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A note on weaving fusion frames

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ABSTRACT. Fusion frames are widely studied for their applications in recovering signals from large data. These are proved to be very useful in many areas, such as, distributed processing, wireless sensor networks, packet encoding. Inspired by the work of Bemrose et al.[12], this paper delves into the properties and characterizations of weaving fusion frames.

CONTENTS

1. Introduction

The concept of Hilbert space frames was first introduced by Duffin and Schaeffer [1] in 1952. After several decades, in 1986, the importance of frame theory was popularized by work as in the groundbreaking work by Daubechies, Grossman and Meyer [2]. Since then frame theory has been widely used by mathematicians and engineers in various fields of mathematics and engineering, namely, operator theory [3], harmonic analysis [4], wavelet analysis [5], signal processing [6], image processing [7], sensor network [8], data analysis [9], Retro Banach Frame [10], etc.

Frame theory literature became richer through several generalizations-fusion frame (frames of subspaces) [13, 15] , G-frame (generalized frames) [16], K-frame (atomic systems) [17], K-fusion frame (atomic subspaces) [18], etc. and these generalizations have been proved to be useful in various applications.

Classical frames have been instrumental in signal processing and functional analysis, providing a stable and redundant way to represent signals. However, they face significant limitations in distributed processing, particularly when it comes to projecting signals onto multidimensional subspaces. This limitation is crucial in applications like wireless sensor networks, where data is collected and processed across multiple sensors, and in packet encoding, where robustness and redundancy are essential. To address these challenges, fusion frames

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were introduced, extending the concept of frames to collections of subspaces with associated weights.

Fusion frames have proven to be highly effective in distributed processing, enabling more flexible and stable signal representations across multiple subspaces. This has made them particularly valuable in practical applications like wireless sensor networks, distributed signal processing, and error-resilient data transmission in packet encoding.

Beyond these practical uses, fusion frames have also emerged as a powerful tool in theoretical research. They play a significant role in the solution of the Kadison-Singer problem, a long-standing question in operator theory, and in optimal subspace packing, which is important for coding theory and communications. The rapid development of fusion frame theory over the years has led to a wide range of applications and a deeper understanding of their mathematical properties.

In this paper, we explore the various properties and characterizations of weaving fusion frames, a concept that extends traditional fusion frames by allowing the interweaving of subspaces. We delve into their structural aspects and provide theoretical insights into their stability and robustness.

Throughout this paper, $\mathcal{H}$ will be a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the space of all bounded linear operators from $\mathcal{H}_1$ into $\mathcal{H}_2$, and we use $\mathcal{L}(\mathcal{H})$ for $\mathcal{L}(\mathcal{H}, \mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, we denote $D(T), N(T)$ and $R(T)$ for domain, null space and range of T, respectively. For a collection of closed subspaces $\mathcal{W}_i$ of $\mathcal{H}$ and scalars w_i, $i \in I$, the weighted collection of closed subspaces $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ is denoted by $\mathcal{W}_w$. We consider I to be countable index set, $\mathcal{I}$ is the identity operator and $P_{\mathcal{V}}$ is the orthogonal projection onto $\mathcal{V}$.

2. Preliminaries

Before diving into the main sections, throughout this section we recall basic definitions and results needed in this paper. For detailed discussion regarding frames and its applications we refer [9, 19].

2.1. Frame. A collection $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2, \tag{1}$$

for all $f \in \mathcal{H}$. The numbers A, B are called *frame bounds*. The supremum over all A's and infimum over all B's satisfying above inequality are called the *optimal frame bounds*. If a collection satisfies only the right inequality in (1), it is called a *Bessel sequence*.

Given a frame $\{f_i\}_{i\in I}$ for $\mathcal{H}$, the *pre-frame operator* or *synthesis operator* is a bounded linear operator $T : l^2(I) \to \mathcal{H}$ and is defined by $T\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i$. The adjoint of T, $T^* : \mathcal{H} \to l^2(I)$, given by $T^* f = \{\langle f, f_i\rangle\}_{i\in I}$, is called the *analysis operator*. The *frame operator*, $S = TT^* : \mathcal{H} \to \mathcal{H}$, is defined by

$$Sf = TT^*f = \sum_{i\in I}\langle f, f_i\rangle f_i.$$

It is well-known that the frame operator is bounded, positive, self adjoint and invertible.

2.2. Fusion frame. Consider a weighted collection of closed subspaces, $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$, of $\mathcal{H}$. Then $\mathcal{W}_w$ is said to be a fusion frame for $\mathcal{H}$, if there exist constants $0 < A \le B < \infty$ satisfying

$$A\|f\|^2 \le \sum_{i\in I} w_i^2\|P_{\mathcal{W}_i}f\|^2 \le B\|f\|^2, \tag{2}$$

where $P_{\mathcal{W}_i}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{W}_i$. The constants A and B are called *fusion frame bounds*. A collection of closed subspaces, satisfying only the right inequality in (2), is called a fusion Bessel sequence.

For a family of closed subspaces, $\{\mathcal{W}_i\}_{i\in I}$, of $\mathcal{H}$, the associated l^2 space is defined by

$$\left(\sum_{i\in I}\bigoplus \mathcal{W}_i\right)_{l^2} = \left\{\{f_i\}_{i\in I} : f_i \in \mathcal{W}_i, \sum_{i\in I}\|f_i\|^2 < \infty\right\}$$

with the inner product

$$\langle\{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle_{\left(\sum_{i\in I}\bigoplus \mathcal{W}_i\right)_{l^2}} = \sum_{i\in I}\langle f_i, g_i\rangle_{\mathcal{H}}$$

and the norm is

$$\|\{f_i\}_{i\in I}\|^2_{\left(\sum_{i\in I}\bigoplus \mathcal{W}_i\right)_{l^2}} = \sum_{i\in I}\|f_i\|^2.$$

It is easy to see that $\left(\sum_{i\in I}\bigoplus \mathcal{W}_i\right)_{l^2}$ is a Hilbert space. In this context the corresponding dense inclusion is defined as

$$\begin{aligned}
\mathcal{L}^{00}_{\mathcal{W}} &= \left(\sum_{i\in I}\bigoplus \mathcal{W}_i\right)_{l^{00}} \\
&= \left\{\{f_i\}_{i\in I} \in \{\mathcal{W}_i\}_{i\in I} : f_i = 0 \text{ for all but finitely many } i\}\right\} \\
&\subseteq \left(\sum_{i\in I}\bigoplus \mathcal{W}_i\right)_{l^2} = \mathcal{L}^2_{\mathcal{W}}.
\end{aligned}$$

For the analogous dense inclusion we have

$$\mathcal{L}^{00}_{\mathcal{H}} = \left(\sum_{i\in I}\bigoplus \mathcal{H}_i\right)_{l^{00}} \subseteq \left(\sum_{i\in I}\bigoplus \mathcal{H}_i\right)_{l^2} = \mathcal{L}^2_{\mathcal{H}}.$$

Let $\mathcal{W}_w$ be a fusion frame. Then the associated synthesis operator, $T_{\mathcal{W}} : D(T_{\mathcal{W}}) \subseteq \mathcal{L}^2_{\mathcal{H}} \to \mathcal{H}$ is defined as $T_{\mathcal{W}}(\{f_i\}_{i\in I}) = \sum\limits_{i\in I} w_i P_{\mathcal{W}_i} f_i$, where

$$D(T_{\mathcal{W}}) = \left\{ \{f_i\}_{i\in I} \in \mathcal{L}^2_{\mathcal{H}} : \sum_{i\in I} w_i^2 P_{\mathcal{W}_i} f_i \text{ convergent} \right\}.$$

Since $\mathcal{L}^{00}_{\mathcal{H}} \subset D(T_{\mathcal{W}})$ and it is dense in $\mathcal{L}^2_{\mathcal{H}}$, hence the synthesis operator is densely defined.

On the other hand for every $f \in \mathcal{H}$ and $\{f_i\}_{i\in I} \in \mathcal{L}^2_{\mathcal{H}}$ we obtain,

$$\begin{aligned} \langle T^*_{\mathcal{W}} f, \{f_i\}_{i\in I}\rangle_{\mathcal{L}^2_{\mathcal{H}}} = \langle f, T_{\mathcal{W}}(\{f_i\}_{i\in I})\rangle_{\mathcal{H}} &= \left\langle f, \sum_{i\in I} w_i P_{\mathcal{W}_i} f_i \right\rangle_{\mathcal{H}} \\ &= \sum_{i\in I} \langle f, w_i P_{\mathcal{W}_i} f_i\rangle \\ &= \sum_{i\in I} \langle w_i P_{\mathcal{W}_i} f, f_i\rangle \\ &= \langle \{w_i P_{\mathcal{W}_i} f\}_{i\in I}, \{f_i\}_{i\in I}\rangle \end{aligned}$$

and hence the adjoint of synthesis operator, $T^*_{\mathcal{W}} : D(T^*_{\mathcal{W}}) \subseteq \mathcal{H} \to \mathcal{L}^2_{\mathcal{H}}$ is defined as $T^*_{\mathcal{W}}(f) = \{v_i P_{\mathcal{W}_i}(f)\}_{i\in I}$, which is known as analysis operator, where $D(T^*_{\mathcal{W}}) = \{f \in \mathcal{H} : \{w_i P_{\mathcal{W}_i} f\}_{i\in I} \in \mathcal{L}^2_{\mathcal{H}}\}$. It is well-known that (see [13]) the synthesis operator $T_{\mathcal{W}}$ of a fusion frame is bounded, linear and onto, whereas the corresponding analysis operator $T^*_{\mathcal{W}}$ is (possibly into) an isomorphism. If we consider the composition of synthesis and analysis operator we obtain the corresponding fusion frame operator which is defined as $S_{\mathcal{W}}(f) = T_{\mathcal{W}} T^*_{\mathcal{W}}(f)$ $= \sum\limits_{i\in I} w_i^2 P_{\mathcal{W}_i}(f)$. $S_{\mathcal{W}}$ is bounded, positive, self adjoint and invertible. Thus every $f \in \mathcal{H}$ can be expressed by its fusion frame measurements $\{w_i P_{\mathcal{W}_i} f\}_{i\in I}$ as

$$f = \sum_{i\in I} v_i S^{-1}_{\mathcal{W}}(v_i P_{\mathcal{W}_i} f). \tag{3}$$

2.3. Weaving fusion frames. Let $\mathcal{V}_v = \{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$ be two fusion frames for $\mathcal{H}$. Then they are called weaving fusion frames for $\mathcal{H}$ if for every $\sigma \subset I$ and every $f \in \mathcal{H}$ there exist finite positive constants $A \le B$ so that $\{(\mathcal{V}_i, v_i)\}_{i\in\sigma} \cup \{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$ is a fusion frame for $\mathcal{H}$ with the universal bounds $A \le B$, i.e. the following inequality is satisfied:

$$A\|f\|^2 \le \sum_{i\in\sigma} v_i^2 \|P_{\mathcal{V}_i} f\|^2 + \sum_{i\in\sigma^c} w_i^2 \|P_{\mathcal{W}_i} f\|^2 \le B\|f\|^2. \tag{4}$$

Example 2.1. *Let us consider an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$. Suppose for every n, $\mathcal{V}_n = span\{e_n\}$ and $\mathcal{W}_n = span\{e_n, e_{n+1}\}$. Since for every $f \in \mathcal{H}$ and $\sigma \subset \{1, 2, \cdots\}$ we have,*

$$\|f\|^2 \le \sum_{n\in\sigma} \|P_{\mathcal{V}_n} f\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{W}_n} f\|^2 \le 2\|f\|^2.$$

Therefore, $\{(\mathcal{V}_n, 1)\}_{n=1}^{\infty}$ and $\{(\mathcal{W}_n, 1)\}_{n=1}^{\infty}$ are weaving fusion frames for $\mathcal{H}$.

In the following example we discuss a non-example of weaving fusion frames.

Example 2.2. *Let us consider an orthonormal basis* $\{e_n\}_{n=1}^{\infty}$ *for* $\mathcal{H}$. *Suppose for every n,* $\mathcal{V}_n = span\{e_n\}$ *and* $\mathcal{W}_1 = span\{e_2\}$, $\mathcal{W}_2 = span\{e_1\}$ *and* $\mathcal{W}_n = span\{e_n\}$ *for* $n \geq 3$. *Since for* $\sigma = \{2\} \subset \{1, 2, \cdots\}$ *we have,*

$$\sum_{n\in\sigma} \|P_{\mathcal{W}_n} e_2\|^2 + \sum_{n\in\sigma^c} \|P_{\mathcal{V}_n} e_2\|^2 = 0$$

Thus, $\{(\mathcal{V}_n, 1)\}_{n=1}^{\infty}$ *and* $\{(\mathcal{W}_n, 1)\}_{n=1}^{\infty}$ *are not weaving fusion frames for* $\mathcal{H}$.

Remark 2.3. *For weaving fusion frames we can define the associated weaving synthesis, analysis and weaving fusion frame operators analogous to the section 2.2.*

For detailed discussion regarding weaving fusion frames we refer [11].

3. Main results

In this section we discuss various characterizations of weaving fusion frames.

Theorem 3.1. *Let* $\mathcal{V}_v = \{(\mathcal{V}_i, v_i)\}_{i\in I}$ *and* $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$ *be two families of weighted closed subspaces of* $\mathcal{H}$. *Suppose* $\{f_{ij}\}_{j\in J_i}$ *and* $\{g_{ij}\}_{j\in J_i}$ *are frames for* $\mathcal{V}_i$ *and* $\mathcal{W}_i$ *with bounds* A_i, B_i *and* C_i, D_i *respectively for every* $i \in I$. *If* $0 < A = \inf_{i\in I} A_i \leq \sup_{i\in I} B_i = B < \infty$ *and* $0 < C = \inf_{i\in I} C_i \leq \sup_{i\in I} D_i = D < \infty$. *Then the following are equivalent:*

1. *(1)* $\{(\mathcal{V}_i, v_i)\}_{i\in I}$ *and* $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ *are weaving fusion frames for* $\mathcal{H}$.
2. *(2)* $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ *and* $\{w_i g_{ij}\}_{i\in I, j\in J_i}$ *are weaving frames for* $\mathcal{H}$.

Proof. Let us suppose $\alpha = \min(A, C)$ and $\beta = \max(B, D)$.
($1 \implies 2$) Since $\{f_{ij}\}_{j\in J_i}$ and $\{g_{ij}\}_{j\in J_i}$ are frames for $\mathcal{V}_i$ and $\mathcal{W}_i$ respectively with the respective bounds, then for every $f \in \mathcal{H}$ and every $\sigma \subset I$ we have,

$$\begin{aligned}
\alpha(\sum_{i\in\sigma} v_i^2 \|P_{\mathcal{V}_i} f\|^2 &+ \sum_{i\in\sigma^c} w_i^2 \|P_{\mathcal{W}_i} f\|^2) \\
&\leq A \sum_{i\in\sigma} v_i^2 \|P_{\mathcal{V}_i} f\|^2 + C \sum_{i\in\sigma^c} w_i^2 \|P_{\mathcal{W}_i} f\|^2 \\
&\leq \sum_{i\in\sigma} v_i^2 A_i \|P_{\mathcal{V}_i} f\|^2 + \sum_{i\in\sigma^c} w_i^2 C_i \|P_{\mathcal{W}_i} f\|^2 \\
&\leq \sum_{i\in\sigma} v_i^2 \sum_{j\in J_i} |\langle P_{\mathcal{V}_i} f, f_{ij}\rangle|^2 + \sum_{i\in\sigma} w_i^2 \sum_{j\in J_i} |\langle P_{\mathcal{W}_i} f, g_{ij}\rangle|^2 \\
&= \sum_{i\in\sigma} \sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 + \sum_{i\in\sigma} \sum_{j\in J_i} |\langle f, w_i g_{ij}\rangle|^2 \\
&\leq \sum_{i\in\sigma} v_i^2 B_i \|P_{\mathcal{V}_i} f\|^2 + \sum_{i\in\sigma^c} w_i^2 D_i \|P_{\mathcal{W}_i} f\|^2 \\
&\leq B \sum_{i\in\sigma} v_i^2 \|P_{\mathcal{V}_i} f\|^2 + D \sum_{i\in\sigma^c} w_i^2 \|P_{\mathcal{W}_i} f\|^2
\end{aligned}$$

$$\leq \quad \beta\left(\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}f\|^2\right).$$

Thus if $\{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ are weaving fusion frames for $\mathcal{H}$ with bounds $A_{\mathcal{V}\mathcal{W}} \leq B_{\mathcal{V}\mathcal{W}}$, then for every $f \in \mathcal{H}$ and every $\sigma \subset I$ we have,

$$\alpha A_{\mathcal{V}\mathcal{W}}\|f\|^2 \leq \sum_{i\in\sigma}\sum_{j\in J_i} |\langle f, v_i f_{ij}\rangle|^2 + \sum_{i\in\sigma^c}\sum_{j\in J_i} |\langle f, w_i f_{ij}\rangle|^2 \leq \beta B_{\mathcal{V}\mathcal{W}}.$$

Therefore, $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ and $\{w_i g_{ij}\}_{i\in I, j\in J_i}$ are weaving frames for $\mathcal{H}$.

($2 \implies 1$) Conversely, if $\{v_i f_{ij}\}_{i\in I, j\in J_i}$ and $\{w_i g_{ij}\}_{i\in I, j\in J_i}$ are weaving frames for $\mathcal{H}$ with bounds $A_{vw} \leq B_{vw}$, then for every $f \in \mathcal{H}$ and every $\sigma \subset I$ we have,

$$\frac{A_{vw}}{\beta} \leq \sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}f\|^2 \leq \frac{B_{vw}}{\alpha}.$$

Consequently, $\{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ are weaving fusion frames for $\mathcal{H}$. □

We characterize weaving fusion frames by means of the associated weaving fusion frame operator.

Lemma 3.2. *Let $\{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ be two fusion frames for $\mathcal{H}$ with bounds $A \leq B$ and $C \leq D$ respectively. Then the following are equivalent:*

1. *$\{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ are weaving fusion frames for $\mathcal{H}$.*
2. *For every $\sigma \subset I$, suppose $S_{\mathcal{V}\mathcal{W}_\sigma}$ is the corresponding weaving fusion frame operator, then for every $f \in \mathcal{H}$ there exist $\alpha > 0$ independent of σ, we have $\|S_{\mathcal{V}\mathcal{W}_\sigma}f\| \geq \alpha\|f\|$.*

Proof. ($1 \implies 2$) If $\{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\{(\mathcal{W}_i, w_i)\}_{i\in I}$ are weaving fusion frames for $\mathcal{H}$ with the universal bounds $\alpha \leq \beta$, then for every $\sigma \subset I$ and every $f \in \mathcal{H}$ we have,

$$\alpha\|f\|^2 \leq \sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}f\|^2 \leq \beta\|f\|^2. \tag{5}$$

Suppose $S_{\mathcal{V}\mathcal{W}_\sigma}$ is the fusion frame operator for the associated weaving, then using inequality (5) we have, $\alpha\|f\|^2 \leq \langle S_{\mathcal{V}\mathcal{W}_\sigma}f, f\rangle \leq \beta\|f\|^2$.

Thus, $\|S_{\mathcal{V}\mathcal{W}_\sigma}f\| = \sup_{\|g\|=1} |\langle S_{\mathcal{V}\mathcal{W}_\sigma}f, g\rangle| \geq \left\langle S_{\mathcal{V}\mathcal{W}_\sigma}f, \frac{f}{\|f\|}\right\rangle \geq \alpha\|f\|$.

($2 \implies 1$) Applying Remark (2.3), let us suppose for every $\sigma \subset I$, $T_{\mathcal{V}\mathcal{W}_\sigma}$ and $T^*_{\mathcal{V}\mathcal{W}_\sigma}$ be the associated weaving synthesis and analysis operators respectively. Then for every $f \in \mathcal{H}$ we have,

$\alpha^2\|f\|^2 \leq \|S_{\mathcal{V}\mathcal{W}_\sigma}f\|^2 = \|T_{\mathcal{V}\mathcal{W}_\sigma}T^*_{\mathcal{V}\mathcal{W}_\sigma}f\|^2 \leq \|T_{\mathcal{V}\mathcal{W}_\sigma}\|^2\|T^*_{\mathcal{V}\mathcal{W}_\sigma}f\|^2$ and hence we obtain,

$\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}f\|^2 = \|T^*_{\mathcal{V}\mathcal{W}_\sigma}f\|^2 \geq \frac{\alpha^2}{B^2+D^2}\|f\|^2$.

On the other hand the universal upper bound for the corresponding weaving will be obtained from the [Proposition 3.1, [12]]. □

Proposition 3.3. *Let $\mathcal{V}_v = \{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$ be two weighted sequences of closed subspaces of $\mathcal{H}$. Then the following are equivalent:*

(1) *$\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion Bessel sequences for $\mathcal{H}$.*
(2) *For every $\sigma \subset I$, the corresponding weaving synthesis operator $T_{\mathcal{V}\mathcal{W}_\sigma}$: $\mathcal{L}^2_{\mathcal{V}\mathcal{W}} = \left(\sum_{i\in\sigma} \bigoplus \mathcal{V}_i + \sum_{i\in\sigma^c} \bigoplus \mathcal{W}_i\right)_{l^2} \to \mathcal{H}$ is bounded.*
(3) *For every $\sigma \subset I$, the associated weaving analysis operator $T^*_{\mathcal{V}\mathcal{W}_\sigma} : \mathcal{H} \to \mathcal{L}^2_{\mathcal{V}\mathcal{W}}$ is bounded.*
(4) *For every $\sigma \subset I$, the corresponding weaving fusion frame operator $S_{\mathcal{V}\mathcal{W}_\sigma}$: $\mathcal{H} \to \mathcal{H}$ is bounded.*

Proof. ($1 \implies 2$) Let us suppose $\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion Bessel sequences for $\mathcal{H}$ with bound $B_{\mathcal{V}\mathcal{W}}$. Then for every $\sigma \subset I$ and every $f \in \mathcal{H}$ we have, $\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i} f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i} f\|^2 \leq B_{\mathcal{V}\mathcal{W}}\|f\|^2$. Thus for any $\{f_i\}_{i\in\sigma} \cup \{g_i\}_{i\in\sigma^c} \in \mathcal{L}^2_{\mathcal{V}\mathcal{W}}$ we obtain,

$$\begin{aligned}
\left\|\sum_{i\in\sigma} v_i P_{\mathcal{V}_i} f_i + \sum_{i\in\sigma^c} w_i P_{\mathcal{W}_i} g_i\right\| &= \sup_{\|g\|=1}\left|\left\langle \sum_{i\in\sigma} v_i P_{\mathcal{V}_i} f_i + \sum_{i\in\sigma^c} w_i P_{\mathcal{W}_i} g_i, g\right\rangle\right| \\
&= \sup_{\|g\|=1}\left|\left\langle \sum_{i\in\sigma} v_i P_{\mathcal{V}_i} f_i, g\right\rangle + \left\langle \sum_{i\in\sigma^c} w_i P_{\mathcal{W}_i} g_i, g\right\rangle\right| \\
&= \sup_{\|g\|=1}\left|\sum_{i\in\sigma} v_i \langle f_i, P_{\mathcal{V}_i} g\rangle + \sum_{i\in\sigma^c} w_i \langle g_i, P_{\mathcal{W}_i} g\rangle\right| \\
&\leq \sup_{\|g\|=1}\left(\sum_{i\in\sigma} v_i^2 \|P_{\mathcal{V}_i} g\|^2 \|f_i\|^2\right)^{\frac{1}{2}} \\
&+ \sup_{\|g\|=1}\left(\sum_{i\in\sigma^c} w_i^2 \|P_{\mathcal{W}_i} g\|^2 \|g_i\|^2\right)^{\frac{1}{2}} \\
&\leq \sup_{\|g\|=1}\left(\sum_{i\in\sigma} v_i^2 \|P_{\mathcal{V}_i} g\|^2\right)^{\frac{1}{2}} \left(\sum_{i\in\sigma} \|f_i\|^2\right)^{\frac{1}{2}} \\
&+ \sup_{\|g\|=1}\left(\sum_{i\in\sigma^c} w_i^2 \|P_{\mathcal{W}_i} g\|^2\right)^{\frac{1}{2}} \left(\sum_{i\in\sigma^c} \|g_i\|^2\right)^{\frac{1}{2}} \\
&\leq 2\sqrt{B_{\mathcal{V}\mathcal{W}}}\|f\|.
\end{aligned}$$

Therefore, $T_{\mathcal{V}\mathcal{W}_\sigma}$ is bounded.

Analogously ($2 \implies 3$) and ($3 \implies 4$) will be satisfied.

($4 \implies 1$) Suppose $S_{\mathcal{V}\mathcal{W}_\sigma}$ is bounded by the bound $B_{\mathcal{V}\mathcal{W}}$, then for $\sigma \subset I$ and every $f \in \mathcal{H}$ we have,

$$\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}f\|^2 = \langle S_{\mathcal{V}\mathcal{W}_\sigma}f, f\rangle \leq \|S_{\mathcal{V}\mathcal{W}_\sigma}f\|\|f\| \leq B_{\mathcal{V}\mathcal{W}}\|f\|^2.$$

Hence $\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion Bessel sequences. □

Theorem 3.4. *Let $\mathcal{V}_v = \{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$ be two weighted sequences of closed subspaces of $\mathcal{H}$. Then the following are equivalent:*

(1) $\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion frames for $\mathcal{H}$.

(2) For every $\sigma \subset I$, the corresponding weaving synthesis operator $T_{\mathcal{V}\mathcal{W}_\sigma}$: $\mathcal{L}^2_{\mathcal{V}\mathcal{W}} = \left(\sum_{i\in\sigma}\bigoplus \mathcal{V}_i + \sum_{i\in\sigma^c}\bigoplus \mathcal{W}_i\right)_{l^2} \to \mathcal{H}$ is bounded, surjective operator.

(3) For every $\sigma \subset I$, the associated weaving analysis operator $T^*_{\mathcal{V}\mathcal{W}_\sigma} : \mathcal{H} \to \mathcal{L}^2_{\mathcal{V}\mathcal{W}}$ is bounded, injective operator and has closed range.

Proof. The proof follows easily from [15]. □

Lemma 3.5. *For every $\sigma \subset I$, let us consider the associated Hilbert direct sum $\left(\sum_{i\in\sigma}\bigoplus \mathcal{V}_i + \sum_{i\in\sigma^c}\bigoplus \mathcal{W}_i\right)_{l^2}$ of the closed subspaces $\{\mathcal{V}_i\}_{i\in I}$ and $\{\mathcal{W}_i\}_{i\in I}$. Suppose $\mathcal{U}_i \subset \mathcal{V}_i$ and $\mathcal{X}_i \subset \mathcal{W}_i$ for every $i \in I$. Then*

$$\left(\sum_{i\in\sigma}\bigoplus \mathcal{U}_i + \sum_{i\in\sigma^c}\bigoplus \mathcal{X}_i\right)^{\perp}_{l^2} = \left(\sum_{i\in\sigma}\bigoplus \mathcal{U}_i^{\perp} + \sum_{i\in\sigma^c}\bigoplus \mathcal{X}_i^{\perp}\right)_{l^2}.$$

Proof. The proof follows from [20]. □

Theorem 3.6. *Let $\mathcal{V}_v = \{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$ be weaving fusion frames for $\mathcal{H}$ with the universal bounds $A \leq B$. Suppose R is a bounded, invertible operator on $\mathcal{H}$. Then $R\mathcal{V}_v = \{(R\mathcal{V}_i, v_i)\}_{i\in I}$ and $R\mathcal{W}_w = \{(R\mathcal{W}_i, w_i)\}_{i\in I}$ are weaving fusion frames for $\mathcal{H}$ with the bounds $\frac{A}{\|R^{-1}\|^2\|R\|^2}$ and $B\|R^{-1}\|^2\|R\|^2$. Furthermore, the associated weaving fusion frame operators satisfies the following inequality:*

$$\frac{RS_{\mathcal{V}\mathcal{W}}R^*}{\|R\|^2} \leq S_{R\mathcal{V}R\mathcal{W}} \leq \|R^{-1}\|^2 RS_{\mathcal{V}\mathcal{W}}R^*.$$

Proof. Since image of a closed set under an invertible operator is closed, then for every $i \in I$, $R\mathcal{V}_i$ and $R\mathcal{W}_i$ are closed.

Therefore, for every $\sigma \subset I$ and every $f \in \mathcal{H}$ we have,

$$\begin{aligned}\langle S_{R\mathcal{V}R\mathcal{W}}f, f\rangle &= \sum_{i\in\sigma} v_i^2\|P_{R\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{R\mathcal{W}_i}f\|^2 \\ &= \sum_{i\in\sigma} v_i^2\|P_{R\mathcal{V}_i}(R^{-1})^*P_{\mathcal{V}_i}R^*f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{R\mathcal{W}_i}(R^{-1})^*P_{\mathcal{W}_i}R^*f\|^2\end{aligned}$$

$$\begin{aligned} &\leq \|R^{-1}\|^2\left(\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}R^*f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}R^*f\|^2\right)\\ &= \|R^{-1}\|^2\langle S_{\mathcal{V}\mathcal{W}}R^*f, R^*f\rangle\\ &= \|R^{-1}\|^2\langle RS_{\mathcal{V}\mathcal{W}}R^*f, f\rangle. \end{aligned}$$

Again for a bounded linear operator Q we have, $f \in (Q\mathcal{V})^\perp = (\overline{Q\mathcal{V}})^\perp$ if and only if $Q^*f \in \mathcal{V}^\perp$. Thus for every $f \in \mathcal{H}$ we obtain,

$$P_{\mathcal{V}}Q^*f = P_{\mathcal{V}}Q^*P_{\overline{Q\mathcal{V}}}f + P_{\mathcal{V}}Q^*P_{(Q\mathcal{V})^\perp}f = P_{\mathcal{V}}Q^*P_{\overline{Q\mathcal{V}}}f. \tag{6}$$

Therefore, for every $\sigma \subset I$, every $f \in \mathcal{H}$ and applying equation 6 we have,

$$\begin{aligned} \left\langle \frac{RS_{\mathcal{V}\mathcal{W}}R^*}{\|R\|^2}f, f\right\rangle &= \frac{1}{\|R\|^2}\left(\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}R^*f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}R^*f\|^2\right)\\ &= \frac{1}{\|R\|^2}\left(\sum_{i\in\sigma} v_i^2\|P_{\mathcal{V}_i}R^*P_{R\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{\mathcal{W}_i}R^*P_{R\mathcal{W}_i}f\|^2\right)\\ &\leq \sum_{i\in\sigma} v_i^2\|P_{R\mathcal{V}_i}f\|^2 + \sum_{i\in\sigma^c} w_i^2\|P_{R\mathcal{W}_i}f\|^2\\ &= \langle S_{R\mathcal{V}R\mathcal{W}}f, f\rangle. \end{aligned}$$

Thus we obtain,

$$\frac{RS_{\mathcal{V}\mathcal{W}}R^*}{\|R\|^2} \leq S_{R\mathcal{V}R\mathcal{W}} \leq \|R^{-1}\|^2 RS_{\mathcal{V}\mathcal{W}}R^*.$$

Furthermore, for every $f \in \mathcal{H}$ we have, $\|f\| \leq \|R^{-1}\|\|R^*f\|$.
Consequently, for every $\sigma \subset I$ and every $f \in \mathcal{H}$ we obtain,

$$\begin{aligned} \frac{A}{\|R^{-1}\|^2\|R\|^2}\|f\|^2 \leq \frac{A}{\|R\|^2}\|R^*f\|^2 &\leq \frac{1}{\|R\|^2}\langle S_{\mathcal{V}\mathcal{W}}R^*f, R^*f\rangle\\ &\leq \langle S_{\mathcal{V}\mathcal{W}}f, f\rangle\\ &\leq \|R^{-1}\|^2\langle RS_{\mathcal{V}\mathcal{W}}R^*f, f\rangle\\ &\leq B\|R^{-1}\|^2\|R\|^2\|f\|^2. \end{aligned}$$

This completes the proof. □

Let us define weaving fusion Riesz bases analogous to fusion Riesz basis [13, 14].

Suppose $\mathcal{V}_v = \{(\mathcal{V}_i, v_i)\}_{i\in I}$ and $\mathcal{W}_w = \{(\mathcal{W}_i, w_i)\}_{i\in I}$ be two weighted sequences of closed subspaces of $\mathcal{H}$. Then they are said to be weaving fusion Riesz bases if for every $\sigma \subset I$ and every $\{f_i\}_{i\in\sigma} \cup \{g_i\}_{i\in\sigma^c} \in \mathcal{L}^{00}_{\mathcal{V}\mathcal{W}}$ there are finite positive constants $A_{\mathcal{V}\mathcal{W}} \leq B_{\mathcal{V}\mathcal{W}}$ such that

$$A_{\mathcal{VW}}\left(\sum_{i\in\sigma}\|f_i\|^2+\sum_{i\in\sigma^c}\|g_i\|^2\right)\le\left\|\sum_{i\in\sigma}v_if_i+\sum_{i\in\sigma^c}w_ig_i\right\|^2$$
$$\le B_{\mathcal{VW}}\left(\sum_{i\in\sigma}\|f_i\|^2+\sum_{i\in\sigma^c}\|g_i\|^2\right). \tag{7}$$

Furthermore, $\{\mathcal{V}_i\}_{i\in\sigma}\cup\{\mathcal{W}_i\}_{i\in\sigma^c}$ is called an orthonormal weaving fusion basis in $\mathcal{H}$ if $\mathcal{I}_{\mathcal{H}}=\sum_{i\in\sigma}P_{\mathcal{V}_i}+\sum_{i\in\sigma^c}P_{\mathcal{W}_i}$ and $\mathcal{V}_i\perp\mathcal{V}_j$, $\mathcal{W}_i\perp\mathcal{W}_j$ for every $i\neq j$.

We characterize weaving fusion Riesz bases using weaving fusion frame synthesis and analysis operators, establishing their structural properties. This characterization leads to the fact that every weaving fusion Riesz basis is also a weaving fusion frame, underscoring their dual role in the frame theory.

Theorem 3.7. *$\mathcal{V}_v=\{(\mathcal{V}_i,v_i)\}_{i\in I}$ and $\mathcal{W}_w=\{(\mathcal{W}_i,w_i)\}_{i\in I}$ be two weighted sequences of closed subspaces of $\mathcal{H}$. Then the following are equivalent:*

1. *$\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion Riesz bases in $\mathcal{H}$.*
2. *For every $\sigma\subset I$, the corresponding weaving synthesis operator $T_{\mathcal{VW}}$: $\mathcal{L}^2_{\mathcal{VW}}=\left(\sum_{i\in\sigma}\bigoplus\mathcal{V}_i+\sum_{i\in\sigma^c}\bigoplus\mathcal{W}_i\right)_{l^2}\to\mathcal{H}$ is bounded with $R(T_{\mathcal{VW}})=\mathcal{H}$ and $N(T_{\mathcal{VW}})=(\mathcal{L}^2_{\mathcal{VW}})^{\perp}$.*
3. *For every $\sigma\subset I$, the associated weaving analysis operator $T^*_{\mathcal{VW}}:\mathcal{H}\to\mathcal{L}^2_{\mathcal{VW}}$ is bounded with $R(T^*_{\mathcal{VW}})=\mathcal{L}^2_{\mathcal{VW}}$ and $N(T^*_{\mathcal{VW}})=\{0\}$.*

Proof. Applying Lemma 3.5, for every $\sigma\subset I$ the orthogonal complement $(\mathcal{L}^2_{\mathcal{VW}})^{\perp}$ in $\mathcal{L}^2_{\mathcal{H}}$ is given by $\left(\sum_{i\in\sigma}\bigoplus\mathcal{V}_i^{\perp}+\sum_{i\in\sigma^c}\bigoplus\mathcal{W}_i^{\perp}\right)_{l^2}$. Moreover, we have $T_{\mathcal{VW}}=T_{\mathcal{VW}}P_{\mathcal{L}^2_{\mathcal{VW}}}$. Thus the condition in (2) is equivalent to the fact that $T_{\mathcal{VW}|\mathcal{L}^2_{\mathcal{VW}}}$ is bounded, bijective operator and the condition in (3) is equivalent to the fact that $T^*_{\mathcal{VW}}$ is also bounded, bijective operator.

Furthermore, since $R(T^*_{\mathcal{VW}})\subset\mathcal{L}^2_{\mathcal{VW}}$, then we have $(T_{\mathcal{VW}|\mathcal{L}^2_{\mathcal{VW}}})^*=T^*_{\mathcal{VW}}$.

$(1\implies 2)$ Let $\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion Riesz bases in $\mathcal{H}$. Then from the right inequality of the inequality 7 we have, $T_{\mathcal{VW}}$ and $T^*_{\mathcal{VW}}$ are bounded.

If possible $T_{\mathcal{VW}|\mathcal{L}^2_{\mathcal{VW}}}$ is not surjective, then there exists $0\neq f\in R(T_{\mathcal{VW}|\mathcal{L}^2_{\mathcal{VW}}})^{\perp}$ with $f\perp(\mathcal{V}_{i\in\sigma}\cup\mathcal{W}_{i\in\sigma^c})$ for every $\sigma\subset I$. Again since $\{\mathcal{V}_i\}_{i\in\sigma}\cup\{\mathcal{W}_i\}_{i\in\sigma^c}$ is complete, then there exists a sequence $\{h_n\}_{n=1}^{\infty}\in\operatorname{span}(\{\mathcal{V}_i\}_{i\in\sigma}\cup\{\mathcal{W}_i\}_{i\in\sigma^c})$ so that we have,

$$\begin{aligned}0&=\lim_{n\to\infty}\|f-h_n\|=\lim_{n\to\infty}(\|f\|^2-\langle f,h_n\rangle-\langle h_n,f\rangle+\|h_n\|^2)\\&=2\|f\|^2>0,\end{aligned}$$

which is a contradiction. Thus $T_{\mathcal{VW}|\mathcal{L}^2_{\mathcal{VW}}}$ is surjective.

Applying Theorem 3.4, for every $\sigma \subset I$, $\{(\mathcal{V}_i, v_i)\}_{i\in\sigma} \cup \{(\mathcal{W}_i, w_i)\}_{i\in\sigma^c}$ is a fusion frame and hence $\left(\sum_{i\in\sigma} v_i f_i + \sum_{i\in\sigma^c} w_i g_i\right)$ converges unconditionally, where $\{f_i\}_{i\in\sigma} \cup \{g_i\}_{i\in\sigma^c} \in \mathcal{L}^2_{\mathcal{V}\mathcal{W}}$.

Furthermore, from the left inequality of the inequality 7 it is easy to see that $T_{\mathcal{V}\mathcal{W}|\mathcal{L}^{00}_{\mathcal{V}\mathcal{W}}}$ is injective. Since $T_{\mathcal{V}\mathcal{W}}$ is bounded, $\mathcal{L}^{00}_{\mathcal{V}\mathcal{W}}$ is dense in $\mathcal{L}^2_{\mathcal{V}\mathcal{W}}$ and $\left(\sum_{i\in\sigma} v_i f_i + \sum_{i\in\sigma^c} w_i g_i\right)$ is unconditionally convergent, then the left inequality of the inequality 7 will hold for every $\{f_i\}_{i\in\sigma} \cup \{g_i\}_{i\in\sigma^c} \in \mathcal{L}^2_{\mathcal{V}\mathcal{W}}$, for every $\sigma \subset I$. Therefore, $T_{\mathcal{V}\mathcal{W}|\mathcal{L}^2_{\mathcal{V}\mathcal{W}}}$ is injective and hence $T_{\mathcal{V}\mathcal{W}|\mathcal{L}^2_{\mathcal{V}\mathcal{W}}}$ is bijective.

($2 \implies 1$) Let $T_{\mathcal{V}\mathcal{W}|\mathcal{L}^2_{\mathcal{V}\mathcal{W}}}$ is bijective operator. Then the inequality 7 will hold for every $\{f_i\}_{i\in\sigma} \cup \{g_i\}_{i\in\sigma^c} \in \mathcal{L}^2_{\mathcal{V}\mathcal{W}}$, for every $\sigma \subset I$.

If possible for every $\sigma \subset I$, $\{\mathcal{V}_i\}_{i\in\sigma} \cup \{\mathcal{W}_i\}_{i\in\sigma^c}$ is not complete, then there exists $0 \neq f \in \mathcal{H}$ so that $f \perp \overline{\text{span}}(\{\mathcal{V}_i\}_{i\in\sigma} \cup \{\mathcal{W}_i\}_{i\in\sigma^c})$. Therefore, $T^*_{\mathcal{V}\mathcal{W}} f = 0$, which is a contradiction. Hence $\mathcal{V}_v$ and $\mathcal{W}_w$ are weaving fusion Riesz bases.

Finally, applying Theorem 3.4, we have every weaving fusion Riesz basis is also weaving fusion frame. Therefore, $T_{\mathcal{V}\mathcal{W}} T^*_{\mathcal{V}\mathcal{W}}$ and $T^*_{\mathcal{V}\mathcal{W}} T_{\mathcal{V}\mathcal{W}}$ have the same non-zero spectrum. Thus weaving fusion frames and weaving fusion Riesz bases have same bounds.

($2 \iff 3$) This will hold for a bounded bijective operator and its Hilbert adjoint (see [21]). □

References

[1] DUFFIN, R.; SCHAEFFER, A. C. A class of non harmonic Fourier series. *Trans. Amer. Math. Soc.* **72** (1952), 341–366. MR0047179, Zbl 0049.32401, doi: 10.2307/1990760. 54

[2] DAUBECHIES, I.; GROSSMANN, A.; MEYER, Y. Painless non orthogonal expansions. *J. Math. Phys.* **27** (1986), 1271–1283. MR0836025, Zbl 0608.46014, doi: 10.1063/1.527388. 54

[3] HAN, D.; LARSON, D. R. Frames, Bases and Group Representations. *Mem. Amer. Math. Soc.*, **147** (2000). x+94 pp. MR1686653, Zbl 0971.42023, doi: 10.1090/memo/0697. 54

[4] GRÖCHENIG, K. Foundations of Time-frequency Analysis. *Appl. Numer. Harmon. Anal., Boston, Birkhäuser*, 2001. xvi+359 pp. MR1843717, Zbl 0966.42020. 54

[5] GUMBER, A.; SHUKLA, N. K. Uncertainty principle corresponding to an orthonormal wavelet system. *Applicable Analysis.* **97** (2018), 486–498. MR3757680, Zbl 1471.42074, doi: 10.1080/00036811.2016.1274025. 54

[6] FERREIRA, P. Mathematics for multimedia signal processing II: discrete finite frames and signal reconstruction. *Signal Processing for Multimedia. J. S. Byrnes (Ed.)*, (1999), 35–54. Zbl 1007.94523. 54

[7] AREFIJAMAAL, A.; ZEKAEE, E. Image processing by alternate dual Gabor frames. *Bull. Iran. Math. Soc.* **42** (2016), 1305–1314. MR3591044, Zbl 1373.42032, 54

[8] CASAZZA, P. G.; KUTYNIOK, G.; LI, S.; ROZELL, C. J. Modeling sensor networks with fusion frames. *Proc. SPIE.* 2007. 6701:67011M-1–67011M-11. doi: 10.1117/12.730719. 54

[9] CASAZZA, P. G.; KUTYNIOK, G. Finite Frames: Theory and Applications. *Applied and Numerical Harmonic Analysis, Boston: Birkhäuser*, 2012. MR2964006, Zbl 1257.42001. 54, 55

[10] VERMA, G.; VASHISHT, L. K.; SINGH, M. On Excess of Retro Banach Frames. *Journal of Contemporary Mathematical Analysis.* **54** (2019), 143–146. MR3979969, Zbl 1437.42051, doi: 10.3103/S1068362319030026. 54

[11] DEEPSHIKHA; GARG, S.; VASHISHT, L. K.; VERMA, G. On Weaving Fusion Frames for Hilbert Spaces. *International Conference on Sampling Theory and Applications (SampTA)*, 2017. doi: 10.1109/SAMPTA.2017.8024363. 58

[12] BEMROSE, T.; CASAZZA, P. G.; GRÖCHENIG, K.; LAMMERS, M. C.; LYNCH, R. G. Weaving frames. *Operators and Matrices.* **10** (2016), 1093–1116. MR3584693, Zbl 1358.42025, doi: 10.7153/oam-10-61. 54, 59

[13] CASAZZA P. G.; KUTYNIOK, G. Frames of subspaces. *Contemporary Math.* **345** (2014), 87–114. MR2964018, Zbl 1058.42019, doi: 10.1090/conm/345/06242. 54, 57, 62

[14] SHAMSABADI, M.; AREFIJAMAAL, A.; BALAZS, P. The invertibility of U-fusion cross Gram matrices of operators. *Mediterr. J. Math.* **17** (2020), 130. MR4126463, Zbl 1454.42032, doi: 10.1007/s00009-020-01536-0. 62

[15] CASAZZA, P. G.; KUTYNIOK, G.; LI, S. Fusion frames and distributed processing. *Appl. Comput. Harmon. Anal.* **25** (2008), 114–132. MR2419707, Zbl 1258.42029, doi: 10.1016/j.acha.2007.10.001. 54, 61

[16] SUN, W. G-frames and G-Riesz bases. *J. Math. Anal. Appl.* **322** (2006), 437–452. MR2239250, Zbl 1129.42017, doi: 10.1016/j.jmaa.2005.09.039. 54

[17] GĂVRUŢA, L. Frames for operators. *Appl. Comput. Harmon. Anal.* **32** (2012), 139–144. MR2854166, Zbl 1230.42038, doi: 10.1016/j.acha.2011.07.006. 54

[18] BHANDARI, A.; MUKHERJEE, S. Atomic subspaces for operators. *Indian J. Pure Appl. Math.* **51** (2020), 1039–1052. MR4159339, Zbl 1456.42037, doi: 10.1007/s13226-020-0448-y. 54

[19] CHRISTENSEN, O. Frames and Bases-An Introductory Course. *Boston: Birkhäuser*, 2008. xx+313 pp. MR2428338, Zbl 1152.42001. 55

[20] KOHLDORFER, L.; BALAZS, P. On the relation of the frame related operators of fusion frame systems. *Sampl. Theory Signal Process. Data Anal.* **21** (2023). MR4549115, Zbl 1529.42026, doi: 10.1007/s43670-023-00049-7. 61

[21] CONWAY, J. B. A Course in Functional Analysis. *Graduate Texts in Mathematics, 2nd Edition, Springer, New York*, 1990. xvi+399 pp. MR1070713, Zbl 0706.46003. 64

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