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Characterizations of local product kernels and Hardy spaces

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ABSTRACT. In Nagel et al. (J Funct Anal 181: 29–118, 2001), Nagel, Ricci and Stein established the relationships between product kernels and flag kernels on the Euclidean space. In this paper, we study the local behavior of product and flag kernels and show that local product kernels are finite sums of local flag kernels. Furthermore, we prove that the local product Hardy spaces are the intersection of local flag Hardy spaces, and local product Carleson measure spaces are the sum of local flag Carleson measure spaces.

CONTENTS

1. Introduction

The theory of singular integral operators is the central topic in harmonic analysis and has extensive applications in studying PDEs. The classical one-parameter theory treats those singular integral operators that have singularity at the origin, and are invariant under the one-parameter dilations

$$\delta x = (\delta x_1, ..., \delta x_n), \qquad \delta > 0.$$

Singular integral operators that are invariant under the multiparameter dilations

$$\delta \cdot x = (\delta_1 x_1, ..., \delta_n x_n), \delta = (\delta_1, \dots, \delta_n), \delta_j > 0, x \in \mathbb{R}^n$$

are the product singular integral operators, including the double Hilbert transforms as prototype. The theory for these type of operators has been developed

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by many authors over the past decades; see, for instance, Gundy-Stein [18], R. Fefferman and Stein [13], R. Fefferman [10, 11], Chang and R. Fefferman [3, 4, 5], Journé [30], and Pipher [37].

More precisely, R. Fefferman and Stein studied the product convolution singular integral operators which satisfy analogous conditions enjoyed by the double Hilbert transform defined on $\mathbb{R} \times \mathbb{R}$ [13], and established the L^p boundedness of such singular operators for $1 < p < \infty$. Chang and R. Fefferman developed a nice theory of product Hardy spaces initially introduced by Gundy and Stein [18], including the atomic decompositions and their dual spaces [3, 4, 5]. Subsequently, Journé introduced non-convolution product singular integral operators in [30], and many works on the L^p boundedness for $1 < p < \infty$ and Hardy spaces H^p boundedness for operators in Journé's class were obtained [12, 22, 26, 37, 46]. We would like to remark that Ricci and Stein also considered the product theory associated with the Zygmund dilations in [38] and see also [23].

In their remarkable work in [31, 32], Müller, Ricci and Stein studied the Marcinkiewicz multipliers on the Heisenberg group. A surprising fact they proved is that the Marcinkiewicz multipliers can be characterized by the convolution operator with flag kernels. Note that these multipliers are invariant under a two parameter group of dilations on $\mathbb{C}^n \times \mathbb{R}$, while there is no two parameter group of automorphic dilations on $\mathbb{H}^n$. The properties of flag kernels and the applications of the corresponding singular integrals were then extended to the higher step case in [33], largely in the Euclidean setting, and then in [34, 35] in the general homogeneous groups. It should be pointed out that Stein and Street [39, 40] established the L^p boundedness of multi-parameter singular Radon transforms, which are sufficiently broad and can include some of the well known multi-parameter structures studied in the literature.

The multi-parameter analysis in the flag setting have been widely used in several complex variables. Applying a type of singular integral operators whose novel features are related to singular integrals with flag kernels, Nagel and Stein [36] provided the optimal estimates for solutions of the Kohn-Laplacian for certain classes of model domains in several complex variables. A different method to the flag kernels on a homogeneous group can be found in the work of Głowacki [15, 16]. For other works related to flag kernels, we refer the reader to [1, 2, 9, 19, 20, 21, 24, 25, 27, 28, 29, 41, 42, 43, 44, 45], among others.

In [33], Nagel, Ricci and Stein established a relationship between product and flag kernels on the Euclidean space, namely, product kernels are finite sums of flag kernels. More recently, Nagel, Ricci, Stein and Wainger [35] considered the problem of characterizing the kernels of composition of operators with different homogeneities which arise naturally in the study of ∂-Newmann problem in several complex variables. It turns out that locally the kernels can be characterized in terms of flag kernels adapted to two opposite flags and with different homogeneities.

These works motivates the present work. We consider similar questions for local product kernels and provide a characterization of this class of kernels via local flag kernels introduced in [2] adapted to opposite flags. We also show that the local product Hardy spaces are the intersection of local flag Hardy spaces, while the product Carleson measure spaces are the sums of local flag Carleson measure spaces. This latter result provides new perspectives of the relations from the viewpoint of function space theory.

For the sake of simplicity, we will restrict ourselves to the case $\mathbb{R}^N := \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and write $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ for any $x \in \mathbb{R}^N$. We begin with the definition of a class of distributions called local product kernel.

Definition 1.1. A local product kernel $\mathcal{K}$ on $\mathbb{R}^N$ is a distribution on $\mathbb{R}^N$ which coincides with a smooth function K away from the coordinate subspaces $x_1 = 0$ and $x_2 = 0$ and satisfies

(1) (Differential Inequalities) For each multi-index $\alpha = (\alpha_1, \alpha_2)$ and every $M > 0$, there exists a constant $C_{M,\alpha} > 0$ such that

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}K(x)| \le C_{M,\alpha}|x_1|^{-n_1-|\alpha_1|}|x_2|^{-n_2-|\alpha_2|}(1 + |x_1| + |x_2|)^{-M}. \tag{1}$$

(2) (Cancellation Conditions)

(i) Given any $\phi_i \in C_0^\infty(\mathbb{R}^{n_i})$ supported in the unit ball, $i = 1, 2$, and any scaling parameter $r > 0$, define a distribution $\mathcal{K}_{\phi_i,r}$ by

$$\langle \mathcal{K}_{\phi_i,r}, \psi_i\rangle = \langle \mathcal{K}, (\phi_i)_r \otimes \psi_i\rangle$$

for any test function $\psi_i \in \mathcal{S}(\mathbb{R}^{N-n_i})$. Then, away from the origin in $\mathbb{R}^{n_i}$, $\mathcal{K}_{\phi_i,r}$ coincides with a smooth function $K_{\phi_i,r}$ satisfying the differential inequalities

$$|\partial_{x_2}^{\alpha_2}K_{\phi_1,r}(x_2)| \le C_{M,\alpha_2}|x_2|^{-n_2-|\alpha_2|}(1 + |x_2|)^{-M}, \tag{2}$$

$$|\partial_{x_1}^{\alpha_1}K_{\phi_2,r}(x_1)| \le C_{M,\alpha_1}|x_1|^{-n_1-|\alpha_1|}(1 + |x_1|)^{-M} \tag{3}$$

uniformly in r.

(ii) For any $r_1, r_2 > 0$, the quantity

$$\int_{\mathbb{R}^N} K(x_1, x_2)\phi(r_1x_1, r_2x_2)dx$$

is bounded independently of ϕ, r_1 and r_2.

The following restricted classes of flag kernels were introduced in [2].

Definition 1.2. A distribution $\mathcal{K}$ on $\mathbb{R}^N$ is said to be a local flag kernel adapted to the flag $\mathcal{F}$, denoted by $\mathcal{K} \in \mathcal{P}_0(\mathcal{F})$, if it coincides with a C^∞ function K away from the coordinate subspace $x_1 = 0$ and satisfies the smoothness condition: for any multi-index $\alpha = (\alpha_1, \alpha_2)$ and every $M > 0$,

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}K(x)| \le \begin{cases} C_\alpha|x_1|^{-n_1-|\alpha_1|}(|x_1| + |x_2|)^{-n_2-|\alpha_2|}, \text{ if } |x_1| + |x_2| \le 1, \\ C_{M,\alpha}(1 + |x_1| + |x_2|)^{-M}, \text{ if } |x_1| + |x_2| \ge 1, \end{cases} \tag{4}$$

and the same cancellation condition as that for a local product kernel.

Similarly, a distribution $\mathcal{K}$ on $\mathbb{R}^N$ is said to be a local flag kernel adapted to the flag $\mathcal{F}^\perp$, denoted by $\mathcal{K} \in \mathcal{P}_0(\mathcal{F}^\perp)$, if it coincides with a C^∞ function K away from the coordinate subspace $x_2 = 0$ and satisfies the smoothness condition: for any multi-index $\alpha = (\alpha_1, \alpha_2)$ and any $M > 0$,

$$|\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}K(x)| \le \begin{cases} C_\alpha(|x_1| + |x_2|)^{-n_1-|\alpha_1|}|x_2|^{-n_2-|\alpha_2|}, \text{ if } |x_1| + |x_2| \le 1, \\ C_{M,\alpha}(1 + |x_1| + |x_2|)^{-M}, \text{ if } |x_1| + |x_2| \ge 1, \end{cases} \tag{5}$$

and the same cancellation condition as that for a local product kernel.

Our first main result is the following theorem, which establishes a relationship among the local product kernels, local product multipliers, and local flag kernels.

Theorem 1.3. The following three statements are equivalent:

(i) $\mathcal{K}$ is a local product kernel on $\mathbb{R}^N$;

(ii) $\widehat{\mathcal{K}} = m$ satisfies

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}m(\xi_1, \xi_2)| \le C_\alpha(1 + |\xi_1|)^{-|\alpha_1|}(1 + |\xi_2|)^{-|\alpha_2|}, \quad \forall\alpha = (\alpha_1, \alpha_2). \tag{6}$$

(iii) $\mathcal{K}$ can be decomposed as a sum $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$, where $\mathcal{K}_1 \in \mathcal{P}_0(\mathcal{F})$ and $\mathcal{K}_2 \in \mathcal{P}_0(\mathcal{F}^\perp)$.

Remark 1.4. Theorem 1.3 can be regarded as a local version of this result in [33].

We also consider the relation between local product and flag Hardy spaces, and that between local product and local flag Carleson measure spaces. Before stating our main results, we need to recall the theory of local flag Hardy spaces established in [2] for the restricted class $\mathcal{P}_0(\mathcal{F})$ and $\mathcal{P}_0(\mathcal{F}^\perp)$. For any $d \in \mathbb{N}_+$, let $\mathcal{S}(\mathbb{R}^d)$ denote the class of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{R}^d)$ be its dual. A couple of functions (ψ_0, ψ) on $\mathbb{R}^d$ is said to be admissible, if

$$\begin{aligned} &\operatorname{supp}\widehat{\psi_0} \subset \{\xi \in \mathbb{R}^d : |\xi| \le 2)\}, \widehat{\psi_0}(\xi) = 1 \text{ if } |\xi| \le 1, \\ &\operatorname{supp}\widehat{\psi} \subset \{\xi \in \mathbb{R}^d : \frac{1}{2} \le |\xi_i| \le 2\}, \\ &|\widehat{\psi_0}(\xi)|^2 + \sum_{j=1}^{\infty} |\widehat{\psi}(2^{-j}\xi)|^2 = 1 \text{ for all } \xi \in \mathbb{R}^d. \end{aligned}$$

Let $(\psi_0^{(1)}, \psi^{(1)})$ and $(\psi_0^{(2)}, \psi^{(2)})$ be admissible couples on $\mathbb{R}^N$ and $\mathbb{R}^{n_2}$, respectively. For any positive integers $j, k \in \mathbb{N}_+$, put

$$\psi_j^{(1)}(x_1, x_2) = 2^{j(n_1+n_2)}\psi_j^{(1)}(2^j x_1, 2^j x_2), \qquad \psi_k^{(2)}(x_2) = 2^{kn_2}\psi_k^{(2)}(2^k x_2).$$

For $j, k \in \mathbb{N}$, set

$$\psi_{j,k}^{\mathcal{F}}(x) = (\psi_j^{(1)} *_2 \psi_k^{(2)})(x) = \int_{\mathbb{R}^N} \psi_j^{(1)}(x_1, x_2 - y)\psi_k^{(2)}(y)dy.$$

Recall from [2] that the following local discrete Calderón reproducing formula associated with the flag $\mathcal{F}$ holds

$$f(x) = \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |R|\psi_{j,k}(x - x_R)\psi_{j,k} * f(x_R), \tag{7}$$

where $\mathcal{R}_{\mathcal{F}}^{j,k}$ is the set of dyadic rectangles $R = I\times J$ with side-length $\ell(I) = 2^{-j}$, $\ell(J) = 2^{-j\wedge k}$, , $x_R = (x_I, x_J)$ denotes the lower-left corner of R, and the series converges in $L^2(\mathbb{R}^N)$, $\mathcal{S}(\mathbb{R}^N)$ and $\mathcal{S}'(\mathbb{R}^N)$.

The local flag discrete Littlewood–Paley square function is defined by

$$g_{\mathcal{F}}(f)(x) = \Big\{ \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_R)|^2\chi_R(x)\Big\}^{\frac{1}{2}}.$$

Hence, the local flag Hardy spaces (adapted to the flag $\mathcal{F}$) can be defined as follows.

Definition 1.5. [2] Let $0 < p < \infty$. The local flag Hardy space $h^p_{\mathcal{F}}(\mathbb{R}^N)$ is defined by

$$h^p_{\mathcal{F}}(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N) : g_{\mathcal{F}}(f) \in L^p(\mathbb{R}^N)\}.$$

The definition of $h^p_{\mathcal{F}}$ is independent of the choice of admissible couples, which is the content of the following

Proposition 1.6. Let $0 < p < \infty$. Suppose that $\phi_{j,k}$ satisfies the same conditions as $\psi_{j,k}$. Then for $f \in \mathcal{S}'(\mathbb{R}^N)$

$$\Big\|\Big\{ \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p(\mathbb{R}^N)}$$
$$\approx \Big\|\Big\{ \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\phi_{j,k} * f(x_R)|^2\chi_R\Big\}^{1/2}\Big\|_{L^p(\mathbb{R}^N)}.$$

Interchanging the first variable with the second, we can define the local flag Hardy space $h^p_{\mathcal{F}^\perp}$. More precisely, let $(\phi_0^{(1)}, \phi^{(1)})$ and $(\phi_0^{(2)}, \phi^{(2)})$ be admissible couples on $\mathbb{R}^N$ and $\mathbb{R}^{n_1}$, respectively. For $j, k \in \mathbb{N}$, set $\phi^{\mathcal{F}^\perp}_{j,k} = \phi^{(1)}_j *_1 \phi^{(2)}_k$, that is, $\phi^{\mathcal{F}^\perp}_{j,k}(x) = \int \phi^{(1)}_j(x_1 - u, x_2)\phi^{(2)}_k(u)du$, where $\phi^{(1)}_j, j \ge 1$ and $\phi^{(2)}_k, k \ge 1$ are dilates of $\phi^{(1)}$ and $\phi^{(2)}$, respectively. We remark that the subtle convolution $*_1$ and $*_2$ reflect the two flag multiparameter structures. The local flag discrete Littlewood-Paley-Stein square function associated with the flag $\mathcal{F}^\perp$ is defined by

$$g_{\mathcal{F}^\perp}(f)(x) = \Big\{ \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}_{\mathcal{F}^\perp}^{j,k}} |\phi_{j,k} * f(x_R)|^2\chi_R(x)\Big\}^{\frac{1}{2}},$$

where $\mathcal{R}^{j,k}_{\mathcal{F}^\perp}$ denotes the set of dyadic rectangles $R = I \times J$ with side-length $\ell(I) = 2^{-j\wedge k}$, $\ell(J) = 2^{-k}$, and $x_R = (x_I, x_J)$ denotes the lower-left corner of R. The local flag Hardy spaces $h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)$, $0 < p < \infty$, consists of all $f \in \mathcal{S}'(\mathbb{R}^N)$ satisfying

$$\|f\|_{h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)} = \|g_{\mathcal{F}^\perp}(f)\|_{L^p(\mathbb{R}^N)} < \infty.$$

Let us now recall the local product Hardy spaces introduced in [7], which extended the local Hardy spaces of Goldberg [17]. For $i = 1, 2$, let $(\varphi^{(i)}_0, \varphi^{(i)})$ be an admissible couple on $\mathbb{R}^{n_i}$. For $j, k \geq 1$, set $\varphi^{(1)}_j(x_1) = 2^{jn_1}\varphi^{(1)}(2^j x_1)$, $\varphi^{(2)}_k(x_2) = 2^{kn_2}\varphi^{(2)}(2^k x_2)$. For $j, k \in \mathbb{N}$, put $\varphi_{j,k} = \varphi^{(1)}_j \otimes \varphi^{(2)}_k$.

For $f \in \mathcal{S}'(\mathbb{R}^N)$, define the local product discrete Littlewood-Paley square function by

$$g(f)(x) = \Big\{ \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}^{j,k}} |\varphi_{j,k} * f(x_R)|^2 \chi_R(x) \Big\}^{1/2},$$

where $\mathcal{R}^{j,k}$ denotes the collection of dyadic rectangles $R = I\times J$ on $\mathbb{R}^N$ of $\ell(I) = 2^{-j}$ and $\ell(J) = 2^{-k}$, and x_R denotes the left-lower corner of R. The local product Hardy space $h^p(\mathbb{R}^N)$, $0 < p < \infty$ consists of all $f \in \mathcal{S}'(\mathbb{R}^N)$ satisfying

$$\|f\|_{h^p(\mathbb{R}^N)} = \|g(f)\|_{L^p(\mathbb{R}^N)} < \infty.$$

We formulate our second main result as follows

Theorem 1.7. Let $0 < p < \infty$. The local product Hardy space $h^p(\mathbb{R}^N)$ is the intersection of the local flag Hardy spaces $h^p_{\mathcal{F}}(\mathbb{R}^N)$ and $h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)$. Moreover,

$$\|f\|_{h^p(\mathbb{R}^N)} \approx \|f\|_{h^p_{\mathcal{F}}(\mathbb{R}^N)} + \|f\|_{h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)}.$$

As is well known that the dual of these multi-parameter Hardy spaces are corresponding Carleson measure spaces, see [8].

Definition 1.8. Let $0 < p \leq 1$ and let $\{\varphi_{j,k}\}$ be defined as above. We say that $f \in \mathcal{S}'(\mathbb{R}^N)$ belongs to the local product Carleson measure spaces $cmo^p(\mathbb{R}^N)$ if

$$\|f\|_{cmo^p(\mathbb{R}^N)} = \sup_{\Omega} \Big\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{N}} \sum_{\substack{R\in\mathcal{R}^{j,k}\\ R\subset\Omega}} |R||\varphi_{j,k} * f(x_R)|^2 \Big\}^{\frac{1}{2}} < \infty$$

for all open sets in $\mathbb{R}^N$ with finite measure.

Definition 1.9. Let $0 < p \leq 1$ and let $\{\psi_{j,k}\}$ be defined as above. We say that $f \in \mathcal{S}'(\mathbb{R}^N)$ belongs to the local flag Carleson measure spaces (adapted to the flag $\mathcal{F}$) $cmo^p_{\mathcal{F}}(\mathbb{R}^N)$ if

$$\|f\|_{cmo^p_{\mathcal{F}}(\mathbb{R}^N)} = \sup_{\Omega} \Big\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{j,k\in\mathbb{N}} \sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{F}}\\ R\subset\Omega}} |R||\psi_{j,k} * f(x_R)|^2 \Big\}^{\frac{1}{2}} < \infty$$

for all open sets in $\mathbb{R}^N$ with finite measure.

Similarly, we can define the local flag Carleson measure spaces (adapted to the flag $\mathcal{F}^\perp$) $cmo^p_{\mathcal{F}^\perp}$. Our last main result of this paper can be formulated as follows:

Theorem 1.10. Let $0 < p \le 1$. The local product Carleson measure space $cmo^p(\mathbb{R}^N)$ is the sum of the local flag Carleson measure spaces $cmo^p_{\mathcal{F}}(\mathbb{R}^N)$ and $cmo^p_{\mathcal{F}^\perp}(\mathbb{R}^N)$. Moreover,

$$\|f\|_{cmo^p(\mathbb{R}^N)} \approx \inf\Big\{\|f_1\|_{cmo^p_{\mathcal{F}}(\mathbb{R}^N)} + \|f_2\|_{cmo^p_{\mathcal{F}^\perp}(\mathbb{R}^N)} : f = f_1 + f_2\Big\},$$

where the infimum is taken over all decompositions $f = f_1 + f_2$ with $f_1 \in cmo^p_{\mathcal{F}}(\mathbb{R}^N)$ and $f_2 \in cmo^p_{\mathcal{F}^\perp}(\mathbb{R}^N)$.

Remark 1.11. Similar relations between global Hardy and Carleson measure spaces were established in [1]. Theorem 1.7 and Theorem 1.10 can be viewed as a natural extension of these results to the local setting. We also mention that the (local) flag Hardy spaces can also be used to characterize the (local) Hardy spaces associated with different homogeneities; see [2] for more details.

This paper is organized as follows. In Section 2, we will show that every local product kernel can be written as a sum of flag kernels. Section 3 is given to the proof that the definition of $h_{\mathcal{F}}(\mathbb{R}^N)$ is independent of the choice of admissible couples, which is rather standard. Section 4 is devoted to the local product Hardy space is the intersection of the two local flag Hardy spaces. Finally, in Section 5, we give the proof of the local product Carleson measure space $cmo^p(\mathbb{R}^N)$ is the sum of two local flag Carleson measure spaces.

Throughout this paper, the letter C and c stand for a positive constant which is independent of the essential variables, but whose value may vary from line to line. We use the notation $A \approx B$ to denote that there exists a positive constant C such that $C^{-1}B \le A \le CB$. For any two positive numbers c_1, c_2, denote $c_1 \vee c_2 = \max\{c_1, c_2\}$, $c_1 \wedge c_2 = \min\{c_1, c_2\}$. Let $-a \wedge b$ mean $-(a \wedge b)$ not $(-a) \wedge b$. Let $\mathbb{N}$ denote the set of all non-negative integers, and $\mathbb{N}_+$ be the set of all positive integers.

2. Proof of Theorem 1.1

We first prove $(i) \Rightarrow (ii)$. Let $\mathcal{K}$ be a local product kernel which coincides with a smooth function K away from subspaces $\{x_1 = 0\}$ and $\{x_2 = 0\}$. As pointed out in [33], we may assume that K has compact support by using a smooth truncation argument. By Theorem 2.1.11 in [33], the Fourier transform of $\mathcal{K}$, $m = \widehat{\mathcal{K}}$, is a product multiplier satisfying

$$|\partial^{\alpha_1}_{\xi_1}\partial^{\alpha_2}_{\xi_2} m(\xi_1, \xi_2)| \le C|\xi_1|^{-|\alpha_1|}|\xi_2|^{-|\alpha_2|}, \qquad \xi_1 \neq 0,\ \xi_2 \neq 0, \tag{8}$$

for all multi-indices (α_1, α_2).

We consider four cases separately.

Case 1: $|\xi_1| > 1, |\xi_2| > 1$. In this case, (6) follows from (8).

Case 2: $|\xi_1| > 1, |\xi_2| \le 1$. In this case, (6) is equivalent to

$$|\xi_1|^{|\alpha_1|}|\partial_{\xi_1}^{|\alpha_1|}\partial_{\xi_2}^{\alpha_2} m(\xi_1,\xi_2)| \le C. \tag{9}$$

We may assume that $|\alpha_2| > 0$, as the case $|\alpha_2| = 0$ follows from (8). Write

$$\begin{aligned}|\xi_1|^{\alpha_1}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1,\xi_2) &= |\xi_1|^{|\alpha_1|}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\int_{\mathbb{R}^N} K(x_1,x_2)e^{-ix\cdot\xi}dx_1dx_2\\ &= \int_{\mathbb{R}^N}(-i|\xi_1|x_1)^{\alpha_1}(-ix_2)^{\alpha_2}K(x_1,x_2)e^{-ix\cdot\xi}dx_1dx_2.\end{aligned}$$

Suppose that ϕ is a smooth function on the positive half line, supported on $[0,1]$, and equal to 1 near the origin. Consider first

$$\int_{\mathbb{R}^N}(-i|\xi_1|x_1)^{\alpha_1}(-ix_2)^{\alpha_2}K(x_1,x_2)\phi(|\xi_1||x_1|)e^{-ix\cdot\xi}dx_1dx_2.$$

Note that for every α_1 the function $x_1^{\alpha_1}e^{-ix_1\cdot\frac{\xi_1}{|\xi_1|}}\phi(|x_1|)$ is a normalized bump function. By the cancellation properties of K, we have

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}(-i|\xi_1|x_1)^{\alpha_1}(-ix_2)^{\alpha_2}K(x_1,x_2)\phi(|\xi_1||x_1|)e^{-ix\cdot\xi}dx_1dx_2\Big|\\ &\le C\int_{\mathbb{R}^{n_2}}\frac{|x_2|^{|\alpha_2|}}{|x_2|^{n_2}(1+|x_2|)^M}dx_2 \le C,\end{aligned}$$

where we have chosen $M > |\alpha_2|$.

Let us now estimate the remainder term

$$\int_{\mathbb{R}^N}(-i|\xi_1|x_1)^{\alpha_1}(-ix_2)^{\alpha_2}K(x_1,x_2)(1-\phi(|\xi_1||x_1|))e^{-ix\cdot\xi}dx_1dx_2.$$

Choose the index ℓ so that $|\xi_1^\ell| \sim |\xi_1|$. Integrating by parts k ($k > |\alpha_1|$) times in x_1^ℓ, we can rewrite it as

$$(i\xi_1^\ell)^{-k}\int_{\mathbb{R}^N}\partial_{x_1^\ell}^k\big((-i|\xi_1|x_1)^{\alpha_1}(-ix_2)^{\alpha_2}K(x_1,x_2)(1-\phi(|\xi_1||x_1|))\big)e^{-ix\cdot\xi}dx_1dx_2.$$

Utilizing Liebniz's rule, we distribute the derivatives on various factors and decompose it as a sum of distributions, which will individually satisfy the required estimates. More precisely, if one of the derivatives fall on the factor $1-\phi$, then the integration takes place on the region where $|x_1| \le |\xi_1|^{-1}$, and we can apply the same argument as the first term. If no derivative falls on the factor $1-\phi$, we can basically reduce ourselves to the single term

$$(i\xi_1^\ell)^{-k}\int_{\mathbb{R}^N}(-i|\xi_1|x_1)^{\alpha_1}(-ix_2)^{\alpha_2}\partial_{x_1^\ell}^k K(x_1,x_2)(1-\phi(|\xi_1||x_1|))e^{-ix\cdot\xi}dx_1dx_2,$$

which converges absolutely and is uniformly bounded. This establishes (9).

Case 3, where $|\xi_1| \le 1$ and $|\xi_2| > 1$, can be treated in the same way, the details being omitted.

Case 4: $|\xi_1| \le 1, |\xi_2| \le 1$. In this case, it is sufficient to show that

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1,\xi_2)| \le C. \tag{10}$$

To this end, we consider four subcases.

If $|\alpha_1| = |\alpha_2| = 0$, then (10) follows from (8).

If $|\alpha_1| > 0, |\alpha_2| > 0$, then the size condition of K gives

$$\begin{aligned}|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1,\xi_2)| &= \left|\int_{\mathbb{R}^N} (-ix_1)^{\alpha_1}(-ix_2)^{\alpha_2} K(x_1,x_2) e^{-ix_1\cdot\xi_1} e^{-ix_2\cdot\xi_2} dx_1 dx_2\right| \\ &\le C\int_{\mathbb{R}^N} \frac{|x_1|^{|\alpha_1|}|x_2|^{|\alpha_2|}}{|x_1|^{n_1}|x_2|^{n_2}(1+|x_1|+|x_2|)^M} dx_1 dx_2 \le C,\end{aligned}$$

where we have chosen $M > |\alpha_1| + |\alpha_2|$.

If $|\alpha_1| > 0, |\alpha_2| = 0$, the cancellation of K is involved. Let ϕ be as before. Hence,

$$\begin{aligned}\partial_{\xi_1}^{\alpha_1} m(\xi_1,\xi_2) &= \int_{\mathbb{R}^N} \phi(|x_2|)(-ix_1)^{\alpha_1} K(x_1,x_2) e^{-ix_1\cdot\xi_1} e^{-ix_2\cdot\xi_2} dx_1 dx_2 \\ &\quad + \int_{\mathbb{R}^N} (1-\phi(|x_2|))(-ix_1)^{\alpha_1} K(x_1,x_2) e^{-ix_1\cdot\xi_1} e^{-ix_2\cdot\xi_2} dx_1 dx_2.\end{aligned}$$

For the first term,

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi(|x_2|)(-ix_1)^{\alpha_1} K(x_1,x_2) e^{-ix_1\cdot\xi_1} e^{-ix_2\cdot\xi_2} dx_1 dx_2 \\ &\qquad = \int_{\mathbb{R}^{n_1}} \tilde{K}(x_1)(-ix_1)^{\alpha_1} e^{-ix_1\cdot\xi_1} dx_1,\end{aligned}$$

where

$$\tilde{K}(x_1) = \int_{\mathbb{R}^{n_2}} K(x_1,x_2)\phi(|x_2|) e^{-ix_2\cdot\xi_2} dx_2.$$

Observe that $\phi(|x_2|)e^{-ix_2\cdot\xi_2}$ is a normalized bump function on $\mathbb{R}^{n_2}$ for every $|\xi_2| \le 1$. By the cancellation condition, we have

$$\left|\int_{\mathbb{R}^{n_1}} \tilde{K}(x_1)(-ix_1)^{\alpha_1} e^{-ix_1\cdot\xi_1} dx_1\right| \le C\int_{\mathbb{R}^{n_1}} \frac{|x_1|^{|\alpha_1|}}{|x_1|^{n_1}(1+|x_1|)^M} dx_1 < \infty,$$

where we have chosen $M > |\alpha_1|$.

For the second term,

$$\begin{aligned}&\left|\int_{\mathbb{R}^N} (1-\phi(|x_2|))(-ix_1)^{\alpha_1} K(x_1,x_2) e^{-ix_1\cdot\xi_1} e^{-ix_2\cdot\xi_2} dx_1 dx_2\right| \\ &\le C\int_{\mathbb{R}^{n_1}}\int_{|x_2|\ge\frac{1}{2}} \frac{|x_1|^{|\alpha_1|}}{|x_1|^{n_1}(1+|x_1|+|x_2|)^M} dx_1 dx_2 < \infty,\end{aligned}$$

where M is chosen to be larger than $|\alpha_1| + n_2$.

The remaining case $|\alpha_1| = 0, |\alpha_2| > 0$ can be treated in a similar way. The details are omitted. This completes the proof of (10).

Next we prove $(ii) \Rightarrow (iii)$. For the simplicity of presentation, we assume that $n_1 = n_2 = 1$. But our proof presented below can be extended to higher dimensions without any essential difficulty. Suppose that $m = \widehat{\mathcal{K}}$ satisfies (6). Take a cutoff function η supported on $[-2, 2]$, and identically equal to 1 on $[-1, 1]$. Let $m_1(\xi_1, \xi_2) = m(\xi_1, \xi_2)(1 - \eta(\frac{1+|\xi_1|}{1+|\xi_2|}))$ and $m_2(\xi_1, \xi_2) = m(\xi_1, \xi_2)\eta(\frac{1+|\xi_1|}{1+|\xi_2|})$, $\mathcal{K}_1 = (m_1)^\vee$ and $\mathcal{K}_2 = (m_2)^\vee$. Then $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$. To prove (iii), it suffices to show that $\mathcal{K}_1 \in \mathcal{P}_0(\mathcal{F})$ and $\mathcal{K}_2 \in \mathcal{P}_0(\mathcal{F}^\perp)$. By Theorem 2.1 in [2], it suffices to show that their Fourier transforms m_1 and m_2 satisfy

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m_1(\xi_1, \xi_2)| \le C(1 + |\xi_1| + |\xi_2|)^{-|\alpha_1|}(1 + |\xi_2|)^{-|\alpha_2|} \tag{11}$$

and

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m_2(\xi_1, \xi_2)| \le C(1 + |\xi_1|)^{-|\alpha_1|}(1 + |\xi_1| + |\xi_2|)^{-|\alpha_2|} \tag{12}$$

for all multi-indices (α_1, α_2).

We only show (11) as (12) can be treated in the same way. Leibniz's rule yields that

$$\begin{aligned}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m_1(\xi_1, \xi_2) \quad &\approx \quad \left(1 - \eta(\frac{1 + |\xi_1|}{1 + |\xi_2|})\right)\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2) \\ &\quad + \sum_{\substack{1\le\beta_1\le\alpha_1, \beta_1+\beta_1'=\alpha_1 \\ 1\le\gamma_2\le\alpha_2, \gamma_2+\gamma_2'=\alpha_2}} \partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\gamma_2}(1 - \eta(\frac{1 + |\xi_1|}{1 + |\xi_2|}))\partial_{\xi_1}^{\beta_1'}\partial_{\xi_2}^{\gamma_2'} m(\xi_1, \xi_2).\end{aligned}$$

For the first part, observe that supp $m_1 \subset \{1 + |\xi_1| \ge 1 + |\xi_2|\}$,

$$\begin{aligned}\left|\left(1 - \eta(\frac{1 + |\xi_1|}{1 + |\xi_2|})\right)\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2} m(\xi_1, \xi_2)\right| \quad &\le \quad C(1 + |\xi_1|)^{-|\alpha_1|}(1 + |\xi_2|)^{-|\alpha_2|} \\ &\le \quad C(1 + |\xi_1| + |\xi_2|)^{-|\alpha_1|}(1 + |\xi_2|)^{-|\alpha_2|}.\end{aligned}$$

For the remaining part, observe in fact that if one of the derivatives falls on the factor $1 - \eta$, then $1 + |\xi_1| \approx 1 + |\xi_2| \approx 1 + |\xi_1| + |\xi_2|$ due to the support property of the derivative of η. Hence,

$$\left|\partial_{\xi_2}^{\gamma_2}\left(\partial_{\xi_1}^{\beta_1}(1 - \eta(\frac{1 + |\xi_1|}{1 + |\xi_2|}))\right)\right| \le C(1 + |\xi_1|)^{-|\beta_1|}(1 + |\xi_2|)^{-|\gamma_2|},$$

which implies that

$$\begin{aligned}&\left|\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\gamma_2}(1 - \eta(\frac{1 + |\xi_1|}{1 + |\xi_2|}))\partial_{\xi_1}^{\beta_1'}\partial_{\xi_2}^{\gamma_2'} m(\xi_1, \xi_2)\right| \\ &\le C(1 + |\xi_1|)^{-|\beta_1|}(1 + |\xi_2|^{-|\gamma_2|}(1 + |\xi_1| + |\xi_2|)^{-|\beta_1'|}(1 + |\xi_2|)^{-|\gamma_2'|} \\ &\le C(1 + |\xi_1| + |\xi_2|)^{-|\alpha_1|}(1 + |\xi_2|)^{-|\alpha_2|}.\end{aligned}$$

Hence, (11) is proved.

Finally, we prove $(iii) \Rightarrow (i)$. It is easy to see that the local flag kernels $\mathcal{K}_1$ and $\mathcal{K}_2$ are local product kernels, and hence so is $\mathcal{K}_1 + \mathcal{K}_2 = \mathcal{K}$.

The proof of Theorem 1.1 is complete.

3. Proof of Proposition 1.6

To prove Proposition 1.6, we need some preliminaries.

For $L \in \mathbb{N}_+$, we denote

$$\mathcal{S}_L(\mathbb{R}^n) = \{\phi \in \mathcal{S}(\mathbb{R}^n) : \int \phi(x)x^\alpha dx = 0 \text{ for all } |\alpha| \leq L-1\}.$$

The following lemma is the classical almost orthogonality estimates.

Lemma 3.1. [14] Let $\psi, \phi \in \mathcal{S}_L(\mathbb{R}^n)$, $j,k \in \mathbb{N}$. Then for any given positive integer M, there exists a constant $C > 0$ depending only ψ, ϕ, n, M and L such that

$$|\psi_j * \phi_k(x)| \leq C2^{-L|j-k|}\frac{2^{(-j\wedge k)M}}{(2^{-j\wedge k}+|x|)^{n+M}}, \quad x \in \mathbb{R}^n.$$

We also need the almost orthogonality estimates in the flag setting.

Lemma 3.2. [24] Let $j, j', k, k' \in \mathbb{N}$. Suppose that $\psi_{j,k}$ and $\phi_{j',k'}$ are defined as in Proposition 1.6. Then, for any $L, M > 0$,

$$\begin{aligned}&|(\psi_{j,k} * \phi_{j',k'})(x_1,x_2)|\\ &\quad \leq C2^{-L(|j-j'|+|k-k'|)}\frac{2^{(-j\wedge j')M}}{(2^{-j\wedge j'}+|x_1|)^{n_1+M}}\frac{2^{(-j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'}+|x_2|)^{n_2+M}}. \quad (13)\end{aligned}$$

We also need the following maximal function estimate.

Lemma 3.3. [24] Let R be a rectangle in $\mathcal{R}_{\mathcal{F}}^{j,k}$ and R' be a rectangle in $\mathcal{R}_{\mathcal{F}}^{j',k'}$. Then for any point $(u_1,u_2) \in R$, and any $\delta \in (\frac{n_1\vee n_2}{(n_1\vee n_2+M)}, 1]$, we have

$$\begin{aligned}&\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{j',k'}}\frac{2^{(-j\wedge j')M}}{(2^{-j\wedge j'}+|x_I-x_{I'}|)^{n_1+M}}\frac{2^{(-j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'}+|x_J-x_{J'}|)^{n_2+M}}|R'||\psi_{j',k'} * f(x_{I'},x_{J'})|\\ &\quad \leq C_1\{\mathcal{M}_S[(\sum_{R'\in\mathcal{R}_{\mathcal{F}}^{j',k'}}|\psi_{j',k'} * f(x_{I'},x_{J'})|^2\chi_{R'})^{\frac{\delta}{2}}](u_1,u_2)\}^{\frac{1}{\delta}}, \quad (14)\end{aligned}$$

where $\mathcal{M}_S$ is the strong maximal operator on $\mathbb{R}^N$ and

$$C_1 = 2^{(n_1[(j'-j)\vee 0]+n_2[(j'\wedge k'-j\wedge k)\vee 0])(\frac{1}{\delta}-1)}.$$

We are now ready to present the

Proof of Proposition 1.6. Let $f \in \mathcal{S}'(\mathbb{R}^{n_1+n_2})$. By the Calderón reproducing formula (7) and Lemma 3.2,

$$\begin{aligned}&|\phi_{j',k'} * f(x_{I'},x_{J'})|\\ &\quad = \Big|\sum_{j,k\in\mathbb{N}}\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}}|R|\psi_{j,k} * f(x_I,y_J)\phi_{j',k'} * \psi_{j,k}(x_{I'}-x_I,x_{J'}-x_J)\Big|\end{aligned}$$

$$\le C \sum_{j,k\in\mathbb{N}} 2^{-L(|j-j'|+|k-k'|)} \sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |R||\psi_{j,k} * f(x_I, x_J)|$$
$$\times \frac{2^{(-j\wedge j')M}}{(2^{-j\wedge j'} + |x_I - x_{I'}|)^{n_1+M}} \frac{2^{(-j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'} + |x_J - x_{J'}|)^{n_2+M}},$$

where M is chosen large enough so that $\frac{N}{N+M} < \min\{p, 1\}$ and $L > 10M$.

Applying Lemma 3.3 with $\frac{N}{N+M} < \delta < \min\{p, 1\}$, we arrive at

$$|\phi_{j',k'} * f(x_{I'}, x_{J'})|$$
$$\le C \sum_{j,k\in\mathbb{N}} 2^{-L(|j-j'|+|k-k'|)} C_1 \Big\{\mathcal{M}_S\Big(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_R\Big)^{\frac{\delta}{2}}(u_1', u_2')\Big\}^{\frac{1}{\delta}}$$
$$\le C \sum_{j,k\in\mathbb{N}} 2^{-L'(|j-j'|+|k-k'|)} \Big\{\mathcal{M}_S\Big(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_R\Big)^{\frac{\delta}{2}}(u_1', u_2')\Big\}^{\frac{1}{\delta}},$$

for any $(u_1', u_2') \in R'$, where $L' := L - N(1/\delta - 1) > 0$. Applying the Cauchy-Schwarz inequality, we have

$$\sum_{j',k'\in\mathbb{N}} \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{j',k'}} |\phi_{j',k'} * f(x_{I'}, x_{J'})|^2 \chi_{R'}(x_1, x_2)$$
$$\le C \sum_{j',k'\in\mathbb{N}} \Big[\sum_{j,k\in\mathbb{N}} 2^{-L'(|j-j'|+|k-k'|)} \{\mathcal{M}_S\big(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_R\big)^{\frac{\delta}{2}}(x_1, x_2)\}^{\frac{1}{\delta}}\Big]^2$$
$$\le C \sum_{j',k'\in\mathbb{N}} \Big\{\Big[\sum_{j,k\in\mathbb{N}} 2^{-L'(|j-j'|+|k-k'|)}\Big]$$
$$\times \Big[\sum_{j,k\in\mathbb{N}} 2^{-L'(|j-j'|+|k-k'|)} \{\mathcal{M}_S\big(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_R\big)^{\frac{\delta}{2}}(x_1, x_2)\}^{\frac{2}{\delta}}\Big]\Big\}$$
$$\le C \sum_{j,k\in\mathbb{N}} \Big\{\mathcal{M}_S\big[\big(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_R\big)^{\frac{\delta}{2}}\big](x_1, x_2)\Big\}^{\frac{2}{\delta}}.$$

From the Fefferman-Stein vector-valued maximal inequality, we know that $\mathcal{M}_S$ is bounded on $L^{\frac{p}{\delta}}(\ell^{\frac{2}{\delta}})$ and hence,

$$\Big\|\Big(\sum_{j',k'\in\mathbb{N}} \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{j',k'}} |\phi_{j',k'} * f(x_{I'}, x_{J'})|^2 \chi_{R'}\Big)^{\frac{1}{2}}\Big\|_{L^p}$$
$$\le C \Big\|\Big(\sum_{j,k\in\mathbb{N}} \Big\{\mathcal{M}_S\big[\big(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}} |\psi_{j,k} * f(x_I, x_J)|^2 \chi_R\big)^{\frac{\delta}{2}}\big]\Big\}^{\frac{2}{\delta}}\Big)^{\frac{1}{2}}\Big\|_{L^p}$$

$$= C\left\|\left(\sum_{j,k\in\mathbb{N}}\left\{\mathcal{M}_S[(\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}}|\psi_{j,k}*f(x_I,x_J)|^2\chi_R)^{\frac{\delta}{2}}]\right\}^{\frac{2}{\delta}}\right)^{\frac{\delta}{2}}\right\|_{L^{\frac{p}{\delta}}}^{\frac{1}{\delta}}$$

$$\leq C\left\|\left(\sum_{j,k\in\mathbb{N}}\sum_{R\in\mathcal{R}_{\mathcal{F}}^{j,k}}|\psi_{j,k}*f(x_I,y_J)|^2\chi_R\right)^{\frac{1}{2}}\right\|_{L^p}.$$

By symmetry, we establish the reverse inequality and hence, Proposition 1.6. □

4. Proof of Theorem 1.7

This section is devoted to proving that the local product Hardy spaces is the intersection of two local flag Hardy spaces. We need the following almost orthogonality estimate.

Lemma 4.1. Let $j, k, j', k' \in \mathbb{N}$. If $j' \geq k' + 5$, or if $j \geq k$, then any $M, L > 0$,

$$|(\varphi_{j,k} * \psi_{j',k'})(x_1, x_2)| \leq C2^{-L(|j-j'|+|k-k'|)}\frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'}+|x_1|)^{n_1+M}}\frac{2^{-(j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'}+|x_2|)^{n_2+M}}. \tag{15}$$

Proof. Observe that if k is small, that is $0 \leq k \leq c$ for some $c > 0$, then by spectral localizations, k' must be small. On the other hand, if k is large, that is $k \geq c$ for some $c > 0$, then by spectral localizations we have $|k' - k| \leq 2$. Hence, $\varphi_{j,k} * \psi_{j',k'}$ does not vanish identically only if $|k' - k| \leq C$.

We prove (15) first under the assumption $j' \geq k' + 5$. Observe that

$$\operatorname{supp}\widehat{\psi_{j'}^{(1)}} \subset \{(\xi_1, \xi_2) \in \mathbb{R}^N : 2^{j'-1} \leq |\xi_1| + |\xi_2| \leq 2^{j'+1}\}.$$

We now consider three cases separately:

Case 1: $|\xi_1| \geq 1$, $|\xi_2| \leq 1$. The support properties of $\widehat{\varphi_j^{(1)}}$ and $\widehat{\psi_{j'}^{(1)}}$ imply that $|j - j'| \leq 2$.

Case 2: $|\xi_1| \leq 1$, $|\xi_2| \geq 1$. By spectral localizations, we have $j = 0$ and $k, k' > 0$. Observe that $\operatorname{supp}\widehat{\psi_{k'}^{(2)}} \subset \{\xi_2 \in \mathbb{R}^{n_2} : 2^{k'-1} \leq |\xi_2| \leq 2^{k'+1}\}$, which, together with our basic assumption $j' \geq k' + 5$, yields

$$|\xi_1| + |\xi_2| \leq 2^{k'+1} + 1 < 1 + 2^{j'-4}.$$

The support properties of $\widehat{\psi_{j'}^{(1)}}$ and $\widehat{\psi_{k'}^{(2)}}$ force $0 < j' < C$ for some certain $C > 0$. Hence, $|j - j'| \leq C$ in this case.

Case 3: $|\xi_1| \geq 1$, $|\xi_2| \geq 1$. In this case, we have $j, j', k, k' \geq 1$. Taking the Fourier transform,

$$(\varphi_{j,k} * \psi_{j',k'})(\xi_1, \xi_2) = \widehat{\varphi^{(1)}}(2^{-j}\xi_1)\widehat{\varphi^{(2)}}(2^{-k}\xi_2)\widehat{\psi^{(1)}}(2^{-j'}\xi_1, 2^{-j'}\xi_2)\widehat{\psi^{(2)}}(2^{-k'}\xi_2).$$

Observe that $\widehat{\varphi^{(1)}}(2^{-j}\xi_1)\widehat{\varphi^{(2)}}(2^{-k}\xi_2)\widehat{\psi^{(1)}}(2^{-j'}\xi_1, 2^{-j'}\xi_2)$ does not vanish only if

$$2^{j'-2} \le \sqrt{2^{2j} + 2^{2k}} \le 2^{j'+2}.$$

The second inequality gives $j \le j' + 2$; The first inequality, together with $j' \ge k' + 5$ and $|k - k'| \le 2$, yields

$$2^{2j} \ge 2^{2j'-4} - 2^{2k} \ge 2^{2j'-4} - 2^{4+2k'} \ge 2^{2j'-4} - 2^{2j'-6} > 2^{2j'-6},$$

which implies $j \ge j' - 2$.

From these considerations, we see that the left hand of (15) does not vanish only if $|j' - j'| \le C$ and $|k - k'| \le C$. Thus,

$$\begin{aligned}
&|\varphi_{j,k} * \psi_{j',k'}(x_1, x_2)| \\
&\quad \le C \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'} + |x_1|)^{n_1+M}} \frac{2^{-(j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'} + |x_2|)^{n_2+M}} \\
&\quad \le C 2^{-L(|j-j'|+|k-k'|)} \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'} + |x_1|)^{n_1+M}} \frac{2^{-(j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'} + |x_2|)^{n_2+M}},
\end{aligned}$$

which concludes the proof of Lemma 4.1 when $j' \ge k' + 5$.

For the case $j \ge k$, the left hand of (15) dose not vanish only if $|k - k'| \le C$. If $j' = 0$, the spectral localizations implies $j = 0$. It is easy to check $\varphi_{0,k} * \psi_{0,k'}$ satisfies the desired estimate. We now assume that $j' > 0$. Let $p_{j,k}^{L-1}$ be the degree $(L-1)$ Taylor polynomial of the function $\varphi_{j,k}$. By the cancellation condition of $\psi_{j'}^{(1)}$, we get that

$$\begin{aligned}
&\left|\varphi_{j,k} * \psi_{j'}^{(1)}(x_1, x_2)\right| \\
&= \left| \int_{\mathbb{R}^{n_1+n_2}} [\varphi_{j,k}(x_1 - y_1, x_2 - y_2) - p_{j,k}^{L-1}(x_1, x_2)] \psi_{j'}^{(1)}(y_1, y_2) dy_1 dy_2 \right| \\
&\le C \int_{\mathbb{R}^{n_1+n_2}} (2^j|y_1| + 2^k|y_2|)^L \frac{2^{jn_1}}{(1 + 2^j|x_1 - \theta y_1|)^{n_1+M}} \frac{2^{kn_2}}{(1 + 2^k|x_2 - \theta y_2|)^{n_2+M}} \\
&\qquad \times \frac{2^{j'n_1}}{(1 + 2^{j'}|y_1|)^{n_1+M'}} \frac{2^{j'n_2}}{(1 + 2^{j'}|y_2|)^{n_2+M'}} dy_1 dy_2 \\
&\le C 2^{-L|j-j'|} \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'} + |x_1|)^{n_1+M}} \frac{2^{-(j\wedge j'\wedge k)M}}{(2^{-j\wedge j'\wedge k} + |x_2|)^{n_2+M}},
\end{aligned}$$

where $\theta \in (0, 1)$ and M' is large enough so that $M' > M + L + n + m$. Thus,

$$\begin{aligned}
&\left|\varphi_{j,k} * \psi_{j',k'}(x_1, x_2)\right| \\
&= \left| \int_{\mathbb{R}^{n_2}} \varphi_{j,k} * \psi_{j'}^{(1)}(x_1, x_2 - u) \psi_{k'}^{(2)}(u) du \right| \\
&\le C 2^{-L|j-j'|} \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'} + |x_1|)^{n_1+M}} \int_{\mathbb{R}^{n_2}} \frac{2^{-(j\wedge j'\wedge k)M}}{(2^{-j\wedge j'\wedge k} + |x_2 - u|)^{n_2+M}} \frac{2^{k'n_2}}{(1 + 2^{k'}|u|)^{n_2+M}} du \\
&\le C 2^{-L(|j-j'|+|k-k'|)} \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'} + |x_1|)^{n_1+M}} \frac{2^{-(j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'} + |x_2|)^{n_2+M}}.
\end{aligned}$$

This finishes the proof of Lemma 4.1. □

We now turn to the

Proof of Theorem 1.7. We begin by writing

$$\begin{aligned} g_{\mathcal{F}}(f)(x) \leq & \Big(\sum_{\substack{j',k' \in \mathbb{N} \\ j'-5<k'<j'+2}} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{j',k'}} |\psi_{j',k'} * f(x_{R'})|^2 \chi_{R'}(x) \Big)^{\frac{1}{2}} \\ & + \Big(\sum_{\substack{j',k' \in \mathbb{N} \\ k' \leq j'-5}} \sum_{R' \in \mathcal{R}_{\mathcal{F}}^{j',k'}} |\psi_{j',k'} * f(x_{R'})|^2 \chi_{R'}(x) \Big)^{\frac{1}{2}} \\ =: & \; g_1(f)(x) + g_2(f)(x). \end{aligned}$$

For the first part, we observe that $\psi_{j',k'}$ satisfy the same smoothness and cancellation conditions as $\psi_{j'}$ and the rectangles R' in $\mathcal{R}_{\mathcal{F}}^{j',k'}$ are almost cubes in $\mathbb{R}^N$, provided $j'-5<k'<j'+2$. Thus, by arguments similar to that given in the proof of Proposition 1.6, we conclude that

$$\|g_1(f)\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{h^p_{classical}(\mathbb{R}^N)} \leq C\|f\|_{h^p(\mathbb{R}^N)},$$

where $h^p_{classical}(\mathbb{R}^N)$ represents the classical one-parameter local Hardy space of Goldberg [17], and the last inequality uses the relation between local Hardy spaces and local product Hardy spaces (see [6, 7]).

We treat next the second part. In view of the almost orthogonality estimate in Lemma 4.1 with $j' \geq k'+5$, by repeating the same argument as used in the proof of Proposition 1.6, we derive

$$\begin{aligned} \|g_2(f)\|_{L^p(\mathbb{R}^N)} & \leq C\Big\|\Big\{ \sum_{j,k\in\mathbb{N}} \Big\{\mathcal{M}_S[(\sum_{R\in\mathcal{R}^{j,k}} |\psi_{j,k} * f(x_R)|^2 \chi_R)^{\frac{\delta}{2}}]\Big\}^{\frac{2}{\delta}} \Big\}^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^N)} \\ & \leq C\Big\|\Big\{ \sum_{j,k\in\mathbb{N}} \sum_{R\in\mathcal{R}^{j,k}} |\psi_{j,k} * f(x_R)|^2 \chi_R \Big\}^{\frac{1}{2}} \Big\|_{L^p(\mathbb{R}^N)} \approx \|f\|_{h^p(\mathbb{R}^N)}. \end{aligned}$$

Thus, we have shown that $\|f\|_{h^p_{\mathcal{F}}(\mathbb{R}^N)} \leq C\|f\|_{h^p(\mathbb{R}^N)}$. The inequality

$$\|f\|_{h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)} \leq C\|f\|_{h^p(\mathbb{R}^N)}$$

can be demonstrated in the same way. We omit the details.

Let us now turn to prove the reverse inequality, namely,

$$\|f\|_{h^p(\mathbb{R}^N)} \leq C\big(\|f\|_{h^p_{\mathcal{F}}(\mathbb{R}^N)} + \|f\|_{h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)}\big).$$

From Theorem 1.3, we see that each local product kernel can be written as a sum of two local flag kernels. By adapting this idea to the vector-valued setting, we shall prove that the product square functions is the sum of two semiproduct square functions, which are essentially local flag square functions in view of Lemma 4.1. Assume that $f \in h^p_{\mathcal{F}_1} \cap h^p_{\mathcal{F}_2}$, define

$$g_{\mathcal{S}}(f)(x) := \Big\{ \sum_{\substack{j,k\in\mathbb{N} \\ j\geq k}} \sum_{R\in\mathcal{R}^{j,k}} |\varphi_{j,k} * f(x_R)|^2 \chi_R(x) \Big\}^{\frac{1}{2}}$$

and

$$g_{\mathcal{S}^\perp}(f)(x) := \Big\{ \sum_{\substack{j,k\in\mathbb{N} \\ j\leq k}} \sum_{R\in\mathcal{R}^{j,k}} |\varphi_{j,k} * f(x_R)|^2 \chi_R(x) \Big\}^{\frac{1}{2}}.$$

Since $g(f)(x)$ and $g_{\mathcal{S}}(f)(x)+g_{\mathcal{S}^\perp}(f)(x)$ are pointwise comparable, it is sufficient to show

$$\|g_{\mathcal{S}}(f)\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{h^p_{\mathcal{F}}(\mathbb{R}^N)} \tag{16}$$

and

$$\|g_{\mathcal{S}^\perp}(f)\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{h^p_{\mathcal{F}^\perp}(\mathbb{R}^N)}. \tag{17}$$

We only need to show (16) taking into account symmetry. Recall from Lemma 4.1 that $\varphi_{j,k} * \psi_{j',k'}$ satisfy the almost orthogonality estimate of flag type. Hence arguing as in the proof of Proposition 1.6, we obtain

$$\begin{aligned} \|g_{\mathcal{S}}(f)\|_{L^p(\mathbb{R}^N)} &\leq C\Big\|\Big\{ \sum_{j',k'\in\mathbb{N}} \Big\{\mathcal{M}_S[(\sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{F}}} |\psi_{j',k'} * f(x_{R'})|^2 \chi_{R'})^{\frac{\delta}{2}}]\Big\}^{\frac{2}{\delta}} \Big\}^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^N)} \\ &\leq C\Big\|\Big\{ \sum_{j',k'\in\mathbb{N}} \sum_{R'\in\mathcal{R}^{j',k'}_{\mathcal{F}}} |\psi_{j',k'} * f(x_{R'})|^2 \chi_{R'} \Big\}^{\frac{1}{2}}\Big\|_{L^p(\mathbb{R}^N)} \\ &\approx \|f\|_{h^p_{\mathcal{F}}}(\mathbb{R}^N). \end{aligned}$$

This concludes the proof of Theorem 1.7. □

5. Proof of Theorem 1.10

Proof of Theorem 1.10. Assume first that $f \in cmo^p_{\mathcal{F}} + cmo^p_{\mathcal{F}^\perp}$. For any partition $f = f_1 + f_2$ with $f_1 \in cmo^p_{\mathcal{F}}$ and $f_2 \in cmo^p_{\mathcal{F}^\perp}$ and any $g \in h^p = h^p_{\mathcal{F}} \cap h^p_{\mathcal{F}^\perp}$, we have

$$\begin{aligned} |\langle f, g\rangle| &\leq |\langle f_1, g\rangle| + |\langle f_2, g\rangle| \\ &\leq C\Big(\|f_1\|_{cmo^p_{\mathcal{F}}} + \|f_2\|_{cmo^p_{\mathcal{F}^\perp}}\Big) \max\Big\{\|g\|_{h^p_{\mathcal{F}}}, \|g\|_{h^p_{\mathcal{F}^\perp}}\Big\} \\ &\leq C\Big(\|f_1\|_{cmo^p_{\mathcal{F}}} + \|f_2\|_{cmo^p_{\mathcal{F}^\perp}}\Big)\|g\|_{h^p}, \end{aligned}$$

which implies $f \in cmo^p$ with $\|f\|_{cmo^p} \le C\Big(\|f_1\|_{cmo^p_{\mathcal{F}}} + \|f_2\|_{cmo^p_{\mathcal{F}^\perp}}\Big)$. Taking the infimum over all possible partitions $f = f_1 + f_2$ with $f_1 \in cmo^p_{\mathcal{F}}$ and $f_2 \in cmo^p_{\mathcal{F}^\perp}$, we obtain

$$\|f\|_{cmo^p} \le C \inf_{\substack{f=f_1+f_2 \\ f_1\in cmo^p_{\mathcal{F}}, f_2\in cmo^p_{\mathcal{F}^\perp}}} \Big(\|f_1\|_{cmo^p_{\mathcal{F}}} + \|f_2\|_{cmo^p_{\mathcal{F}^\perp}}\Big).$$

Conversely, given $f \in cmo^p$, we need to construct a partition $f = f_1 + f_2$ with $f_1 \in cmo^p_{\mathcal{F}}$, $f_2 \in cmo^p_{\mathcal{F}^\perp}$ and

$$\|f_1\|_{cmo^p_{\mathcal{F}}} + \|f_2\|_{cmo^p_{\mathcal{F}^\perp}} \le C\|f\|_{cmo^p}. \tag{18}$$

For $i = 1, 2$, we choose $\phi_0^{(i)}, \phi^{(i)} \in \mathcal{S}(\mathbb{R}^{n_i})$ with

$$\begin{aligned}
&\operatorname{supp} \widehat{\phi_0^{(i)}} \subset \{\xi_i \in \mathbb{R}^{n_i} : |\xi_i| \le 2\}, \quad \widehat{\phi_0^{(i)}}(\xi_i) = 1 \text{ for } |\xi_i| \le 1,\\
&\operatorname{supp} \widehat{\phi^{(i)}} \subset \{\xi_i \in \mathbb{R}^{n_i} : \tfrac{1}{2} \le |\xi_i| \le 2\},\\
&\widehat{\phi_0^{(i)}}(\xi_i) + \sum_{j=1}^{\infty} \widehat{\phi^{(i)}}(2^{-j}\xi_i) = 1 \text{ for all } \xi_i \in \mathbb{R}^{n_i},
\end{aligned}$$

and set $\phi_{j'',k''} = \phi_{j''}^{(1)} \otimes \phi_{k''}^{(2)}$ for every $j'', k'' \in \mathbb{N}$. For $f \in cmo^p$, define

$$f_1 = \mathcal{T}_1(f) = \mathcal{K}_1 * f = \sum_{\substack{j'',k''\in\mathbb{N} \\ j''\ge k''}} \phi_{j'',k''} * f$$

and

$$f_2 = \mathcal{T}_2(f) = \mathcal{K}_2 * f = \sum_{\substack{j'',k''\in\mathbb{N} \\ j''< k''}} \phi_{j'',k''} * f.$$

Then $f = f_1 + f_2$, and (18) will follow if we can prove

$$\|\mathcal{T}_1(f)\|_{cmo^p_{\mathcal{F}}} \le C\|f\|_{cmo^p}$$

and

$$\|\mathcal{T}_2(f)\|_{cmo^p_{\mathcal{F}^\perp}} \le C\|f\|_{cmo^p}.$$

We only consider $\mathcal{T}_1$ as the other can be similarly treated. For $R \in \mathcal{R}^{j,k}_{\mathcal{F}}$ and $R' \in \mathcal{R}^{j',k'}$, set

$$S_R = |[\psi_{j,k} * \mathcal{T}_1(f)](x_R)|^2 \text{ and } T_{R'} = |(\varphi_{j',k'} * f)(x_{R'})|^2.$$

As observed before, $\psi_{j,k}$ does not identically vanish only if $k \le j + 2$. The Calderón reproducing formula in (7) and the support properties of $\widehat{\varphi_{j',k'}}$ and

$\widehat{\phi_{j'',k''}}$ yield that

$$
\begin{aligned}
S_R^{\frac{1}{2}}
&= \Big| \sum_{j',k'\in\mathbb{N}} \sum_{\substack{j'',k''\in\mathbb{N}\\ j''\geq k''}} \sum_{R'\in\mathcal{R}^{j',k'}} |R'|(\psi_{j,k} * \phi_{j'',k''} * \varphi_{j',k'})(x_R - x_{R'})(\varphi_{j',k'} * f)(x_{R'}) \Big| \\
&= \Big| \sum_{\substack{j',k'\in\mathbb{N}\\ k'\leq j'+4}} \sum_{\substack{j'',k''\in\mathbb{N}\\ |j'-j''|\leq 2, |k'-k''|\leq 2}} \sum_{R'\in\mathcal{R}_{\mathcal{F}}^{j'}} |R'|(\psi_{j,k} * \phi_{j'',k''} * \varphi_{j',k'})(x_R - x_{R'})(\varphi_{j',k'} * f)(x_{R'}) \Big| \\
&= \Big| \sum_{\substack{j',k'\in\mathbb{N}\\ k'\leq j'+4}} \sum_{R'\in\mathcal{R}^{j',k'}} |R'|(\psi_{j,k} * \tilde{\phi}_{j',k'} * \varphi_{j',k'})(x_R - x_{R'})(\varphi_{j',k'} * f)(x_{R'}) \Big|,
\end{aligned}
$$

where

$$
\tilde{\phi}_{j',k'} = \sum_{\substack{j'',k''\in\mathbb{N}\\ |j'-j''|\leq 2, |k'-k''|\leq 2}} \phi_{j'',k''}
$$

for every $j', k' \in \mathbb{N}$ with $k'' \leq j'' + 4$. We observe that $\tilde{\phi}_{j',k'} * \varphi_{j',k'}$ satisfies the same smoothness and cancellation conditions as $\varphi_{j',k'}$. It then follows from Lemma 4.1 that

$$
\begin{aligned}
S_R^{\frac{1}{2}} &\leq C \sum_{\substack{j',k'\in\mathbb{N}\\ k'\leq j'+4}} \sum_{R'\in\mathcal{R}^{j',k'}} 2^{-(n_1|j-j'|+n_2|k-k'|)L} 2^{-j'n_1} 2^{-k'n_2} \\
&\quad \times \frac{2^{-(j\wedge j')M}}{(2^{-j\wedge j'} + |x_I - x_{I'}|)^{n_1+M}} \frac{2^{-(j\wedge j'\wedge k\wedge k')M}}{(2^{-j\wedge j'\wedge k\wedge k'} + |x_J - x_{J'}|)^{n_2+M}} |(\varphi_{j',k'} * f)(x_{R'})| \\
&\leq C \sum_{\substack{j',k'\in\mathbb{N}\\ k'\leq j'+4}} \sum_{R'\in\mathcal{R}^{j',k'}} r(R,R')P(R,R')T_{R'}^{\frac{1}{2}},
\end{aligned}
$$

where

$$
r(R,R') = r_L(R,R') = \left(\frac{|I|}{|I'|} \wedge \frac{|I'|}{|I|}\right)^L \left(\frac{|J|}{|J'|} \wedge \frac{|J'|}{|J|}\right)^L
$$

and

$$
P(R,R') = P_M(R,R') = \left(1 + \frac{|x_I - x_{I'}|}{2^{(-j\wedge j')}}\right)^{-n_1+M} \left(1 + \frac{|x_J - x_{J'}|}{2^{(-k\wedge k')}}\right)^{-n_2+M}.
$$

Squaring both sides, multiplying by $|R|$, adding up all the terms over $j, k \in \mathbb{N}$, $k \leq j + 2$ $R \in \mathcal{R}_{\mathcal{F}}^{j,k}$, $R \subset \Omega$, taking the supremum over all open sets Ω with finite measure in $\mathbb{R}^N$, and applying Hölder's inequality, we get

$$
\sup_{\Omega} \left\{ \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{\substack{j,k\in\mathbb{N}\\ k\leq j+2}} \sum_{\substack{R\in\mathcal{R}_{\mathcal{F}}^{j,k}\\ R\subset\Omega}} |R| S_R \right\}
$$

$$\leq C\sup_{\Omega}\Big\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{\substack{j,k\in\mathbb{N}\\k\leq j+2}}\sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{F}}\\R\subset\Omega}}|R|\big[\sum_{\substack{j',k'\in\mathbb{N}\\k'\leq j'+4}}\sum_{R'\in\mathcal{R}^{j',k'}}r(R,R')P(R,R')\big]$$

$$\times\big[\sum_{\substack{j',k'\in\mathbb{N}\\k'\leq j'+4}}\sum_{R'\in\mathcal{R}^{j',k'}}r(R,R')P(R,R')T_{R'}\big]\Big\}$$

$$\leq C\sup_{\Omega}\Big\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{\substack{j,k\in\mathbb{N}\\k\leq j+2}}\sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{F}}\\R\subset\Omega}}\sum_{\substack{j',k'\in\mathbb{N}\\k'\leq j'+4}}\sum_{R'\in\mathcal{R}^{j',k'}}|R|r(R,R')P(R,R')T_{R'}\Big\}$$

$$\leq C\sup_{\Omega}\Big\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{\substack{j,k\in\mathbb{N}\\k\leq j+2}}\sum_{\substack{R\in\mathcal{R}^{j,k}_{\mathcal{F}}\\R\subset\Omega}}\sum_{\substack{j',k'\in\mathbb{N}\\k'\leq j'+4}}\sum_{R'\in\mathcal{R}^{j',k'}}|R'|\tilde{r}(R,R')P(R,R')T_{R'}\Big\}$$

$$\leq C\sup_{\Omega}\Big\{\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{j,k\in\mathbb{N}}\sum_{\substack{R\in\mathcal{R}^{j,k}\\R\subset\Omega}}\sum_{j',k'\in\mathbb{N}}\sum_{R'\in\mathcal{R}^{j',k'}}|R'|\tilde{r}(R,R')P(R,R')T_{R'}\Big\},$$

where $\tilde{r}(R,R')=r_{L-1}(R,R')$ and the last inequality follows from the fact that, for any fixed $j\in\mathbb{N}$,

$$\mathcal{R}^{j,k}_{\mathcal{F}}=\begin{cases}\mathcal{R}^{j,k}, & \text{if } k\leq j;\\ \mathcal{R}^{j,j}, & \text{if } k=j+1 \text{ or } j+2.\end{cases}$$

To finish the proof, we only need to bound the last term by

$$C\sup_{\bar{\Omega}}\Big\{\frac{1}{|\bar{\Omega}|^{\frac{2}{p}-1}}\sum_{j',k'\in\mathbb{N}}\sum_{\substack{R\in\mathcal{R}^{j',k'}\\R\subset\bar{\Omega}}}|R'|T_{R'}\Big\}.$$

For $j,k\in\mathbb{N}$ and every dyadic rectangle $R'=I'\times J'$, denote $R'_{j,k}=2^jI'\times 2^kJ'$. Set $\Omega^{j,k}=\bigcup_{R'=I'\times J'\subset\Omega}3R'_{j,k}$. Given any $R\subset\Omega$, let $\mathcal{A}_{i,l}$ be the collection of dyadic rectangles R' defined as follows

$\mathcal{A}_{0,0}(R)=\{R' : 3R'_{0,0}\cap 3R\neq\varnothing\}$,

$\mathcal{A}_{i,0}(R)=\{R' : 3R'_{i,0}\cap 3R\neq\varnothing \text{ and } 3R'_{i-1,0}\cap 3R=\varnothing\}$,

$\mathcal{A}_{0,l}(R)=\{R' : 3R'_{0,l}\cap 3R\neq\varnothing \text{ and } 3R'_{0,l-1}\cap 3R=\varnothing\}$,

$\mathcal{A}_{i,l}(R)=\{R' : 3R'_{i,l}\cap 3R\neq\varnothing \text{ and } 3R'_{i-1,l}\cap 3R=\varnothing \text{ and } 3R'_{i,l-1}\cap 3R=\varnothing\}$.

Clearly, every dyadic rectangle R' belongs to exactly one $\mathcal{A}_{i,l}$. Consequently,

$$\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{j,k\in\mathbb{N}}\sum_{\substack{R\in\mathcal{R}^{j,k}\\R\subset\Omega}}\sum_{j',k'\in\mathbb{N}}\sum_{R'\in\mathcal{R}^{j',k'}}|R'|r(R,R')P(R,R')T_{R'}$$

$$\le \Big(\sum_{R'\in\mathscr{A}_{0,0}(R)} + \sum_{i\ge1}\sum_{R'\in\mathscr{A}_{i,0}(R)} + \sum_{l\ge1}\sum_{R'\in\mathscr{A}_{l,0}(R)} + \sum_{i,l\ge1}\sum_{R'\in\mathscr{A}_{j,k}(R)} \Big) \sum_{j,k\in\mathbb{N}} \sum_{\substack{R\in\mathscr{R}^{j,k}\\ R\subset\Omega}} |R'| r(R,R')P(R,R')T_{R'}$$
$$=: I_1 + I_2 + I_3 + I_4.$$

We only consider I_4 as the other three terms can be treated similarly; see [24] for more Details. For $i,l\ge1$, set

$$a_{i,l} = \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{R\subset\Omega}\sum_{R'\in\mathscr{A}_{i,l}(R)} |R'| r(R,R')P(R,R')T_{R'}$$

and $\mathscr{B}_{i,l} = \{R' : R'_{i,l}\cap\Omega^{0,0}\ne\emptyset\}$. Then $\cup_{R\subset\Omega}\mathscr{A}_{i,l}\subset\mathscr{B}_{i,l}$. Denote

$$\mathscr{F}_h^{i,l} = \{R'\in\mathscr{B}_{i,l} : |R'_{i,l}\cap\Omega^{0,0}| \ge \frac{1}{2^h}|R'_{i,l}|\}, \quad h\in\mathbb{N}, \quad h\ge0,$$

$$\mathscr{D}_h^{i,l} = \mathscr{F}_h^{i,l}\backslash\mathscr{F}_{h-1}^{i,l} \text{ and } \Omega_h^{i,l} = \cup_{R'\in\mathscr{D}_h^{i,l}} R', \quad h\ge1.$$

Thus,

$$a_{i,l} \le \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{h\ge1}\sum_{R'\in\mathscr{D}_h^{i,l}} \sum_{\substack{R:R\subset\Omega\\ R'\in\mathscr{A}_{i,l}(R)}} |R'| r(R,R')P(R,R')T_{R'}. \tag{19}$$

Note that $R'\in\mathscr{A}_{i,l}(R)$ implies $|x_I - x_{I'}| \ge \ell(I)\vee 2^i\ell(I')$ and $|x_J - x_{J'}| \ge \ell(J)\vee 2^l\ell(J')$. We consider the following four cases: Case 1. $2^iI'| \ge |I|, |2^lJ'| \ge |J|$; Case 2. $|2^iI'| \ge |I|, |2^lJ'| < |J|$; Case 3. $|2^iI'| < |I|$, $|2^lJ'| \ge |J|$; Case 4. $|2^iI'| < |I|, |2^lJ'| < |J|$. Hence,

$$a_{i,l} \le \frac{1}{|\Omega|^{\frac{2}{p}-1}} \sum_{h\ge1}\sum_{R'\in\mathscr{D}_h^{i,l}} |R'|T_{R'} \Big(\sum_{R\in Case\ 1} + \cdots + \sum_{R\in Case\ 4} \Big) r(R,R')P(R,R')$$
$$=: a_{i,l,1} + \cdots + a_{i,l,4}.$$

We first estimate $a_{i,l,2}$. Observe that

$$|I\times(2^iJ')| \le |3R'_{i,l}\cap 3R| \le |3R'_{i,l}\cap\Omega_{0,0}| \le \frac{1}{2^{h-1}}|3R'_{i,l}|,$$

which implies $2^{h-1}|I| \le 3^{n_1}2^{in_1}|I'| \le 2^{(i+2)n_1}|I'|$. We consider two subcases.

Category 1: $|I'|\ge|I|$. In this category, one has $|I'| = 2^{h-1-i(n_1+2)+\eta}$ for some $\eta>0$. For each fixed η, the number of such I's must be less than $(\eta+h)^{n_1+n_2}2^{\eta+h}$.

Category 2: $|I'|<|I|$. In this category, we have $|I'|<|I|\le|2^iI'|$. Then $2^\lambda\ell(I') = \ell(I)$ for some positive integer λ satisfying $1\le\lambda\le i$. For each λ, the number of I's must be finite. Moreover, $2^{h-1}2^{\lambda n_1}|I'| = 2^{h-1}|I| \le 2^{(i+2)n_1}|I'|$, which yields $h \le 3n_1 i$. Also, observe that

$$\frac{|x_I - x_{I'}|}{\ell(I)} = \frac{|x_I - x_{I'}|}{\ell(I')}\frac{\ell(I')}{\ell(I)} \ge C2^{i-\lambda}.$$

In Case 2, $|2^l J'| < |J|$ implies $2^{ln_2+\kappa}|J'| = |J|$ for some $\kappa \ge 0$. For each fixed κ, the number of such J's must be finite. Given a sufficiently large $M > n_1 L$, we have

$$\begin{aligned}
&\sum_{Category\ 1} r(R,R')P(R,R') \\
&= \sum_{Category\ 1}\left(\frac{|I|}{|I'|}\right)^L\left(\frac{|J|}{|J'|}\right)^L\left(1+\frac{|x_I-x_{I'}|}{\ell(I')}\right)^{-(n_1+M)}\left(1+\frac{|x_J-x_{J'}|}{\ell(J')}\right)^{-(n_2+M)} \\
&\le C\sum_{\eta,\kappa\ge 0}(h+\eta)^{n_1+n_2}2^{\eta+h}2^{-(h+\eta-in_1)L}2^{-(ln_2+\kappa)L}2^{-(n_1+M)i} \\
&\le C2^{-h(L-n_1-n_2-1)}2^{-i(M-n_1L)}2^{-ln_2L}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{Category\ 2} r(R,R')P(R,R') \\
&= \sum_{Category\ 2}\left(\frac{|I|}{|I'|}\right)^L\left(\frac{|J|}{|J'|}\right)^L\left(1+\frac{|x_I-x_{I'}|}{\ell(I')}\right)^{-(n_1+M)}\left(1+\frac{|x_J-x_{J'}|}{\ell(J')}\right)^{-(n_2+M)} \\
&\le C\sum_{\lambda=1}^{i}\sum_{\kappa\ge 0}2^{-n_1\lambda L}2^{-(ln_2+\kappa)L}2^{-M(i-\lambda)} \\
&\le C2^{-in_1L}2^{-ln_2L}.
\end{aligned}$$

We decompose $a_{i,l,2}$ as

$$a_{i,l,2} = \frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{h\ge 1}\sum_{R'\in\mathscr{D}_h^{i,l}}|R'|T_{R'}\Big(\sum_{Category\ 1}+\sum_{Category\ 2}\Big)r(R,R')P(R,R') =:\, b_1+b_2$$

For $x \in \Omega_h^{i,l}$, we have $x \in R$ for some dyadic rectangle $R \subset \Omega_h^{i,l}$, and therefore

$$\mathcal{M}_S(\chi_{\Omega^{0,0}})(x) \ge |R'_{i,l} \cap \Omega^{0,0}|/|R'_{i,l}| \ge 2^{-h},$$

which yields

$$|\Omega_h^{i,l}| \le |\{x : \mathcal{M}_S(\chi_{\Omega^{0,0}})(x) \ge 2^{-h}\}| \le C2^{2h}|\Omega^{0,0,0}| \le C2^{2h}|\Omega|.$$

Consequently,

$$b_1 \le C\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{h\ge 1}2^{-h(L-(n_1+n_2)-1)}2^{-i(M-n_1L)}2^{-ln_2L}|\Omega_h^{i,l}|^{\frac{2}{p}-1}\frac{1}{|\Omega_h^{i,l}|^{\frac{2}{p}-1}}\sum_{R'\in\Omega_h^{i,l}}|R'|T_{R'}$$

$$\begin{aligned}
&\le C\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{h\ge1}2^{-h(L-(n_1+n_2)-1)}2^{-i(M-n_1L)}2^{-ln_2L}(2^{2h}|\Omega|)^{\frac{2}{p}-1}\\
&\quad\times\sup_{\bar\Omega}\left\{\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R'|T_{R'}\right\}\\
&\le C2^{-i(M-n_1L)}2^{-ln_2L}\cdot\sup_{\bar\Omega}\left\{\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R'|T_{R'}\right\}
\end{aligned}$$

and

$$\begin{aligned}
b_2&\le C\frac{1}{|\Omega|^{\frac{2}{p}-1}}\sum_{1\le h\le 3n_1i}2^{-in_1L}2^{-ln_2L}|\Omega_h^{i,l}|^{\frac{2}{p}-1}\frac{1}{|\Omega_h^{i,l}|^{\frac{2}{p}-1}}\sum_{R'\in\Omega_h^{i,l}}|R'|T_{R'}\\
&\le C\frac{1}{|\Omega|^{\frac{2}{p}-1}}2^{-i(n_1L-\frac{6n_1}{p})}2^{-ln_2L}|\Omega|^{\frac{2}{p}-1}\sup_{\bar\Omega}\left\{\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R'|T_{R'}\right\}\\
&\le C2^{-i(n_1L-\frac{6n_1}{p})}2^{-ln_2L}\cdot\sup_{\bar\Omega}\left\{\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R'|T_{R'}\right\}.
\end{aligned}$$

Choosing $M > n_1L > \frac{6n_1}{p}$, we obtain

$$\sum_{i,l\ge1}a_{i,l,2}\le\sum_{i,l\ge1}b_1+\sum_{i,l\ge1}b_2\le C\sup_{\bar\Omega}\left\{\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R'|T_{R'}\right\}.$$

Similar arguments apply to other three terms, the details being omitted. Putting together these estimates, we derive that

$$IV=\sum_{i,l\ge1}\left(a_{i,l,1}+\cdots+a_{i,l,4}\right)\le C\sup_{\bar\Omega}\left\{\frac{1}{|\bar\Omega|^{\frac{2}{p}-1}}\sum_{R'\subset\bar\Omega}|R'|T_{R'}\right\}.$$

This establishes the $cmo^p\to cmo^p_{\mathcal F}$ boundedness of $\mathcal{T}_1$ and hence Theorem 1.10 follows.

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