathbb{A}^1$-chain connected components $\mathcal{S}(X)$(see Definition 2.3). Both of these notions may not coincide even for smooth and proper schemes [2]. However, if we take infinite iterations of $\mathcal{S}$ and subsequently form the direct limit, the resulting sheaf $\mathcal{L}(X)$(also known as the universal $\mathbb{A}^1$-invariant quotient) will coincide with $\pi_0^{\mathbb{A}^1}(X)$, provided that the latter is $\mathbb{A}^1$-invariant [2, Theorem 1].

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The $\mathbb{A}^1$-invariance of the sheaf of $\mathbb{A}^1$-connected components for a general space $\mathcal{X}$ in $\mathcal{H}(k)$ has recently been disproved [1]. Nevertheless, there are various examples of schemes where the equality of $\pi_0^{\mathbb{A}^1}$ and $\mathcal{L}$ has been established. It is known to coincide for $\mathbb{A}^1$-rigid schemes, proper curves [2], smooth projective surfaces over an algebraically closed field [5], smooth projective retract rational varieties over an infinite field [3], etc. Moreover, $\mathcal{L}(X)$ provides a complete geometric description of $\pi_0^{\mathbb{A}^1}(X)$ for sections over finitely generated, separable field extensions of k [4, Theorem 1.1].

In all the above examples, $\mathcal{L}$ has been shown to stabilise at some finite stage. In other words, $\mathcal{L}$ is shown to be equal to $\mathcal{S}^n$ for some n in all these cases. This leads to a natural question: are these iterations really necessary? More specifically, does there exist an n such that $\mathcal{S}^n(X) = \mathcal{L}(X)$ for any scheme X? For a general space $\mathcal{X}$, it has already been answered in the negative by Balwe-Rani-Sawant [4, Theorem 1.2]. For each n, they have constructed a sheaf of sets for which the iterations of naive $\mathbb{A}^1$-connected components do not stabilise before the nth stage. Moreover, they have remarked on the possibility of suitably modifying their construction to produce schemes X_n with the same property [4, Remark 4.7].

The purpose of this note is to show that the infinite iterations of naive $\mathbb{A}^1$-connected components in the construction of $\mathcal{L}$ are certainly required in the case of varieties as well and that the suggested example in op. cit. indeed works. We prove the following:

Theorem 1.1. *For each $n \in \mathbb{N}$, there exists a variety X_n over $\mathbb{C}$ of dimension $n+1$ such that $\mathcal{S}^n(X_n) \neq \mathcal{S}^{n+1}(X_n)$.*

The first example of a variety for which $\mathcal{S}(X) \neq \mathcal{S}^2(X)$ is of a singular surface S_1 over $\mathbb{C}$ [2, Construction 4.3]. Taking $X_1 = S_1$, we have inductively constructed a sequence of varieties X_n having two points, α_n and β_n, in $X_n(\mathbb{C})$ such that α_n and β_n have the same images in $\mathcal{S}^{n+1}(X_n)(\mathbb{C})$ but distinct images in $\mathcal{S}^n(X_n)(\mathbb{C})$. We also show that these varieties X_n are $\mathbb{A}^1$-connected and that $\pi_0^{\mathbb{A}^1}(X_n) = \mathcal{S}^{n+2}(X_n) = *$.

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2. Preliminaries

In this section, we recall relevant material from [2, 8] to make our exposition self-contained. We fix a base field k. Let Sm/k denote the Grothendieck site of smooth schemes of finite type over k equipped with the Nisnevich topology.

Notation 2.1. For any smooth scheme U over k and $t \in k$, s_t^U denotes the morphism $U \to \mathbb{A}_k^1 \times U$ given by $u \mapsto (t, u)$. For any $H \in \mathcal{F}(\mathbb{A}_k^1 \times U)$, define $H(t) := H \circ s_t^U$.

Definition 2.2. Let $\mathcal{F}$ be a sheaf of sets in Nisnevich topology. For any smooth scheme U in Sm/k and x_0, x_1 in $\mathcal{F}(U)$, we say x_0 and x_1 are $\mathbb{A}^1$-*homotopic* if there exists $h \in \mathcal{F}(\mathbb{A}_k^1 \times U)$ such that $h(0) = x_0$ and $h(1) = x_1$. Moreover, h is called an $\mathbb{A}^1$-*homotopy* connecting x_0 and x_1.

Definition 2.3. The sheaf of *naive* $\mathbb{A}^1$-*connected components* of $\mathcal{F}$, denoted by $\mathcal{S}(\mathcal{F})$ is defined as the Nisnevich sheafication of the presheaf $\mathcal{S}^{pre}(\mathcal{F})$,

$$\mathcal{S}^{pre}(\mathcal{F})(U) := \frac{\mathcal{F}(U)}{\sim},$$

where $\sim$ is the equivalence relation generated by $\mathbb{A}^1$-homotopy. Equivalently, $\mathcal{S}(\mathcal{F})$ is the Nisnevich sheafication of the presheaf

$$U \mapsto \pi_0 \mathrm{Sing}_*^{\mathbb{A}^1}(\mathcal{F})(U),$$

where $\mathrm{Sing}_{\mathbb{A}^1}^*(\mathcal{F})$ is the *Morel-Voevodsky singular construction* on $\mathcal{F}$[8, p.87].

For any sheaf of sets $\mathcal{F}$, it is immediate from the definition of $\mathcal{S}$, that $\mathcal{S}(F)$ satisfies the following universal property.

Lemma 2.4. *Let $\mathcal{F}$, $\mathcal{G} \in Shv(Sm/k)_{Nis}$ be sheaves of sets. Suppose $\psi : \mathcal{F} \to G$ is a morphism such that for any $\mathbb{A}^1$-homotopy $h \in \mathcal{F}(\mathbb{A}_k^1 \times U)$, and for any $s, t \in k$, the morphisms $(\psi \circ h)(s)$ and $(\psi \circ h)(t)$ are identical. Then ψ factors through the canonical morphism $\mathcal{F} \to \mathcal{S}(\mathcal{F})$.*

Proof. View the morphism $\psi : \mathcal{F} \to G$ as a morphism of presheaves. By the definition of $\mathcal{S}^{pre}(\mathcal{F})$, for any smooth scheme U, $\psi(U)$ factors through the morphism $\mathcal{F}(U) \to \mathcal{S}^{pre}(\mathcal{F})(U)$:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\psi(U)} & \mathcal{G}(U) \\ \downarrow & \nearrow & \\ \mathcal{S}^{pre}(\mathcal{F})(U) & & \end{array}$$

Since $\mathcal{F}$ and $\mathcal{G}$ are sheaves of sets, after Nisnevich sheafication, the lemma follows. $\square$

Definition 2.5. A sheaf $\mathcal{F} \in Shv(Sm/k)_{Nis}$ is called $\mathbb{A}^1$-*invariant* if the maps $\mathcal{F}(U) \to \mathcal{F}(\mathbb{A}_k^1 \times U)$, induced by the projections $\mathbb{A}_k^1 \times U \to U$, are bijections. We say a scheme X is $\mathbb{A}^1$-*rigid* if, when viewed as a sheaf of sets, X is $\mathbb{A}^1$-invariant.

Iterating the construction of $\mathcal{S}$ infinitely many times yields a sequence of epimorphisms

$$\mathcal{F} \to \mathcal{S}(F) \to \mathcal{S}^2(\mathcal{F}) \dots .$$

After taking the direct limit, we arrive at the *universal* $\mathbb{A}^1$*-invariant quotient* $\mathcal{L}(\mathcal{F})$,

$$\mathcal{L}(F) := \lim_{\to n} \mathcal{S}^n(\mathcal{F}).$$

Definition 2.6. For any scheme X over k, an *elementary Nisnevich covering* comprises of the following two maps:

(1) An open immersion $j : U \to X$.
(2) An etale map $p : V \to X$ where its restriction to $p^{-1}(X \setminus j(U))$ is an isomorphism onto $X \setminus j(U)$.

The resulting cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\ j\ } & X \end{array}$$

is called an *elementary distinguished square*.

One of the significant aspects of using an elementary Nisnevich covering is illustrated by the following result in [8, §3, Lemma 1.6].

Lemma 2.7. *An elementary distinguished square is a cocartesian square in the category* $Shv(Sm/k)_{Nis}$.

The lemma mentioned in [8] is originally stated for smooth schemes; however, the same proof holds for general schemes without any modifications. We will recall some results from [9, 4, 7] that will be used to prove the $\mathbb{A}^1$-connectedness of X_n. The following lemma is a standard result from [7, Lemma 6.1.3].

Lemma 2.8. *A sheaf of sets* $\mathcal{F}$ *on* Sm/k *is* $\mathbb{A}^1$*-connected if* $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K) = *$ *for any finitely generated separable extension* K *of* k.

The following theorem from [4, Theorem 2.2] provides an explicit formula for computing $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K)$ for any field K/k.

Theorem 2.9. *Let* $\mathcal{F}$ *be a sheaf of sets. For any finitely generated field extension* K/k*, the natural map* $\pi_0^{\mathbb{A}^1}(\mathcal{F})(K) \to \mathcal{L}(\mathcal{F})(K)$ *is a bijection.*

The analogue of Lemma 2.8 for $\mathcal{S}$ is given by the following result from [9, Theorem 3.2].

Theorem 2.10. *Suppose* $\mathcal{F}$ *is a sheaf on sets such that* $\mathcal{S}(\mathcal{F})(K) = *$ *for any finitely generated separable* K/k*. Then,* $\mathcal{S}^2(\mathcal{F}) = *$.

Notation 2.11. For any sheaf of sets $\mathcal{F}$ and x in $\mathcal{F}(U)$, we will use $[x]_j$ to denote the image of x in $\mathcal{S}^j(\mathcal{F})(U)$.

Notation 2.12. From now on, all schemes are defined over $\mathbb{C}$. For any two schemes X and Y, and for any $x \in X(\mathbb{C})$, we will denote the morphism $Y \to \operatorname{Spec} \mathbb{C} \xrightarrow{x} X$ by $Y \xrightarrow{x} X$.

3. Proof of Theorem 1.1

We divide the proof into three parts. First, we construct the required sequence X_n of varieties and fix two $\mathbb{C}$-valued points α_n and β_n in X_n. Second, we provide geometric arguments to show that the images of α_n and β_n cannot be equal in $\mathcal{S}^i(X_n)(\mathbb{C})$ for $i \leq n$. Finally, we use appropriate elementary Nisnevich covers of X_n to construct maps from X_n to $\mathcal{S}(X_{n+1})$ that map α_n and β_n to $[\beta_{n+1}]_1$ and $[\alpha_{n+1}]_1$(see Notation 2.11), respectively. This allows us to construct an $\mathbb{A}^1$-homotopy connecting $[\alpha_n]_n$ and $[\beta_n]_n$ in $\mathcal{S}^n(X_n)$, thereby ensuring that the images of α_n and β_n are equal in $\mathcal{S}^{n+1}(X_n)$.

3.1. Construction of the varieties X_n. For $n \geq 0$, the variety X_n is quasi affine and has dimension $n+1$. In proving the theorem, it will be useful to have the explicit equations defining X_n. Therefore, we will construct the affine varieties Y_n in $\mathbb{A}^{2n+1}_{\mathbb{C}}$, such that X_n is an open subvariety of Y_n. We begin by constructing the varieties X_n. Set $X_0 = Y_0 := \mathbb{A}^1_{\mathbb{C}}$.

Construction 3.1. We now recall the construction of surface S_1 from [2, Construction 4.3] which will serve as our X_1.

(1) Let $\lambda_i \in \mathbb{C} \setminus 0$ for $i = 1, 2, 3$, and let $f(x_1) = (x_1 - \lambda_1)(x_1 - \lambda_2)(x_1 - \lambda_3)$ with $\lambda = \sqrt{-\lambda_1\lambda_2\lambda_3}$. Define E as the following planar curve,
$$E := \operatorname{Spec} \mathbb{C}[x_1, y_1]/\langle y_1^2 - f(x_1)\rangle.$$
Let $\pi : E \to \mathbb{A}^1$ be the projection onto x_1-axis. Thus, $\pi^{-1}(0) = \{(0, \pm\lambda)\}$.

(2) Define Y_1 and X_1 as the following surfaces in $\mathbb{A}^3_{\mathbb{C}}$,
$$Y_1 := \operatorname{Spec} \mathbb{C}[x_0, x_1, y_1]/\langle y_1^2 - x_0^2 f(x_1)\rangle \,, X_1 := Y_1 \setminus \{(0,0,0)\}.$$
Let i_1 denote the inclusion of X_1 into Y_1.

(3) Let $\bar{\phi}_1$ and $\bar{\psi}_1 : Y_1 \to X_0$ be the projection onto the x_0-axis and x_1-axis, respectively. Define ϕ_1 and ψ_1 as the restrictions of $\bar{\phi}_1$ and $\bar{\psi}_1$ to X_1. The surface X_1 can be viewed as a family of curves parametrized by $\mathbb{A}^1_{\mathbb{C}}$ via ϕ_1, where the fiber over 0 is $\mathbb{G}_m$ and the fiber over any nonzero point is E.

(4) Let $\alpha_1 = (1, 0, \lambda)$ and $\beta_1 = (0, 1, 0)$. Then, α_1 is contained in the copy of E in X_1 corresponding to $x_0 = 1$, while β_1 is contained in the copy of $\mathbb{G}_m$ in X_1 corresponding to $x_0 = 0$. $\mathbb{A}^1$-rigidity of E and $\mathbb{G}_m$ will be used to show that α_1 and β_1 cannot be connected by a chain of $\mathbb{A}^1_{\mathbb{C}}$ in X_1.

(5) Let $\mathbb{A}^1_{\mathbb{C}} \times E$ denote the surface $\operatorname{Spec}\mathbb{C}[x_0, x_1, y_1]/\langle y_1^2 - f(x_1)\rangle$, and let $\rho_1 : \mathbb{A}^1_{\mathbb{C}} \times E \to Y_1$ be the morphism given by $(x_0, x_1, y_1) \mapsto (x_0, x_1, x_0 y_1)$. Then, ρ_1 is an isomorphism outside $\bar{\phi}_1^{-1}(\{0\})$. ρ_1 will be used to construct an $\mathbb{A}^1$-homotopy in $\mathcal{S}(X_1)$ connecting $[\alpha_1]_1$ and $[\beta_1]_1$.

Next, we inductively define the varieties X_n and the morphisms $\phi_n, \psi_n : X_n \to X_{n-1}$.

Construction 3.2. For $n \geq 2$, assuming that X_{n-1}, ϕ_{n-1} and ψ_{n-1} are defined, we define X_n as the pullback of the following diagram.

$$\begin{array}{ccc} X_n & \xrightarrow{\psi_n} & X_{n-1} \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle \phi_{n-1}} \\ X_{n-1} & \xrightarrow[\psi_{n-1}]{} & X_{n-2} \end{array}$$

We now define α_n and β_n. These are the $\mathbb{C}$-valued points of X_n whose images are not equal in $\mathcal{S}^n(X_n)(\mathbb{C})$ but will be equal in $\mathcal{S}^{n+1}(X_n)(\mathbb{C})$. We start by defining α_2 and β_2. Since $\psi_1(\alpha_1) = \phi_1(\beta_1) = 0$, the pair (α_1, β_1) induces the morphism

$$\operatorname{Spec} \mathbb{C} \xrightarrow{(\alpha_1,\beta_1)} X_1 \times_{\psi_1, X_0, \phi_1} X_1.$$

Similarly, since $\psi_1(\beta_1) = \phi_1(\alpha_1) = 1$, the pair (β_1, α_1) will induce the morphism

$$\operatorname{Spec} \mathbb{C} \xrightarrow{(\beta_1,\alpha_1)} X_1 \times_{\psi_1, X_0, \phi_1} X_1.$$

Define $\alpha_2, \beta_2 : \operatorname{Spec} \mathbb{C} \to X_2$ by the following morphisms,

$$\alpha_2 := (\alpha_1, \beta_1) \quad \text{and} \quad \beta_2 := (\beta_1, \alpha_1).$$

To define α_n and β_n for $n \geq 3$, an alternative definition of X_n will be more convenient.

Remark 3.3. X_n can be realised as the n-fold fiber product of X_1 over $\mathbb{A}^1_{\mathbb{C}}$ as follows: For $i \geq 2$, we see immediately that $X_i = X_1 \times_{\mathbb{A}^1_{\mathbb{C}}} X_{i-1}$ via the following pullback square.

$$\begin{array}{ccc} X_i & \xrightarrow{\psi_i} & X_{i-1} \\ {\scriptstyle \phi_2 \circ \ldots \phi_i}\downarrow & & \downarrow{\scriptstyle \phi_1 \circ \ldots \phi_{i-1}} \\ X_1 & \xrightarrow[\psi_1]{} & \mathbb{A}^1_{\mathbb{C}} \end{array}$$

Applying this definition of X_i repeatedly, we will find that X_n can also be expressed as the n-fold fiber product, specifically, $X_n = X_1 \times_{\psi_1, \mathbb{A}^1_{\mathbb{C}}, \phi_1} X_1 \cdots \times_{\psi_1, \mathbb{A}^1_{\mathbb{C}}, \phi_1} X_1$.

For all $n \geq 3$, α_n and β_n are defined using the above n-fold fiber product description of X_n. The morphisms are given by

$$\alpha_n : \operatorname{Spec} \mathbb{C} \xrightarrow{(a_1,\ldots,a_n)} X_n \quad \text{and} \quad \beta_n : \operatorname{Spec} \mathbb{C} \xrightarrow{(b_1,\ldots,b_n)} X_n,$$

where $a_i, b_i \in \{\alpha_1, \beta_1\}$ such that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ form alternating sequences of α_1 and β_1 with $a_1 = \alpha_1$ and $b_1 = \beta_1$. More precisely,

$$\alpha_n := (\alpha_1, \beta_1, \alpha_1, \ldots, a_n) \quad \text{and} \quad \beta_n := (\beta_1, \alpha_1, \beta_1, \ldots, b_n).$$

Since $\psi_1(\beta_1) = \phi_1(\alpha_1) = 1$ and $\psi_1(\alpha_1) = \phi_1(\beta_1) = 0$, the above definitions are well defined. Similar to the definition of X_n, we define Y_n inductively.

Construction 3.4. For $n \geq 2$, assuming Y_{n-1}, $\bar{\phi}_{n-1}$ and $\bar{\psi}_{n-1}$ are defined, we define Y_n using the following pullback square.

$$\begin{array}{ccc} Y_n & \xrightarrow{\bar{\psi}_n} & Y_{n-1} \\ {\scriptstyle \bar{\phi}_n}\downarrow & & \downarrow{\scriptstyle \bar{\phi}_{n-1}} \\ Y_{n-1} & \xrightarrow[\bar{\psi}_{n-1}]{} & Y_{n-2} \end{array}$$

The following simple lemma provides a geometric description of X_n and Y_n.

Lemma 3.5. *Let $n \geq 1$.*

(1) X_n is an open subscheme of Y_n. Moreover, ϕ_n and ψ_n are the restrictions of $\bar{\phi}_n$ and $\bar{\psi}_n$ to X_n.

(2) $Y_n = Spec\, \mathbb{C}[x_0, x_1, y_1 \ldots, x_n, y_n]/\langle \{y_i^2 - x_{i-1}^2 f(x_i)\}_{i=1}^n \rangle$. The morphism $\bar{\phi}_n$ is given by

$$(x_0, x_1, y_1, \ldots x_n, y_n) \mapsto (x_0, x_1, y_1 \ldots, x_{n-1}, y_{n-1}),$$

and the morphism $\bar{\psi}_n$ is given by

$$(x_0, x_1, y_1, \ldots x_n, y_n) \mapsto (x_1, x_2, y_2 \ldots, x_n, y_n).$$

Proof. For (1), we first claim that similar to X_n, Y_n can be obtained as the n-fold fiber product of Y_1 over $\mathbb{A}^1_{\mathbb{C}}$. Indeed, by replacing the roles of ϕ_i and ψ_i with $\bar{\phi}_i$ and $\bar{\psi}_i$ in Remark 3.3, we find that

$$Y_n = Y_1 \times_{\bar{\psi}_1, \mathbb{A}^1_{\mathbb{C}}, \bar{\phi}_1} Y_1 \cdots \times_{\bar{\psi}_1, \mathbb{A}^1_{\mathbb{C}}, \bar{\phi}_1} Y_1.$$

Now, since $i_1 : X_1 \to Y_1$ is an open immersion, and since ϕ_1 and ψ_1 are simply the restrictions of $\bar{\phi}_1, \bar{\psi}_1$ to X_1 respectively, the following morphism

$$X_1 \times_{\psi_1, \mathbb{A}^1_{\mathbb{C}}, \phi_1} X_1 \cdots \times_{\psi_1, \mathbb{A}^1_{\mathbb{C}}, \phi_1} X_1 \xrightarrow{i_1 \times \cdots \times i_1} Y_1 \times_{\bar{\psi}_1, \mathbb{A}^1_{\mathbb{C}}, \bar{\phi}_1} Y_1 \cdots \times_{\bar{\psi}_1, \mathbb{A}^1_{\mathbb{C}}, \bar{\phi}_1} Y_1$$

must be an open immersion. Now, $\phi_n, \psi_n : X_{n-1} \times_{\psi_{n-1}, X_{n-2}, \phi_{n-1}} X_{n-1} \to X_{n-1}$ are the first and second projections onto X_{n-1}, respectively. Therefore, in the n-fold fiber product description, ϕ_n and ψ_n will project X_n onto the first and last $n-1$ factors of X_n, respectively. Similarly, $\bar{\phi}_n$ and $\bar{\psi}_n$ will project Y_n onto the first and last $n-1$ factors of Y_n, respectively. It follows that the morphisms ϕ_n, ψ_n are precisely the restrictions of $\bar{\phi}_n, \bar{\psi}_n$ to X_n.

For (2), because $Y_n = Y_1 \times_{\bar{\psi}_1, \mathbb{A}^1_{\mathbb{C}}, \bar{\phi}_1} Y_1 \cdots \times_{\bar{\psi}_1, \mathbb{A}^1_{\mathbb{C}}, \bar{\phi}_1} Y_1$, the coordinate ring A_n of Y_n is given by

$$A_n = \frac{\mathbb{C}[x_0, x_1, y_1]}{\langle y_1^2 - x_0^2 f(x_1) \rangle} \otimes_{\bar{\psi}_1^*, \mathbb{C}[x], \bar{\phi}_1^*} \frac{\mathbb{C}[x_0, x_1, y_1]}{\langle y_1^2 - x_0^2 f(x_1) \rangle} \cdots \otimes_{\bar{\psi}_1^*, \mathbb{C}[x], \bar{\phi}_1^*} \frac{\mathbb{C}[x_0, x_1, y_1]}{\langle y_1^2 - x_0^2 f(x_1) \rangle}.$$

Since $\bar{\phi}_1$ and $\bar{\psi}_1$ project onto the x_0 and x_1 axis, respectively, this tensor product equals

$$A_n = \text{Spec}\, \mathbb{C}[x_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n]/\langle \{y_i^2 - x_{i-1}^2 f(x_i)\}_{i=1}^n \rangle.$$

Moreover, since $\bar{\phi}_n$ and $\bar{\psi}_n$ project Y_n onto the first and last $n-1$ factors of Y_n, respectively, $\bar{\phi}_n$ is the projection onto the coordinates $(x_0, x_1, y_1 \dots, x_{n-1}, y_{n-1})$, and $\bar{\psi}_n$ is the projection onto the coordinates $(x_1, x_2, y_2 \dots, x_n, y_n)$. □

3.2. Geometric properties of X_n. In this subsection, we apply the universal property of $\mathcal{S}$ from Lemma 2.4 to the morphism ψ_n in order to construct maps $\mathcal{S}(X_n) \to X_{n-1}$. Then, using an inductive argument, we show that the images of α_n and β_n cannot be equal in $\mathcal{S}^i(X_n)(\mathbb{C})$ for $i \leq n$.

Lemma 3.6. *Let $n \geq 1$. Then the fibers of closed points under ϕ_n are $\mathbb{A}^1$-rigid.*

Proof. From Lemma 3.5, it follows that ϕ_n is the restriction of the affine map $Y_n \to Y_{n-1}$ corresponding to the natural ring homomorphism of coordinate rings:

$$\frac{\mathbb{C}[x_0, x_1, y_1, \dots, x_{n-1}, y_{n-1}]}{\langle \{y_i^2 - x_{i-1}^2 f(x_i)\}_{i=1}^{n-1} \rangle} \xrightarrow{\bar{\phi}_n^*} \frac{\mathbb{C}[x_0, x_1, y_1, \dots, x_{n-1}, y_{n-1}][x_n, y_n]}{\langle \{y_i^2 - x_{i-1}^2 f(x_i)\}_{i=1}^{n-1} \rangle + \langle y_n^2 - x_{n-1}^2 f(x_n) \rangle}.$$

Now, let Q be a closed point of X_{n-1}. Thus, $Q = (a_0, a_1, b_1, \dots, a_{n-1}, b_{n-1})$ where $a_i, b_i \in \mathbb{C}$. Hence, $\phi_n^{-1}(Q)$ is isomorphic to Spec $\frac{\mathbb{C}[x_n, y_n]}{\langle y_n^2 - a_{n-1}^2 f(x_n) \rangle} \setminus (0,0,0)$. If $a_{n-1} \neq 0$, then $\phi_n^{-1}(Q)$ is isomorphic to E, and otherwise, it is isomorphic to Spec $\frac{\mathbb{C}[x_n, y_n]}{\langle y_n^2 \rangle} \setminus (0,0,0)$. Since both of these varieties are $\mathbb{A}^1$-rigid, this completes the proof. □

Lemma 3.7. *Let $n \geq 1$ and let γ be any morphism $\mathbb{A}^1_{\mathbb{C}} \to X_n$. Then $\psi_n \circ \gamma$ is a constant morphism.*

Proof. We prove this by induction on n. Let's verify the base case for $n = 1$. Let γ be a morphism $\mathbb{A}^1_{\mathbb{C}} \to X_1$. Recall that $\rho_1 : \mathbb{A}^1_C \times E \to Y_1$(see Construction 3.1,(5)) defined by $(x_0, x_1, y_1) \mapsto (x_0, x_1, x_0 y_1)$ is an isomorphism outside the fiber $\bar{\phi}_1^{-1}(0)$. Therefore, it induces a rational map $X_1 \dashrightarrow \mathbb{A}^1_{\mathbb{C}} \times E$. Since ψ_1 is the projection onto the x_1- axis, ψ_1 is the same as the morphism induced by the rational map $X_1 \dashrightarrow \mathbb{A}^1_{\mathbb{C}} \times E \to E \xrightarrow{\pi} \mathbb{A}^1_{\mathbb{C}}$. Now, either the image of $\psi_1 \circ \gamma$ lies completely in the fiber $\phi_1^{-1}(0)$, or $\psi_1 \circ \gamma$ factors through the rational map $\mathbb{A}^1_{\mathbb{C}} \dashrightarrow E$, which can be completed to a morphism $\mathbb{A}^1_{\mathbb{C}} \to \bar{E}$, where $\bar{E}$ is the projective closure of E . Since both $\phi_1^{-1}(0)$ and $\bar{E}$ are $\mathbb{A}^1$-rigid, this completes the argument for the case $n = 1$.

Assuming the lemma holds for $n-1$, we will prove it for n. Let $\gamma : \mathbb{A}^1_{\mathbb{C}} \to X_n$ be fixed. Recall that X_n is defined by the following Cartesian square:

$$\begin{array}{ccc} X_n & \xrightarrow{\psi_n} & X_{n-1} \\ {\scriptstyle \phi_n}\downarrow & & \downarrow{\scriptstyle \phi_{n-1}} \\ X_{n-1} & \xrightarrow[\psi_{n-1}]{} & X_{n-2} \end{array}$$

Define $\gamma_1 := \phi_n \circ \gamma$ and $\gamma_2 := \psi_n \circ \gamma$. We aim to show that γ_2 is a constant morphism. By the induction hypothesis, $\psi_{n-1} \circ \gamma_1$ is constant. From the commutativity of the square above, it follows that $\phi_{n-1} \circ \gamma_2$ is constant. This implies that the image of γ_2 lies in a fiber of ϕ_{n-1}, which is $\mathbb{A}^1$-rigid by Lemma 3.6. Therefore, it follows that γ_2 must be a constant morphism. □

Lemma 3.8. *The morphism $\psi_n : X_n \to X_{n-1}$ in $Shv(Sm/k)_{Nis}$ factors through the epimorphism $X_n \to \mathcal{S}(X_n)$.*

Proof. By Lemma 2.4, it suffices to show that for any smooth scheme U and any $\mathbb{A}^1$-homotopy $F \in X_n(\mathbb{A}^1 \times U)$, $\psi_n \circ F$ is a constant $\mathbb{A}^1$-homotopy. Let $G := \psi_n \circ F$ and let $s, t \in \mathbb{C}$. We need to show that the morphisms $G(t), G(s) : U \to X_{n-1}$ are identical. Since X is separated, the set $S := \{x \in U | G(t)(x) = G(s)(x)\}$ forms a closed subscheme of U. From Lemma 3.7, we know that $U(\mathbb{C}) \subset S$, which further implies that $U = S$. Hence, $G(s) = G(t)$ for any $s, t \in \mathbb{C}$, and the result follows. □

Theorem 3.9. $[\alpha_n]_n \neq [\beta_n]_n$ *for all n.*

Proof. We prove the theorem by induction on n. For $n = 1$, we need to show that $[\alpha_1]_1$ and $[\beta_1]_1$ cannot be connected by an $\mathbb{A}^1$-chain homotopy. To establish this, it suffices to show that any morphism $\gamma : \mathbb{A}^1_{\mathbb{C}} \to X_1$ containing $\alpha_1 = (1, 0, \lambda)$ in its image is a constant morphism. Recall that the morphism $\rho_1 : \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} E \to Y_1$ defined by $(x_0, x_1, y_1) \to (x_0, x_1, x_0 y_1)$ is an isomorphism outside $\bar{\phi}_1^{-1}(\{0\})$. Since $\alpha_1 \notin \bar{\phi}_1^{-1}(\{0\})$, ρ_1^{-1} induces a rational map $\gamma' : \mathbb{A}^1_{\mathbb{C}} \dashrightarrow \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} E \to E$. This rational map can be completed to a morphism $\mathbb{A}^1_{\mathbb{C}} \to \bar{E}$. Consequently, γ' is constant, implying that the image of γ is contained in affine line corresponding to $\rho_1(\mathbb{A}^1_{\mathbb{C}} \times (0, \lambda))$. Since $\rho((0, 0, \lambda)) = (0, 0, 0)$ is not in X_1, the image of γ must be contained in affine line excluding origin, which is $\mathbb{A}^1$-rigid. Thus, γ must be a constant morphism.

Assuming the theorem holds for $n-1$, we will prove it for n. On the contrary, assume that $[\alpha_n]_n = [\beta_n]_n$ in $\mathcal{S}^n(X_n)(\mathbb{C})$. Since the morphism $\psi_n : X_n \to X_{n-1}$ factors through the morphism $X_n \to \mathcal{S}(X_n)$ by Lemma 3.8, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{S}^{n-1}(X_n)(\mathbb{C}) & \xrightarrow{\mathcal{S}^{n-1}(\psi_n)} & \mathcal{S}^{n-1}(X_{n-1})(\mathbb{C}) \\ \downarrow & \nearrow & \\ \mathcal{S}^{n}(X_n)(\mathbb{C}) & & \end{array}$$

Since $\psi_n(\alpha_n) = \beta_{n-1}$ and $\psi_n(\beta_n) = \alpha_{n-1}$, and we have assumed that $[\alpha_n]_n = [\beta_n]_n$, it follows from the commutativity of the above diagram that $[\alpha_{n-1}]_{n-1} = [\beta_{n-1}]_{n-1}$. This conclusion contradicts the induction hypothesis. Therefore, the theorem holds. □

3.3. $\mathbb{A}^1$-homotopies in $\mathcal{S}^n(X_n)$. An explicit $\mathbb{A}^1$-homotopy between $[\alpha_1]_1$ and $[\beta_1]_1$ in $\mathcal{S}(X_1)$ has been constructed in [2, Construction 4.3]. This will be the key input in the following construction of $\mathbb{A}^1$-homotopies in $\mathcal{S}^n(X_n)$.

Theorem 3.10. *$[\alpha_n]_n$ and $[\beta_n]_n$ are $\mathbb{A}^1$-homotopic in $\mathcal{S}^n(X_n)$.*

Proof. For every $n \geq 1$, we will construct an $\mathbb{A}^1$-homotopy $\mathbb{A}^1_{\mathbb{C}} \to \mathcal{S}^n(X_n)$ such that $[\alpha_n]_n$ and $[\beta_n]_n$ are contained in its image. We begin by constructing an elementary Nisnevich cover of X_n for all $n \geq 0$. Let $V = V_1 \sqcup V_2$, where

$$V_1 = E \setminus \{(\lambda_i, 0)_{i=1}^{3}(0, -\lambda)\} \quad \text{and} \quad V_2 = \mathbb{A}^1_{\mathbb{C}} \setminus \{0\}.$$

Define $p_1 := \pi|_{V_1}$ and p_2 to be the inclusion $V_2 \to \mathbb{A}^1_{\mathbb{C}}$. The morphism p_1 is etale and $p_1^{-1}(0) = (0, \lambda)$, thus the map $p_1 \sqcup p_2$ forms a Nisnevich cover of $\mathbb{A}^1_{\mathbb{C}}$. Now, for $n \geq 1$, consider X_n as schemes over $\mathbb{A}^1_{\mathbb{C}}$ through $\Phi_n := \phi_1 \circ \dots \phi_n$. We then obtain the following elementary distinguished square,

$$\begin{array}{ccc} W \times_{\mathbb{A}^1_{\mathbb{C}}} X_n & \xrightarrow{pr_2} & V_2 \times_{\mathbb{A}^1_{\mathbb{C}}} X_n \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle p_2 \times id} \\ V_1 \times_{\mathbb{A}^1_{\mathbb{C}}} X_n & \xrightarrow[p_1 \times id]{} & \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{A}^1_{\mathbb{C}}} X_n \end{array}$$

where $W = V_1 \times_{\mathbb{A}^1_{\mathbb{C}}} V_2$.

We now construct maps from X_n to $\mathcal{S}(X_{n+1})$ that send α_n and β_n to $[\beta_{n+1}]_1$ and $[\alpha_{n+1}]_1$, respectively. Since the above square is cocartesian in $Shv(Sm/k)_{Nis}$ by Lemma 2.7, it suffices to construct morphisms $h_i^n : V_i \times_{\mathbb{A}^1_{\mathbb{C}}} X_n \to X_{n+1}$ for $i = 1$ and 2, such that the following two compositions are identical:

$$W \times_{\mathbb{A}^1_{\mathbb{C}}} X_n \xrightarrow{pr_i \circ h_i^n} X_{n+1} \to \mathcal{S}(X_{n+1}) \quad \text{for } i = 1, 2.$$

By Remark 3.3, $X_{n+1} = X_1 \times_{\psi_1, \mathbb{A}^1_{\mathbb{C}}, \Phi_n} X_n$. Thus, we will define $h_i^n : V_i \times_{\mathbb{A}^1_{\mathbb{C}}} X_n \to X_{n+1}$ as $h_i \times id$, where

- $h_1 : V_1 \to X_1 \,; (x_1, y_1) \mapsto (1, x_1, y_1)$,
- $h_2 : V_2 \to X_1 \,; (x_1) \mapsto (0, x_1, 0)$.

The maps h_i^n are well defined because $\psi_1 \circ h_i = p_i$.

Now, we will define $\mathcal{H}^n : \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} (W \times_{\mathbb{A}^1_{\mathbb{C}}} X_n) \to X_{n+1}$ such that $\mathcal{H}^n(0) = pr_2 \circ h_2^n$ and $\mathcal{H}^n(1) = pr_1 \circ h_1^n$. Similar to h_i^n, $\mathcal{H}^n$ is a product of two morphisms, $\mathcal{H} \times id : (\mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} W) \times_{\mathbb{A}^1_{\mathbb{C}}} X_n \to X_{n+1}$, where $\mathcal{H}$ is the restriction of ρ_1:

$$\mathcal{H} : \mathbb{A}^1_{\mathbb{C}} \times_{\mathbb{C}} W \to X_1 \quad ; \quad (x_0, x_1, y_1) \mapsto (x_0, x_1, x_0 y_1).$$

Clearly, $\mathcal{H}^n(0) = pr_2 \circ h_2^n$ and $\mathcal{H}^n(1) = pr_1 \circ h_1^n$. Therefore, the morphisms $pr_2 \circ h_2^n$ and $pr_1 \circ h_1^n$ become identical in $\mathcal{S}(X_{n+1})$. Thus, h_1^n and h_2^n can be glued together to obtain the maps $\mathcal{F}_n : X_n \to \mathcal{S}(X_{n+1})$ for all $n \geq 0$.

Finally, for $m \geq 1$, define the required $\mathbb{A}^1$-homotopy in $\mathcal{S}^m(X_m)$ as the following composition:

$$\mathbb{A}^1_{\mathbb{C}} \xrightarrow{\mathcal{F}_0} \mathcal{S}(X_1) \xrightarrow{\mathcal{S}(\mathcal{F}_1)} \mathcal{S}^2(X_2) \xrightarrow{\mathcal{S}^2(\mathcal{F}_2)} \mathcal{S}^3(X_3) \to \cdots \to \mathcal{S}^m(X_m).$$

To ensure that the above $\mathbb{A}^1$-homotopy indeed connects $[\alpha_m]_m$ and $[\beta_m]_m$ in $\mathcal{S}^m(X_m)$, what remains is to show that:

(1) $\mathcal{F}_0(0) = [\alpha_1]_1$ and $\mathcal{F}_0(1) = [\beta_1]_1$,
(2) For all $n \geq 1$, $\mathcal{F}_n(\beta_n) = [\alpha_{n+1}]_1$ and $\mathcal{F}_n(\alpha_n) = [\beta_{n+1}]_1$.

Since $h_1(0, \lambda) = \alpha_1$ and $h_2(1) = \beta_1$, we obtain (1). For (2), consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & (V_1 \times_{\mathbb{A}^1_{\mathbb{C}}} X_n) \sqcup (V_2 \times_{\mathbb{A}^1_{\mathbb{C}}} X_n) & \xrightarrow{h_1^n \sqcup h_2^n} & X_{n+1} \\
 & \nearrow^{((0,\lambda),\beta_n)\sqcup(1,\alpha_n)} & \downarrow & & \downarrow \\
\operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C} & \xrightarrow[\beta_n \sqcup \alpha_n]{} & X_n & \xrightarrow[\mathcal{F}_n]{} & \mathcal{S}(X_{n+1})
\end{array}$$

Since $h_1^n((0, \lambda), \beta_n) = (\alpha_1, \beta_n) = \alpha_{n+1}$ and $h_2^n((1), \alpha_n) = (\beta_1, \alpha_n) = \beta_{n+1}$, we are done. □

Proof of Theorem 1.1. By Theorem 3.9 and Theorem 3.10, we have

$$[\alpha_n]_n \neq [\beta_n]_n \quad \text{and} \quad [\alpha_n]_{n+1} = [\beta_n]_{n+1}.$$

Hence, $\mathcal{S}^n(X_n)(\mathbb{C}) \neq \mathcal{S}^{n+1}(X_n)(\mathbb{C})$. □

Remark 3.11. For any scheme X/k, the field value sections of the sheaf of $\mathbb{A}^1$-connected components of X can be computed by the following formula [4]:

$$\pi_0^{\mathbb{A}^1}(X)(F) = \mathcal{L}(X)(F) := \lim_{\to n} \mathcal{S}^n(\mathcal{X})(F) \quad \text{for any } F/k.$$

If X is proper, then $\mathcal{S}(X)(F) = \mathcal{S}^2(X)(F)$. However, for non-proper X, the proof of Theorem 1.1 shows that the infinite iterations of $\mathcal{S}$ in the above formula are essential.

In [6, Question 2.16], it is asked whether $\mathcal{S}(X)(F) = \mathcal{S}^2(X)(F)$ for non-proper smooth schemes over k when $k = \bar{k}$. This question was already answered in the negative in [2, Construction 4.5], where X_1 was embedded in a smooth variety T using [2, Lemma 4.4], such that $\mathcal{S}(T)(\mathbb{C}) \neq \mathcal{S}^2(T)(\mathbb{C})$. We will use a slight generalisation of the same lemma(see Lemma 3.13) to embed X_n in the smooth varieties Z_n such that $\mathcal{S}^n(Z_n)(\mathbb{C}) \neq \mathcal{S}^{n+1}(Z_n)(\mathbb{C})$. This shows that the infinite iterations of $\mathcal{S}$ are required even for field value points of non-proper smooth schemes over an algebraically closed field.

The following lemma is a reformulation of [5, Lemma 2.12] and will be used in the construction of Z_n.

Lemma 3.12. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of sets on Sm/k. Assume that $\mathcal{G}$ is $\mathbb{A}^1$-invariant. Then, for any n and any $\mathbb{A}^1$-homotopy $h : \mathbb{A}^1 \times U \to$*

$\mathcal{S}^n(\mathcal{F})$, there exists $\gamma : U \to \mathcal{G}$ such that the given $\mathbb{A}^1$-homotopy factors through $\mathcal{S}^n(\mathcal{F} \times_{\mathcal{G},\gamma} U) \to \mathcal{S}^n(\mathcal{F})$.

The proof of the next lemma runs along the same lines as in [2, Lemma 4.4].

Lemma 3.13. *Let X be an affine scheme over a field k. Then there exists a closed embedding of X into a smooth scheme T over k such that for any n, if $H : \mathbb{A}^1_k \to \mathcal{S}^n(T)$ is an $\mathbb{A}^1$-homotopy containing $[x]_n$ in its image for some $x \in X(k)$, then H factors through $\mathcal{S}^n(X) \to \mathcal{S}^n(T)$.*

Proof. Suppose $X \subset \mathbb{A}^n_k$ is defined by the ideal $\langle f_1, \dots, f_r\rangle \subset k[x_1, \dots x_n]$. Consider the map $f : \mathbb{A}^n_k \to \mathbb{A}^r_k$ which is given by $(x_1, \dots, x_n) \mapsto (f_1, \dots, f_r)$. Then the fiber of f at $(0, \dots, 0)$ is X. Now, choose some etale map $g : C \to \mathbb{A}^1_k$ such that C is a smooth curve of positive genus and $0 \in \mathbb{A}^1_k$ has a unique preimage say c. Then, $C^r \xrightarrow{g^r} \mathbb{A}^r_k$ is etale. Define $T := \mathbb{A}^n_k \times_{f,\mathbb{A}^r_k,g^r} V^r_1$. Then the fiber of the map $T \to V^r_1$ over $(c, \dots, c)$ is X. Clearly, T is a smooth scheme containing X.

We claim that T is the required smooth scheme. Suppose that H is an $\mathbb{A}^1$-homotopy in $\mathcal{S}^n(T)$ whose image contains $[x]_n$ for some $x \in X(k)$. Since C^r is $\mathbb{A}^1$-invariant, by Lemma 3.12, there exists Q : Spec $k \to C^r$ such that the $\mathbb{A}^1$-homotopy H factors through $\mathcal{S}^n(T \times_{C^r,Q} \operatorname{Spec} k) \to \mathcal{S}^n(T)$. Since $[x]_n$ is contained in the image of H and $x \in X(k)$, the point $(c, \dots, c)$ belongs to the image of the composition $\mathbb{A}^1_k \xrightarrow{H} \mathcal{S}^n(T) \to C^r$. Hence, $Q = (c, \dots, c)$, and H factors through the map $\mathcal{S}^n(X) \to \mathcal{S}^n(T)$. □

Proposition 3.14. *For every $n \in N$, there exists a smooth variety Z_n over $\mathbb{C}$, such that $\mathcal{S}^n(Z_n)(\mathbb{C}) \neq \mathcal{S}^{n+1}(Z_n)(\mathbb{C})$.*

Proof. For every n and Y_n(see Construction 3.4), let T_n be the smooth variety corresponding to Y_n arising from Lemma 3.13. Then there exists a morphism γ_n from T_n to an $\mathbb{A}^1$-rigid scheme V_n and a point $P_n \in V_n(\mathbb{C})$ such that fiber of the morphism γ_n over P_n is Y_n. Choose a suitable open subscheme Z_n of T_n such that $\gamma_n|^{-1}_{Z_n}(P_n) = X_n$. Since any $\mathbb{A}^1_k \to \mathcal{S}^n(Z_n)$ whose image contains $[\alpha_n]_n$ factors through the map $\mathcal{S}^n(X_n) \to \mathcal{S}^n(Z_n)$, we have $[\alpha_n]_n \neq [\beta_n]_n$ in $\mathcal{S}^n(Z_n)(\mathbb{C})$, while $[\alpha_n]_{n+1} = [\beta_n]_{n+1}$ in $\mathcal{S}^{n+1}(Z_n)(\mathbb{C})$. □

3.4. $\mathbb{A}^1$-connectedness of X_n.

In this subsection, we show that the sequence of sheaves $(\mathcal{S}^m(X_n))_{m\geq 1}$ stabilises at the $n + 2$ stage, and that

$$\pi_0^{\mathbb{A}^1}(X_n) = \mathcal{S}^{n+2}(X_n) = * \, .$$

Theorem 3.15. *Let k be any finitely generated field extension of $\mathbb{C}$ and let $n \geq 1$. Then $\mathcal{S}^{n+1}(X_n)(k) = *$.*

Proof. It suffices to show that $\mathcal{S}^{n+1}(X_n \times_{\mathbb{C}} \operatorname{Spec} k)(k) = *$. The maps $\mathcal{F}_n$ constructed in Theorem 3.10 will be used to prove the theorem. We will abuse the notation and write X_n for the schemes $X_n \times_{\mathbb{C}} \operatorname{Spec} k$, and $\mathcal{F}_n$ for the maps

$\mathcal{F}_n \times_{\mathbb{C}} \operatorname{Spec} k : X_n \times_{\mathbb{C}} \operatorname{Spec} k \to \mathcal{S}(X_{n+1} \times_{\mathbb{C}} \operatorname{Spec} k)$, applying same conventions to all the maps involved in construction of $\mathcal{F}_n$. We prove that $\mathcal{S}^{n+1}(X_n)(k) =*$ by induction on n.

Case $(n = 1)$**.** We claim that $[Q]_2 = [\beta_1]_2$ for any $Q \in X_1(k)$. If $\mathcal{F}_0(k) : \mathbb{A}^1_k(k) \to \mathcal{S}(X_1)(k)$ is surjective, there would be nothing to prove. However, since this is not the case, we will first list all k-valued points of X_1, that do not admit a lift to $\mathbb{A}^1_k(k)$ via $\mathcal{F}_0(k)$. We will then slightly modify $\mathcal{F}_0$ to $\mathcal{F}^a_0$ or $\tilde{\mathcal{F}}^a_0$ such that their images contain $[\beta_1]_1$, and union of their images contain all k-valued points of $\mathcal{S}(X_1)$.

Recall that the map $\mathcal{F}_0$ includes the following morphisms:

- $h_i : V_i \to X_1$, for $i = 1, 2$, and
- $\mathcal{H} : \mathbb{A}^1_k \times W \to X_1$, where $\mathcal{H}$ is the restriction of ρ_1.

If $Q \in X_1(k)$ is contained in the image of any of these maps, then $[Q]_1$ is in the image of $\mathcal{F}_0$, and we have nothing to prove.

Now, the map $\rho_1 : \mathbb{A}^1_k \times E \to Y_1$ is an isomorphism when restricted to $\mathbb{A}^1_k \setminus \{0\} \times E$, while the image of $\rho_1(0) \setminus \{(0,0,0)\}$ coincides with $h_2(V_2)$. Since $V_1 = E \setminus \{(\lambda_1, 0)^3_{i=1}, (0, -\lambda)\}$, $h_1 = \rho(1)|_{V_1}$, and $W = V_1 \setminus \{(0, \lambda)\}$, the remaining k-valued points of X_1 must belong to one of the following sets:

- $\rho_1(\mathbb{A}^1_k \setminus \{0, 1\} \times \{(0, \lambda)\})$,
- $\rho_1(\mathbb{A}^1_k \setminus \{0\} \times \{(0, -\lambda)\})$, or
- $\rho_1(\mathbb{A}^1_k \setminus \{0\} \times \{(\lambda_i, 0)^3_{i=1}\})$.

Let Q belong to any of the sets above. Then, the case $n = 1$ will be completed by proving $[Q]_2 = [\beta_1]_2$. Let $a \in k^*$.

If $Q = \rho_1((a, 0, \lambda))$, define the map $h^a_1 : V_1 \to X_1$ to be $\rho_1(a)|_{V_1}$. Then, $h^a_1(a)(0, \lambda) = Q$. Now, replace h_1 by h^1_a in the construction of $\mathcal{F}_0$, while keeping all the other maps the same. Since $\mathcal{H}(a) = h^a_1|_W$ and $\mathcal{H}(0) = h_2|_W$, the maps h^a_1 and h_2 can be glued together to produce the map $\mathcal{F}^a_0 : \mathbb{A}^1_k \to S(X_1)$. Since $h_2(1) = \beta_1$, $\mathcal{F}^a_0$ contains both $[Q]_1$ and $[\beta_1]_1$ in its image. Hence, we have $[Q]_2 = [\beta_1]_2$ as desired.

If $Q = \rho_1((a, 0, -\lambda))$, define $\tilde{V_1} = E \setminus \{(\lambda_1, 0)^3_{i=1}, (0, \lambda)\}$ and let $\tilde{h}^a_1 = \rho(a)|_{\tilde{V_1}}$. Then $\tilde{h}^a_1(a)(0, -\lambda) = Q$. Use $\tilde{V_1} \sqcup V_2$ as the Nisnevich cover instead of $V_1 \sqcup V_2$. Thus, $\tilde{V_1} \times_{\mathbb{A}^1_k} V_2 = W$, $\mathcal{H}(a) = \tilde{h}^a_1|_W$ and $\mathcal{H}(0) = h_2|_W$. Hence, similar to $\mathcal{F}^a_0$, we obtain $\tilde{\mathcal{F}}^a_0$ by gluing $\tilde{h}^a_1$ and h_2, which will contain both $[Q]_1$ and $[\beta_1]_1$ in its image.

If $Q = \rho_1((a, \lambda_i, 0))$, then $[Q]_1 = [(0, \lambda_i, 0])_1$ via the $\mathbb{A}^1_k$ corresponding to $\rho_1|_{\mathbb{A}^1_k \times \{(\lambda_i, 0)\}}$. Since $(0, \lambda_i, 0) \in h_2(V_2)$, we have $[(0, \lambda_i, 0)]_2 = [\beta_1]_2$ via $\mathcal{F}_0$, hence the claim is proved.

Case $(n > 1)$**.** Assuming the statement of theorem for n, we will prove it for $n + 1$. It is sufficient to prove the following claim.

Claim: $[Q]_{n+2} = [\beta_{n+1}]_{n+2}$ for any $Q \in X_{n+1}(k)$.

Since $X_{n+1} = X_1 \times_{\mathbb{A}^1_k} X_n$, we can write Q as (Q_1, Q_2), for some $Q_1 \in X_1(k)$ and $Q_2 \in X_n(k)$. Using notations from the case $n = 1$, we see that

$$Q_1 \in \rho\left(\mathbb{A}^1_k \times \{(\lambda_i, 0)\}_{i=1}^3\right) \cup \left(\bigcup_{a \in k^*} \left(h_1^a(V_1) \cup h_1^a(\tilde{V}_1)\right)\right) \cup h_2(V_2).$$

If $[Q_1]_1 \in h_1^a(V_1)$, we can modify $\mathcal{F}_n$ to $\mathcal{F}'_n : X_n \to \mathcal{S}(X_{n+1})$ such that $[Q]_1$ and $[\beta_{n+1}]_1$ are in the image of $\mathcal{F}'_n$. The map $\mathcal{F}_n$ was constructed by gluing $h_i \times id : V_i \times_{\mathbb{A}^1_k} X_n \to X_1 \times_{\mathbb{A}^1_k} X_n$ for $i = 1, 2$. Replacing $h_1 \times id$ with $h_1^a \times id$ in the construction of $\mathcal{F}_n$, $h_1^a \times id$ and $h_2 \times id$ can be glued to give the required $\mathcal{F}'_n$. Then $[\beta]_1$ and $[Q]_1$ can be lifted to $X_n(k)$ via $\mathcal{F}'_n$ and since $\mathcal{S}^{n+1}(X_n)(k) = *$ by the induction hypothesis, we obtain $[\beta_{n+1}]_{n+2} = [Q]_{n+2}$.

For $[Q_1]_1 \in \tilde{h}_1^a(V_1)$, replace the Nisnevich cover $V_1 \sqcup V_2$ with $\tilde{V}_1 \sqcup V_2$ and replace $h_1 \times id$ with $\tilde{h}_1^a \times id$ in the construction of $\mathcal{F}_n$ and the rest of the argument proceeds similarly as for the case $[Q_1]_1 \in h_1^a(V_1)$. If $[Q_1]_1 \in h_2^a(V_2)$, then $[Q]_1$ is contained in the image of $\mathcal{F}_n$ and we are done.

Finally, suppose Q_1 is contained in the image of $\rho_1|_{\mathbb{A}^1_k \times \{(\lambda_i, 0)\}}$ for some $i \in \{1, 2, 3\}$. Then (Q_1, Q_2) and $((0, \lambda_i, 0), Q_2)$ are in the image of the map

$$\mathbb{A}^1_k \xrightarrow{(\rho_1|_{\mathbb{A}^1_k \times \{(\lambda_i, 0)\}}, Q_2)} X_1 \times_{\mathbb{A}^1_k} X_n.$$

Since $((0, \lambda_i, 0), Q_2)$ is in the image of $h_2 \times id$, we obtain

$$[((0, \lambda_i, 0), Q_2)]_{n+2} = [\beta_{n+1}]_{n+2},$$

which further implies that $[\beta_{n+1}]_{n+2} = [Q]_{n+2}$.

□

Theorem 3.16. $\pi_0^{\mathbb{A}^1}(X_n) = \mathcal{S}^{n+2}(X_n) = *$.

Proof. Since $\mathcal{S}^{n+1}(X_n)(k) = *$ for any finitely generated and separable extension k of $\mathbb{C}$, we have $\mathcal{S}^{n+2}(X_n) = *$ from Theorem 2.10. Combining Theorem 2.9 with Lemma 2.8, we conclude that $\pi_0^{\mathbb{A}^1}(X_n) = *$. □

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