New York Journal of Mathematics

New York J. Math. **30** (2024) 93–186.

Hopf-Galois structures on extensions of degree p^2q and skew braces of order p^2q: the elementary abelian Sylow p-subgroup case

E. Campedel, A. Caranti and I. Del Corso

ABSTRACT. Let p, q be distinct primes, with $p > 2$. In a previous paper we classified the Hopf-Galois structures on Galois extensions of degree p^2q, when the Sylow p-subgroups of the Galois group are cyclic. This is equivalent to classifying the skew braces of order p^2q, for which the Sylow p-subgroups of the multiplicative group are cyclic. In this paper we complete the classification by dealing with the case when the Sylow p-subgroups of the Galois group are elementary abelian.

According to Greither and Pareigis, and Byott, we will do this by classifying, for the groups $(G, \cdot)$ of order p^2q, the regular subgroups of their holomorphs whose Sylow p-subgroups are elementary abelian.

We rely on the use of certain gamma functions $\gamma : G \to \mathrm{Aut}(G)$. These functions are in one-to-one correspondence with the regular subgroups of the holomorph of G, and are characterised by the functional equation $\gamma(g^{\gamma(h)} \cdot h) = \gamma(g)\gamma(h)$, for $g, h \in G$. We develop methods to deal with these functions, with the aim of making their enumeration easier and more conceptual.

CONTENTS

Received May 10, 2023.

2010 *Mathematics Subject Classification.* 12F10, 16W30, 20B35, 20D45.

Key words and phrases. Hopf-Galois extensions, Hopf-Galois structures, holomorph, regular subgroups, braces, skew braces.

This paper is part of the thesis submitted by E. Campedel as a requirement for the PhD degree at the University of Milano–Bicocca. The authors are members of INdAM—GNSAGA. The authors gratefully acknowledge support from the Departments of Mathematics of the Universities of Milano–Bicocca, Pisa, and Trento.

ISSN 1076-9803/2024

1. Introduction

1.1. The general problem, and the classical approach. Let L/K be a finite Galois field extension, and let $\Gamma = \mathrm{Gal}(L/K)$. The group algebra $K[\Gamma]$ is a K-Hopf algebra, and its natural action on L endows L/K with a Hopf–Galois structure. In general this is not the only Hopf–Galois structure on L/K, and the study of the Hopf–Galois structures other than the classical one is relevant, for example in the context of algebraic number theory where different Hopf–Galois structures may behave differently at integral level (see Child's book [10] for an overview and [5] for a specific result).

This motivated the study of the Hopf-Galois structures on a finite Galois field extension L/K and their classification.

On the other hand, the celebrated result by Greither and Pareigis [12, Theorem 2.1] showed that all Hopf–Galois structures on L/K can be described in a purely group theoretic way. In the reformulation due to Byott [4] this result states that to each Hopf–Galois structure on L/K one can associate a group G, with the same cardinality as Γ, and such that the holomorph $\mathrm{Hol}(G)$, regarded as a subgroup of $\mathrm{Perm}(G)$, contains a regular subgroup isomorphic to Γ. We will refer to the isomorphism class of G as the *type* of the corrisponding Hopf–Galois structures.

As first noticed by Bachiller in [3], and clearly explained in the appendix to [18] by Byott and Vendramin, classifying the regular subgroups of $\mathrm{Hol}(G)$ is equivalent to determining the operations "$\circ$" on G such that $(G, \cdot, \circ)$ is a (right) skew brace (in the relevant literature it is more common to use left skew braces; we have translated the statements in the literature from left to right). Therefore, the Hopf–Galois structures on an extension with Galois group isomorphic to a group Γ correspond to the skew braces $(G, \cdot, \circ)$ with $(G, \circ) = \Gamma$; see also the recent work [17]. This has further motivated the study of this context, which in recent years has been deeply investigated.

Definition 1.1. Let Γ, $G = (G, \cdot)$ be finite groups with $|G| = |\Gamma|$. We define the following numbers.

(1) $e(\Gamma, G)$, the number of Hopf-Galois structures of type G on a Galois extension with group Γ,
(2) $e'(\Gamma, G)$, the number of regular subgroups of $\mathrm{Hol}(G)$ isomorphic to Γ,

(3) $e''(\Gamma, G)$, the total number of (right) skew braces $(G, \cdot, \circ)$ such that $\Gamma \cong (G, \circ)$.
(4) $f'(\Gamma, G)$, the number of classes of regular subgroups of $\mathrm{Hol}(G)$ isomorphic to Γ, under conjugation by elements of $\mathrm{Aut}(G)$,
(5) $f''(\Gamma, G)$, the number of isomorphism classes of skew braces $(G, \cdot, \circ)$ such that $\Gamma \cong (G, \circ)$.

Remark. Recall that, given a group $(G, \cdot)$, by the *(total) number of skew braces on* $(G, \cdot)$ we mean the number of distinct operations "$\circ$" on the set G such that $(G, \cdot, \circ)$ is a skew brace.

Theorem 1.2. *Let L/K be a finite Galois field extension with Galois group Γ. For any group $G = (G, \cdot)$ with $|G| = |\Gamma|$, we have*

[13, Theorem 4.2]: $e'(\Gamma, G) = e''(\Gamma, G)$*;*
[18, Proposition A.3]: $f'(\Gamma, G) = f''(\Gamma, G)$.

The number $e(\Gamma, G)$ is given by

[4, Corollary p. 3220]:

$$e(\Gamma, G) = \frac{|\mathrm{Aut}(\Gamma)|}{|\mathrm{Aut}(G)|}\, e'(\Gamma, G). \tag{1.1}$$

Moreover $e(\Gamma)$, the total number of Hopf-Galois structures on L/K, is given by $\sum_G e(\Gamma, G)$ where the sum is over all isomorphism types G of groups of order $|\Gamma|$.

In the paper [7], to the introduction of which we refer for more details on the literature, we classified the *Hopf-Galois structures* on a Galois extension L/K of order p^2q, where p, q are distinct primes, p is odd, and the Sylow p-subgroups of $\Gamma = \mathrm{Gal}(L/K)$ are cyclic. In the same paper we also computed, for $(G, \cdot)$ a group of order p^2q with cyclic Sylow p-subgroups, the *number of skew braces* $(G, \cdot, \circ)$, that is, the number of group operations "$\circ$" on the set G, such that $(G, \cdot, \circ)$ is a skew brace. We also computed the number of *isomorphism classes* of such skew braces, together with the cardinality of each such class.

Acri and Bonatto in [1, 2], using a different method, determine the number of *isomorphism classes of* all *skew braces* $(G, \cdot, \circ)$ of order p^2q; for the case of groups $(G, \cdot)$ with cyclic Sylow p-subgroups, their results coincide with ours.

The classification of [7] has been extended to all groups of order p^2q in the PhD thesis of the first author [6]. The present paper completes the work of [7], by determining the classification of Hopf–Galois structures on a Galois extension L/K of order p^2q, where p, q are distinct primes, in the remaining cases where $p > 2$ and the Sylow p-subgroups of $\Gamma = \mathrm{Gal}(L/K)$ are elementary abelian. Our methods work also in the case $p = 2$ (see [6]), but we do not include this case here, since the classification in the case $4q$ has already appeared in [15, 18].

In [7, Theorem 3.3 and Corollary 3.4] we have shown that for $p > 2$ a Galois field extension L/K of order p^2q with Galois group Γ admits Hopf–Galois structures of type G only for those G such that G and Γ have isomorphic Sylow p-subgroups.

As in [7], we explicitly determine the number of Hopf–Galois structures on L/K for each type, showing in particular that each G with elementary abelian Sylow p-subgroups defines some structure.

We accomplish this as follows. For any given group $G = (G, \cdot)$ of order p^2q, with $p > 2$, and for each group Γ of order p^2q with elementary abelian Sylow p-subgroups, we determine the following numbers.

(1) The total number of regular subgroups of $\mathrm{Hol}(G)$ isomorphic to Γ (Theorem 1.6).
This is the same, according to Theorem 1.2 [13, Theorem 4.2], as the total number of (right) skew braces $(G, \cdot, \circ)$ such that $\Gamma \cong (G, \circ)$.
(2) The number of isomorphism classes of (right) skew braces $(G, \cdot, \circ)$ such that $\Gamma \cong (G, \circ)$ (Theorem 1.7).
This is the same, according to Theorem 1.2 [18, Proposition A.3], as the number of conjugacy classes in $\mathrm{Hol}(G)$ of regular subgroups isomorphic to Γ; our numbers here coincide with the numbers found by Acri and Bonatto in [2].
Additionally, in Theorem 1.7 we also determine the length of each such conjugacy class.
(3) The number of Hopf–Galois structures of type G on a Galois extension with Galois group isomorphic to Γ (Theorem 1.5).

Remark 1.3. *Frattini's argument states that if a group X acts on a set, and N is a transitive subgroup of X, then $X = NS$, where S is any one-point stabiliser. In our situation, a regular subgroup $N \leq \mathrm{Hol}(G) = X$ acts transitively on G, so that* $\mathrm{Hol}(G) = N \mathrm{Aut}(G)$*, as* $\mathrm{Aut}(G)$ *is the stabiliser of* 1*. It follows that the conjugacy class of N in* $\mathrm{Hol}(G)$ *is the same as the orbit of N under the action of* $\mathrm{Aut}(G)$*.*

1.2. The methods. As in our previous paper [7], we follow Byott's approach, that is, for each group $G = (G, \cdot)$ of order p^2q with elementary abelian Sylow p-subgroups, we determine the regular subgroups of $\mathrm{Hol}(G)$ isomorphic to Γ. As we noted above, this is in turn equivalent to determining the right skew braces $(G, \cdot, \circ)$ such that $(G, \circ) \cong \Gamma$.

Our method relies on the use of the alternative brace operation $\circ$ on G mainly through the use of the function

$$\begin{aligned} \gamma : G &\to \mathrm{Aut}(G) \\ g &\mapsto (x \mapsto (x \circ g) \cdot g^{-1}), \end{aligned}$$

which is characterised by the functional equation

$$\gamma(g^{\gamma(h)} \cdot h) = \gamma(g)\gamma(h). \tag{1.2}$$

(See [7, Theorem 2.2] and the ensuing discussion for the details.) The functions γ satisfying (1.2) are called *gamma functions (GF)* and we will refer to (1.2) as the *gamma functional equation (GFE)*. The GF's are in one-to-one correspondence with the regular subgroups of $\mathrm{Hol}(G)$, and occur naturally in the theory of skew braces. It follows that to determine the number $e'(\Gamma, G)$ defined in Definition 1.1 we can count the number of functions $\gamma : G \to \mathrm{Aut}(G)$ verifying (1.2) and such that, for the operation $\circ$ defined on G by

$$g \circ h = g^{\gamma(h)} h,$$

we have $(G, \circ) \cong \Gamma$.

The classification of the groups of order p^2q is known after Hölder [14]; we have recorded the classification of these groups and of their automorphism groups in [8].

To enumerate the gamma functions we use the general results listed in Section 2 below; some of these had been developed in [7]. In the course of our discussion, we will appeal to some ad-hoc arguments; indeed, counting the gamma functions in the case of groups with elementary abelian Sylow p-subgroups presents several additional difficulties compared to the cyclic case.

The first is due to the sheer number of groups involved. Following the notation in Section 3, we distinguish the groups into *types* indexed by the numbers 5, 6, 7, 8, 9, 10 and 11 (see Table 3). Each type corresponds to an isomorphism class of groups of order p^2q, except for type 8, which correspond to $\frac{q-3}{2}$ isomorphism classes G_k, where $k \in \mathbb{Z}/q\mathbb{Z}$, $k \neq 0, \pm 1$ and $G_k \simeq G_{k^{-1}}$.

Our analysis is further complicated by the difficulty of proving the existence of a Sylow q-subgroup which is invariant under its image under γ (see Subsections 8.6, 9.5, 10.5, 10.6), and by having to deal with the case when $\gamma(G)$ is not contained in the group of inner automorphisms of G (e.g. see Section 10).

We now sketch the main tools and arguments we will use in counting the gamma functions.

We make use of an argument of duality, as introduced by A. Koch and P.J. Truman in [16], which we employ in the form spelled out in [7]. Each gamma function γ can be paired with a gamma function $\tilde{\gamma}$, which defines Hopf-Galois structures of the same type ([7, Subsection 2.8]). Under suitable assumptions, we can use this pairing to halve the number of GF we have to consider. Moreover, in some circumstances, the duality argument allows us to choose a GF with a kernel that is more suitable for calculations (Lemma 2.7 and Proposition 2.9).

The theory developed in Section 2 offers some methods to build gamma functions on G piecewise. The first tool is Proposition 2.6, which is sort of a homomorphism theorem for gamma functions. Under suitable assumptions, it gives a one-to-one correspondence between certain gamma functions defined on G and the (relatives) gamma functions defined on a quotient of G, which is smaller and then easier to investigate. We refer to this method as *lifting and restriction*.

A further tool is Proposition 2.8 (*gluing*), which is a generalisation of Proposition 2.6, and describes a way to construct GF's on G, when G is of the form $G = AB$, for $A, B \leq G$, starting from a *relative gamma function* defined on A

(see Definition 2.1) and a relative gamma function defined on B.

Let $r \in \{p, q\}$. To apply the tools above it will be useful to know when there exists a Sylow r-subgroup H which is $\gamma(H)$-invariant (*invariant* for short).

In [7, Theorem 3.3] we prove that for G a group of order p^2q and γ a GF on G, there always exists an invariant Sylow p-subgroup.

If G is a group of order p^2q, then either G has a unique Sylow q-subgroup or it has p^f Sylow q-subgroups, where $f = 1, 2$.

In the first case, since the unique Sylow q-subgroup B is characteristic, it is invariant.

In the second case, there are p Sylow q-subgroups when G is of type 6, and p^2 when G is of type 7, 8, 9, 10 and also 4 (for the last one we refer to [8],[7]).

Let γ be a GF on G, and consider the action of $\gamma(G)$ on the set Ω of the Sylow q-subgroups of G. If $p^2 \mid |\ker(\gamma)|$, then $|\gamma(G)| = 1$ or q, so that there exists at least one orbit of length 1, namely there exists $B \in \Omega$ which is $\gamma(G)$-invariant.

Moreover, if $q \mid |\ker(\gamma)|$, then there exists a Sylow q-subgroup B contained in $\ker(\gamma)$, therefore it is $\gamma(B)$-invariant.

In the remaining cases, namely when $|\ker(\gamma)| = 1$ or p, we will prove for some specific type of group G that we can find such a Sylow q-subgroup (see 6.2, 7.1, 8.7, 10.6, and 11.4). (For the type 4 see [7, Subsection 4.4]).

Therefore, we obtain the following.

Proposition 1.4. *If G is a group of order p^2q and γ is a GF on G, then there always exist both an invariant Sylow p-subgroup and an invariant Sylow q-subgroup of G.*

1.3. Hopf-Galois structures of order p^2q.

Theorem 1.5. *Let L/K be a Galois field extension of order p^2q, where p and q are two distinct primes with $p > 2$, and let $\Gamma = \mathrm{Gal}(L/K)$.*

Let G be a group of order p^2q.

If the Sylow p-subgroups of G and Γ are not isomorphic, then there are no Hopf-Galois structures of type G on L/K.

If the Sylow p-subgroups of Γ and G are elementary abelian, then the numbers $e(\Gamma, G)$ of Hopf-Galois structures of type G on L/K are given in the following tables.

(i) *For $q \nmid p^2 - 1$:*

Γ \ G	*5*	*11*
5	p^2	$2p(p^2-1)$
11	p^2q	$2p(1+qp^2-2q)$

where the upper left sub-tables of sizes 1×1 *and* 2×2 *give respectively the cases $p \nmid q-1$ and $p \mid q-1$.*

(ii) *For $q \nmid p-1$ and $q \mid p+1$:*

G \ Γ	5	10
5	p^2	$p(p-1)(q-1)$
10	p^2	$2+2p^2(q-3)-p^3+p^4$

(iii) For $q \mid p-1$:
If $q=2$,

G \ Γ	5	6	7
5	p^2	$2p(p+1)$	$p(3p+1)$
6	p^2	$2p(p+1)$	$p(3p+1)$
7	p^2	$2p^2(p+1)$	$2+p(p+1)(2p-1)$

If $q=3$,

G \ Γ	5	6	7	9
5	p^2	$4p(p+1)$	$2p(3p+1)$	$4p(p+1)$
6	p	$2p(p+3)$	$4p(p+1)$	$p(3p+5)$
7	p^2	$2p^2(p+1)^2$	$2+p^2(2p^2+3p+2)$	$p(p+1)^3$
9	$p^2(2p-1)$	$4p(p^2+1)$	$2(2p^3+3p^2-2p+1)$	$2+2p+p^3(p+3)$

If $q>3$,

G \ Γ	5	6
5	p^2	$2p(p+1)(q-1)$
6	p	$2p(p+2q-3)$
7	p^2	$2p^2(p+1)(pq-2p+1)$
8, G_2	p^3	$4p(p^2+pq-3p+1)$
8, $G_k \not\simeq G_2$	p^2	$4p(p^2+pq-3p+1)$
9	p^2	$4p(p^2+pq-3p+1)$

G \ Γ	7	9
5	$p(3p+1)(q-1)$	$2p(p+1)(q-1)$
6	$4(p^2+pq-2p)$	$p(4q+3p-7)$
7	$2+p^2(2p^2+pq+2q-4)$	$p(p+1)(p^2(2q-5)+2p+1)$
8, G_2	$2p(p^2q-4p+pq+2)$	$p(p^3+3p^2-14p+4pq-6)$
8, $G_k \not\simeq G_2$	$4p(2p^2-5p+pq+2)$	$p(p^3+5p^2-18p+4pq+8)$
9	$2(4p^3-9p^2+2p^2q+2p+1)$	$2+4p+p^2(p^2+5p+4q-16)$

$G8$ \ Γ	$G \not\simeq G_{\pm 2}$	$G \simeq G_{\pm 2}$, $q>5$	$G \simeq G_2$, $q=5$
5	$4p(p+1)(q-1)$	$4p(p+1)(q-1)$	$16p(p+1)$
6	$8p(q+p-2)$	$8p(q+p-2)$	$8p(p+3)$
7	$4p^2(p+1)(pq-3p+2)$	$4p^2(p+1)(pq-3p+2)$	$8p^2(p+1)^2$
8	*Table* 1	*Table* 2	$4(1+p+3p^2(p+1))$
9	$8p(2p^2+pq-5p+2))$	$4p(3p^2+2pq-8p+3)$	$16p(2p^3-2p+p+1)$

TABLE 1. G and Γ of type 8, $G \simeq G_k \not\simeq G_{\pm 2}$

Γ	*if either k or k^{-1} is a solution of $x^2 - x - 1 = 0$:*
G_k, G_{1-k} G_{1+k} $G_s \not\simeq G_k, G_{1+k}, G_{1-k}$	$2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$ $4(3p + 2p^2q - 8p^2 + 3p^3)$ $8(2p + p^2q - 5p^2 + 2p^3)$
Γ	*if k and k^{-1} are the solutions of $x^2 + x + 1 = 0$:*
G_k G_{1-k}, $G_{1-k^{-1}}$ G_{1+k} $G_s \not\simeq G_k, G_{1+k}, G_{1-k}, G_{1-k^{-1}}$	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ $2(7p + 4p^2q - 18p^2 + 7p^3)$ $2(1 + 4p + 4p^2q - 15p^2 + 6p^3)$ $8(2p + p^2q - 5p^2 + 2p^3)$
Γ	*if k and k^{-1} are the solutions of $x^2 - x + 1 = 0$:*
G_{-k} G_{1+k}, $G_{1+k^{-1}}$ G_{1-k} $G_s \not\simeq G_{-k}, G_{1-k}, G_{1+k}, G_{1+k^{-1}}$	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ $2(7p + 4p^2q - 18p^2 + 7p^3)$ $2(1 + 4p + 4p^2q - 15p^2 + 6p^3)$ $8(2p + p^2q - 5p^2 + 2p^3)$
Γ	*if k and k^{-1} are the solutions of $x^2 + 1 = 0$:*
G_k G_{1+k}, G_{1-k} $G_s \not\simeq G_k, G_{1+k}, G_{1-k}$	$4(1 + 2p + 2p^2q - 9p^2 + 4p^3)$ $4(3p + 2p^2q - 8p^2 + 3p^3)$ $8(2p + p^2q - 5p^2 + 2p^3)$
Γ	*if $k^2 \neq \pm k \pm 1, -1$:*
G_k, G_{-k} G_{1+k}, $G_{1+k^{-1}}$, G_{1-k}, $G_{1-k^{-1}}$ $G_s \not\simeq G_{\pm k}, G_{1\pm k}, G_{1\pm k^{-1}}$	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ $2(7p + 4p^2q - 18p^2 + 7p^3)$ $8(2p + p^2q - 5p^2 + 2p^3)$

TABLE 2. G and Γ of type 8, $G \simeq G_k$ for $k = \pm 2$,

Γ	*if $q > 7$:*
G_2 G_3, $G_{\frac{3}{2}}$ G_{-2} $G_s \not\simeq G_2, G_3, G_{\frac{3}{2}}, G_{-2}$	$2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$ $2(7p + 4p^2q - 18p^2 + 7p^3)$ $2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$ $8(2p + p^2q - 5p^2 + 2p^3)$
Γ	*if $q = 7$:*
G_2 G_3	$2(1 + 5p + 11p^2 + 7p^3)$ $2(1 + 4p + 13p^2 + 6p^3)$

Theorem 1.6. *Let $G = (G, \cdot)$ be a group of order p^2q, where p, q are distinct primes, with $p > 2$.*

If Γ is a group of order p^2q and the Sylow p-subgroups of G and Γ are not isomorphic, then no regular subgroup of $\mathrm{Hol}(G)$ *is isomorphic to Γ.*

If G and Γ have elementary abelian Sylow p-subgroups, then the following tables give equivalently

(1) *the number* $e'(\Gamma, G)$ *of regular subgroups of* $\mathrm{Hol}(G)$ *isomorphic to* Γ*;*
(2) *the number of (right) skew braces* $(G, \cdot, \circ)$ *such that* $\Gamma \cong (G, \circ)$.

(i) *For* $q \nmid p^2 - 1$*:*

Γ \ G	5	11
5	p^2	$2pq$
11	$p^2(p^2-1)$	$2p(1+qp^2-2q)$

where the upper left sub-tables of sizes 1×1 *and* 2×2 *give respectively the cases* $p \nmid q-1$ *and* $p \mid q-1$.

(ii) *For* $q \nmid p-1$ *and* $q \mid p+1$*:*

Γ \ G	5	10
5	p^2	$2p^2$
10	$\frac{1}{2}p(p-1)(q-1)$	$2+2p^2(q-3)-p^3+p^4$

(iii) *For* $q \mid p-1$*:*
If $q = 2$,

Γ \ G	5	6	7
5	p^2	$2p$	$p^3(3p+1)$
6	$p^2(p+1)$	$2p(p+1)$	$p^3(p+1)(3p+1)$
7	1	2	$2+p(p+1)(2p-1)$

If $q = 3$,

Γ \ G	5	6	7	9
5	p^2	$2p$	$p^3(3p+1)$	$4p^2$
6	$2p(p+1)$	$2p(p+3)$	$4p^3(p+1)^2$	$2p^2(3p+5)$
7	2	$2(p+1)$	$2+p^2(2p^2+3p+2)$	$2p^2+4p+2$
9	$2p^3+p^2-p$	$2(p^2+1)$	$2p^5+5p^4+p^3-p^2+p$	p^4+3p^3+2p+2

If $q > 3$,

Γ \ G	5	7
5	p^2	$p^3(3p+1)$
6	$p(p+1)(q-1)$	$4p^2(p+1)(p^2+pq-2p)$
7	$q-1$	$2+p^2(2p^2+pq+2q-4)$
8, G_2	$p^2(p+1)(q-1)$	$2p^2(p+1)(p^2q-4p+pq+2)$
8, $G_k \ncong G_2$	$p(p+1)(q-1)$	$4p^2(p+1)(2p^2-5p+pq+2)$
9	$\frac{1}{2}p(p+1)(q-1)$	$4p^5+p^4(q-2)+p^3(2q-7)+3p^2+p$

Γ \ G	*6*	*9*
5	$2p$	$4p^2$
6	$2p(p+2q-3)$	$2p^2(4q+3p-7)$
7	$2+2p(q-2)$	$2+4p+2p^2(2q-5)$
8, G_2	$4(1+p(p+q-3))$	$2p(p^3+3p^2-14p+4pq-6)$
8, $G_k \not\simeq G_2$	$4(1+p(p+q-3))$	$2p(p^3+5p^2-18p+4pq+8)$
9	$2+2p(p+q-3)$	$2+4p+p^2(p^2+5p+4q-16)$

Γ \ *G8*	$G \not\simeq G_{\pm 2}$	$G \simeq G_{\pm 2}$, $q > 5$	$G \simeq G_2$, $q = 5$
5	$4p^2$	$4p^2$	$4p^2$
6	$8p^2(q+p-2)$	$8p^2(q+p-2)$	$8p^2(p+3)$
7	$8p+4p^2(q-3)$	$8p+4p^2(q-3)$	$8p+8p^2$
8	*Table* 1	*Table* 2	$4(1+p+3p^2(p+1))$
9	$4p(2+p(q+2p-5))$	$2p(3+p(2q+3p-8))$	$8p(1+p+2p(p^2-1))$

As a consequence, we are able to compute the numbers of isomorphism classes of skew braces of size p^2q; these numbers coincide with those given in [1, 2] for $q > 2$, and [11] for $q = 2$.

Theorem 1.7. *Let $G = (G, \cdot)$ be a group of order p^2q, where p, q are distinct primes, with $p > 2$. For each group Γ of order p^2q with elementary abelian Sylow p-subgroups the following tables give equivalently*

(1) the number of conjugacy classes within $\mathrm{Hol}(G)$ *of regular subgroups isomorphic to Γ;*

(2) the number of isomorphism classes of skew braces $(G, \cdot, \circ)$ such that $\Gamma \cong (G, \circ)$.

(i) For $q \nmid p^2 - 1$:

Γ \ G	*5*	*11*
5	2	4
11	4	$6p-4$

where the upper left sub-tables of sizes 1×1 *and* 2×2 *give respectively the cases $p \nmid q - 1$ and $p \mid q - 1$.*

(ii) For $q \nmid p - 1$ and $q \mid p + 1$:

Γ \ G	*5*	*10*
5	2	2
10	1	$p+2q-4$

(iii) For $q \mid p - 1$:

If $q = 2$,

Γ \ G	*5*	*6*	*7*
5	2	2	5
6	2	8	$p+10$
7	1	2	5

If $q = 3$,

$\Gamma \backslash G$	*5*	*6*	*7*	*9*
5	2	2	5	3
6	1	12	16	$p+14$
7	1	4	8	4
9	2	6	10	$p+8$

If $q > 3$,

$\Gamma \backslash G$	*5*	*6*	*7*	*8*, G_k	*9*
5	2	2	5	4	3
6	1	$4q$	$4(q+1)$	$8(q+1)$	$4q+p+2$
7	1	$2(q-1)$	$3q-1$	$4(q-1)$	$2(q-1)$
8, $G_s \not\simeq G_2$	1	$4q$	$4(q+1)$	$8(q+1)$	$4q+p+2$
8, G_2	2	$4q$	$6q$	$8(q+1)$	$4q+p+2$
9	1	$2q$	$2(q+1)$	$4(q+1)$	$3q+p-1$

The lengths of the conjugacy classes are spelled out in Propositions 5.2, 6.1, 8.3, 9.3, 10.3, 11.2 and 12.1.

Acknowledgements

We are indebted to the referee for their careful reading of a bulky manuscript, and a number of extremely useful corrections and suggestions.

2. Tools

Definition 2.1. Let G be a group, $A \leq G$, and $\gamma : A \to \operatorname{Aut}(G)$ a function.

γ is said to satisfy the *gamma functional equation* (or *GFE* for short) if

$$\gamma(g^{\gamma(h)}h) = \gamma(g)\gamma(h), \tag{2.1}$$

for all $g, h \in A$.

We will say that A is *invariant* if it is invariant under the action of $\gamma(A)$.

γ is said to be a *relative gamma function* (or *RGF* for short) on A if it satisfies the gamma functional equation, and A is $\gamma(A)$-invariant.

If $A = G$, a relative gamma function is simply called a *gamma function* (or *GF* for short) on G.

For later use, we note that (2.1) can be rephrased, setting $k = g^{\gamma(h)}$, as

$$\gamma(kh) = \gamma(k^{\gamma(h)^{-1}})\gamma(h). \tag{2.2}$$

We will make use of the following results from [7]; some of these results were stated in [7] under the assumption that the relevant group G is finite, but the proofs stand verbatim for arbitrary groups.

Given a group G, denote by $\operatorname{Perm}(G)$ the group of all permutations on the underlying set G. The *right regular representation* of G is the homomorphism

$$\begin{aligned} \rho : G &\to \operatorname{Perm}(G) \\ g &\mapsto (x \mapsto xg). \end{aligned}$$

Theorem 2.2 ([7, Theorem 2.2]).

Let $(G, \cdot)$ be a group. The following data are equivalent.

(1) A regular subgroup $N \leq \mathrm{Hol}(G)$.
(2) A gamma function $\gamma : G \to \mathrm{Aut}(G)$.
(3) A group operation $\circ$ on G such that $(G, \cdot, \circ)$ is a (right) skew brace.

The data of (1)-(3) *are related as follows.*

(i) $g \circ h = g^{\gamma(h)} h$ for $g, h \in G$.
(ii) Each element of N can be written uniquely in the form $\nu(h) = \gamma(h)\rho(h)$, for some $h \in G$.
(iii) For $g, h \in G$ one has $g^{\nu(h)} = g \circ h$.
(iv) The map

$$\gamma : (G, \circ) \to \mathrm{Aut}(G)$$

is a morphism, in particolar $\ker(\gamma) \triangleleft (G, \circ)$.

(v) The map

$$\begin{aligned} \nu : (G, \circ) &\to N \\ h &\mapsto \gamma(h)\rho(h) \end{aligned}$$

is an isomorphism.

We report two useful simple facts concerning inverses and conjugacy in the group $(G, \circ)$ (see Lemma 2.10 [7]). We write $a^{\circ k}$ for the k-th power of a in $(G, \circ)$, and $a^{\ominus k}$ for the inverse of $a^{\circ k}$ in $(G, \circ)$. In the notation of Theorem 2.2, we have, for $a, b \in G$,

$$a^{\ominus 1} = a^{-\gamma(a)^{-1}},$$

and

$$a^{\ominus 1} \circ b \circ a = a^{-\gamma(a)^{-1}\gamma(b)\gamma(a)} b^{\gamma(a)} a.$$

Proposition 2.3 ([7, Proposition 2.6]).

Let G be a group, let $H \subseteq G$ and let γ be a GF on G.
Any two of the following conditions imply the third one:

(1) $H \leq G$;
(2) $(H, \circ) \leq (G, \circ)$;
(3) H is $\gamma(H)$-invariant.

If these conditions hold, then $(H, \circ)$ is isomorphic to a regular subgroup of $\mathrm{Hol}(H)$.

From Theorem 2.2 and Proposition 2.3 we have the following.

Corollary 2.4. *Let G be a group and let γ be a GF on G.*

(1) $\ker(\gamma) \leq G$, *and*
(2) $\gamma(G) \leq \mathrm{Aut}(G)$ of order $[G : \ker(\gamma)]$.

Lemma 2.5 ([7, Lemma 2.13]).

Let G be a group, $A \leq G$ and $\gamma : A \to \mathrm{Aut}(G)$ be a function such that A is invariant under $\gamma(A)$.

Then any two of the following conditions imply the third one.

(1) $\gamma([A, \gamma(A)]) = \{1\}$.
(2) $\gamma : A \to \operatorname{Aut}(G)$ *is a morphism of groups.*
(3) γ *satisfies the GFE.*

We write

$$\begin{aligned} \iota : & G \to \operatorname{Aut}(G) \\ & g \mapsto (x \mapsto g^{-1}xg). \end{aligned}$$

Proposition 2.6 ([7, Proposition 2.14])**.**
Let G be a group and let A, B be subgroups of G such that $G = AB$.
If γ is a GF on G, and $B \leq \ker(\gamma)$, then

$$\gamma(ab) = \gamma(a), \text{ for } a \in A, b \in B, \tag{2.3}$$

so that $\gamma(G) = \gamma(A)$.
Moreover, if A is $\gamma(A)$-invariant, then

$$\gamma' = \gamma_{\restriction A} : A \to \operatorname{Aut}(G) \tag{2.4}$$

is a RGF on A and $\ker(\gamma)$ *is invariant under the subgroup*

$$\left\{\gamma'(a)\iota(a) : a \in A\right\}$$

of $\operatorname{Aut}(G)$*.*
Conversely, let $\gamma' : A \to \operatorname{Aut}(G)$ be a RGF such that

(1) $\gamma'(A \cap B) \equiv 1$,
(2) *B is invariant under* $\{\gamma'(a)\iota(a) : a \in A\}$.

Then the map

$$\gamma(ab) = \gamma'(a), \text{ for } a \in A, b \in B,$$

is a well defined GF on G, and $\ker(\gamma) = \ker(\gamma')B$.

In this situation we will say that γ is a *lifting* of γ'.
The following is a slightly different version of [7, Lemma 2.23].

Lemma 2.7. *Let G be a group. Let C be a subgroup of G such that:*

(1) *C is abelian;*
(2) *C is characteristic in G;*
(3) $C \cap Z(G) = \{1\}$.

Let $\gamma : G \to \operatorname{Aut}(G)$ be a GF, and suppose that for every $c \in C$ we have $\gamma(c) = \iota(c^{-\sigma})$ for some function $\sigma : C \to C$.
Then $\sigma \in \operatorname{End}(C)$, and the following relations hold in $\operatorname{End}(C)$*:*

$$\sigma\, \gamma(g)_{\restriction C}\, (\sigma - 1) = (\sigma - 1)\, \gamma(g)_{\restriction C}\, \iota(g)_{\restriction C}\, \sigma, \quad \text{for } g \in G. \tag{2.5}$$

Note that $\gamma(g)\iota(g) = \iota(g^{\gamma(g)^{-1}})\gamma(g)$. Setting $g' = g^{\ominus 1} = g^{-\gamma(g)^{-1}}$, we see that $\gamma(g)\iota(g) = \iota(g')^{-1}\gamma(g')^{-1}$. Therefore, (2.5) can be rewritten as

$$\sigma\, \gamma(g)^{-1}_{\restriction C}\, (\sigma - 1) = (\sigma - 1)\, \iota(g)^{-1}_{\restriction C}\, \gamma(g)^{-1}_{\restriction C}\, \sigma, \quad \text{for } g \in G. \tag{2.6}$$

Let G be a group, and γ a gamma function on G. Suppose $G = AB$, where A, B are subgroups of G, such that $A \cap B = \{1\}$, the subgroup A is $\gamma(B)$-invariant, and there is $\sigma \in \operatorname{End}(A)$ such that $\gamma(a) = \iota(a^{-\sigma})$ for $a \in A$. Then

$$\gamma(ab) = \iota(a^{-\gamma(b)^{-1}\sigma})\gamma(b), \text{ for } a \in A \text{ and } b \in B. \tag{2.7}$$

We are interested in recording the following situation in which gamma functions of this form arise.

Proposition 2.8. *Let $G = AB$ be a group, where A, B are subgroups of G, such that*

(1) $A \cap B = \{1\}$,
(2) A is abelian,
(3) A is characteristic in G,
(4) $A \cap Z(G) = \{1\}$.

If there exist a RGF $\gamma : B \to \operatorname{Aut}(G)$ and $\sigma \in \operatorname{End}(A)$ which satisfy

$$\sigma\,\gamma(b)_{\restriction A}\,(\sigma - 1) = (\sigma - 1)\,\gamma(b)_{\restriction A}\,\iota(b)_{\restriction A}\,\sigma, \quad \text{for } b \in B, \tag{2.8}$$

then the extension of γ to the function $\gamma : G \to \operatorname{Aut}(G)$ defined as in (2.7) *is a GF on G.*

In this situation we will say that γ is the *gluing* of the two RGF on A and B.

Proof. Let γ be a RGF on B and $\sigma \in \operatorname{End}(A)$ such that (2.8) is satisfied. We now show that the function defined in (2.7) satisfies the GFE. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

We have

$$\begin{aligned}\gamma(a_1b_1)\gamma(a_2b_2) &= \iota(a_1^{-\gamma(b_1)^{-1}\sigma})\gamma(b_1)\iota(a_2^{-\gamma(b_2)^{-1}\sigma})\gamma(b_2)\\ &= \iota(a_1^{-\gamma(b_1)^{-1}\sigma}a_2^{-\gamma(b_2)^{-1}\sigma\gamma(b_1)^{-1}})\gamma(b_1)\gamma(b_2).\end{aligned}$$

On the other hand

$$\begin{aligned}&\gamma((a_1b_1)^{\gamma(a_2b_2)}a_2b_2) =\\ &\gamma(a_1^{\gamma(b_2)}b_1^{\iota(a_2^{-\gamma(b_2)^{-1}\sigma})\gamma(b_2)}a_2b_2) =\\ &\gamma(a_1^{\gamma(b_2)}(a_2^{\gamma(b_2)^{-1}\sigma}b_1a_2^{-\gamma(b_2)^{-1}\sigma})^{\gamma(b_2)}a_2b_2) =\\ &\gamma(a_1^{\gamma(b_2)}a_2^{\gamma(b_2)^{-1}\sigma\gamma(b_2)}b_1^{\gamma(b_2)}a_2^{-\gamma(b_2)^{-1}\sigma\gamma(b_2)}a_2b_2) =\\ &\gamma(a_1^{\gamma(b_2)}a_2^{\gamma(b_2)^{-1}\sigma\gamma(b_2)}a_2^{-\gamma(b_2)^{-1}\sigma\iota(b_1)^{-1}\gamma(b_2)}a_2^{\gamma(b_2)^{-1}\iota(b_1)^{-1}\gamma(b_2)}b_1^{\gamma(b_2)}b_2) =\\ &\iota((a_1^{\gamma(b_2)}a_2^{\gamma(b_2)^{-1}\sigma\gamma(b_2)}a_2^{-\gamma(b_2)^{-1}\sigma\iota(b_1)^{-1}\gamma(b_2)}a_2^{\gamma(b_2)^{-1}\iota(b_1)^{-1}\gamma(b_2)})^{-\gamma(b_2)^{-1}\gamma(b_1)^{-1}\sigma})\\ &\qquad\gamma(b_1)\gamma(b_2) =\\ &\iota(a_1^{-\gamma(b_1)^{-1}\sigma}a_2^{-\gamma(b_2)^{-1}\sigma\gamma(b_1)^{-1}\sigma+\gamma(b_2)^{-1}\sigma\iota(b_1)^{-1}\gamma(b_1)^{-1}\sigma-\gamma(b_2)^{-1}\iota(b_1)^{-1}\gamma(b_1)^{-1}\sigma})\\ &\qquad\gamma(b_1)\gamma(b_2).\end{aligned}$$

Now (2.8) (in the form of (2.6)) shows that the two expressions

$$-\sigma\gamma(b_1)^{-1}_{\restriction A}$$

and

$$-\sigma\gamma(b_1)^{-1}_{\restriction A}\sigma + \sigma\iota(b_1)^{-1}_{\restriction A}\gamma(b_1)^{-1}_{\restriction A}\sigma - \iota(b_1)^{-1}_{\restriction A}\gamma(b_1)^{-1}_{\restriction A}\sigma$$

coincide. □

Let γ be a GF on G, N the associated regular subgroup of $\mathrm{Hol}(G)$ and $\circ$ the associated operation. Write $\mathrm{inv} : g \mapsto g^{-1}$ for the inversion map on G. Clearly $\mathrm{inv} \in \mathrm{Perm}(G)$. Then N^{inv}, the conjugate of N under inv, is another regular subgroup of $\mathrm{Hol}(G)$, with corresponding gamma function

$$\begin{aligned} \tilde{\gamma} : G &\to \mathrm{Aut}(G) \\ x &\mapsto \gamma(x^{-1})\iota(x^{-1}), \end{aligned} \tag{2.9}$$

and circle operation

$$x \mathbin{\tilde{\circ}} y = (x^{-1} \circ y^{-1})^{-1}$$

(see [7, Proposition 2.22]).

The following is essentially [7, Proposition 2.24], in which we replaced the hypothesis that the subgroup C is characteristic with the slightly more general hypothesis that C is normal and $\gamma(G)$-invariant. In that case $\gamma(g)_{\restriction C} \in \mathrm{Aut}(C)$. The proof in [7, Proposition 2.24] still stands, as $c^{\iota(g)}, c^{\gamma(g)} \in C$.

Proposition 2.9. *Let G be a non-abelian group. Let C be a subgroup of G such that:*

(1) $C = \langle c \rangle$ is cyclic, of order a power of the prime r,
(2) C is normal in G,
(3) $C \cap Z(G) = \{1\}$, and
(4) there is $a \in G$ which induces by conjugation on C an automorphism whose order is not a power of r.

Let $\gamma : G \to \mathrm{Aut}(G)$ be a GF, and suppose that C is $\gamma(G)$-invariant and $\gamma(C) \leq \iota(C)$, so that for every $c \in C$ we have $\gamma(c) = \iota(c^{-\sigma})$, for some function $\sigma : C \to C$.

Then

(1) either $\sigma = 0$, that is, $C \leq \ker(\gamma)$,
(2) or $\sigma = 1$, that is, $\gamma(c) = \iota(c^{-1})$, so that $C \leq \ker(\tilde{\gamma})$.

Corollary 2.10. *Let G be a non-abelian group. Suppose that G contains a subgroup $C \neq \{1\}$ which satisfies the hypotheses (1)-(4) of Proposition 2.9, and suppose that for every GF γ on G, C is $\gamma(G)$-invariant and $\gamma(C) \leq \iota(C)$.*

For each group $\mathcal{G}$ of the same order as G, let

$$n_C(\mathcal{G}) = \left|\{\gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \text{ and } C \leq \ker(\gamma)\}\right|.$$

Then

$$e'(\mathcal{G}, G) = \left|\{\gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G}\}\right| = 2n_C(\mathcal{G}).$$

Proof. Let $X = \{\gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G}\}$, and

$$X_1 = \{\gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \text{ and } C \leq \ker(\gamma)\},$$
$$X_2 = \{\gamma \text{ GF on } G : (G, \circ) \cong \mathcal{G} \text{ and } C \leq \ker(\tilde{\gamma})\},$$

where $\tilde{\gamma}$ is as in (2.9).

Proposition 2.9 shows that $X = X_1 \cup X_2$. We claim that $X_1 \cap X_2 = \emptyset$. Indeed, if $\gamma \in X_1 \cap X_2$, then for all $c \in C$ we have $\gamma(c) = 1 = \gamma(c^{-1})\iota(c^{-1})$, so that $C \leq Z(G)$, a contradiction.

We have

$$e'(\mathcal{G}, G) = |X| = |X_1| + |X_2| = n_C(\mathcal{G}) + |X_2|.$$

Now we show that there is a bijection between X_1 and X_2, so that $e'(\mathcal{G}, G) = 2n_C(\mathcal{G})$. Consider

$$\psi : X \to X$$
$$\gamma \mapsto \tilde{\gamma}.$$

The map ψ is well defined, indeed $\tilde{\gamma}$ is a GF on G and $(G, \tilde{\circ}) \cong (G, \circ) \cong \mathcal{G}$ (see [7, Proposition 2.22]); moreover

$$\psi^2(\gamma) = \psi(\tilde{\gamma}) = \tilde{\tilde{\gamma}}.$$

It is immediate from the formula for $\tilde{\gamma}$, or from its definition in terms of regular subgroups, that $\tilde{\tilde{\gamma}} = \gamma$, that is, $\psi^2 = 1$, so that ψ is bijective Now, using Proposition 2.9, we obtain $\psi(X_2) = X_1$, and so $|X_2| = |X_1|$. □

Lemma 2.11 ([7, Lemma 2.9])**.**

Let G be a group, N a regular subgroup of $\mathrm{Hol}(G)$*, and γ the associated gamma function.*

Let $\varphi \in$ $\mathrm{Aut}(G)$*.*

(1) The gamma function γ^{φ} associated to the regular subgroup N^{φ} is given by

$$\gamma^{\varphi}(g) = \gamma(g^{\varphi^{-1}})^{\varphi} = \varphi^{-1}\gamma(g^{\varphi^{-1}})\varphi, \tag{2.10}$$

for $g \in G$.

(2) If $H \leq G$ is invariant under $\gamma(H)$, then H^{φ} is invariant under $\gamma^{\varphi}(H^{\varphi})$.

We will refer to the action (2.10) of $\mathrm{Aut}(G)$ on γ of the Lemma as *conjugation*.

Lemma 2.12. *Let $(G, \cdot)$ be a group of order p^2q, $p > 2$, and assume that the Sylow p-subgroup A of G is normal. Let γ be a GF on G and let $(G, \cdot, \circ)$ the associated skew brace.*

(1) If $A = \langle a \rangle$ is cyclic, then a is also a generator for $(A, \circ)$.

(2) If $A = \langle a_1, a_2 \rangle$ is elementary abelian and $\langle a_1 \rangle$ is $\gamma(\langle a_1 \rangle)$-invariant, then $\{a_1, a_2\}$ is also a set of generators for $(A, \circ)$.

Proof. The Sylow p-subgroup A is characteristic, and so $\gamma(A)$-invariant. Therefore, A is also a subgroup of $(G, \circ)$. Moreover, the condition $p > 2$ ensures that $A \simeq (A, \circ)$ (see [7, Theorem 3.3]).

If $A = \langle a \rangle$ is cyclic then $\mathrm{ord}_A(a) = \mathrm{ord}_{(A,\circ)}(a)$ (take $\gamma_{|A}$ in [7, Corollary 2.18]), therefore a is also a generator of $(A, \circ)$.

If A is elementary abelian then every non-trivial element of $(A, \circ)$ has order p. Moreover if $A_1 := \langle a_1 \rangle$ is $\gamma(A_1)$-invariant then $a_2 \notin A_1 = (A_1, \circ)$, so $\{a_1, a_2\}$ generate $(A, \circ)$. □

3. Groups of order p^2q

We briefly describe the groups of order p^2q with elementary abelian Sylow p-subgroups, and list them and their automorphisms in the table below, referring to [8] for the details.

We will say that two groups have the same *type* if they have isomorphic automorphism groups. For groups of order p^2q each type corresponds to an isomorphism class, except for type 8, which corresponds to $\frac{q-3}{2}$ isomorphism classes.

We use the notation $\mathcal{C}_n$ for a cyclic group of order n.

Type 5: Abelian group.

Type 6: This is the non-abelian group with centre of order p for $q \mid p-1$, which we denote by $\mathcal{C}_p \times (\mathcal{C}_p \rtimes \mathcal{C}_q)$. It can be described as

$$\left\langle a_1, a_2, b \,:\, a_1a_2 = a_2a_1,\ a_1^p = a_2^p = b^q = 1,\ a_1^{\iota(b)} = a_1,\ a_2^{\iota(b)} = a_2^{\lambda} \right\rangle,$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$.

Type 7: This is the non-abelian group for $q \mid p-1$ in which a generator of $\mathcal{C}_q$ acts on $\mathcal{C}_p \times \mathcal{C}_p$ as a non-identity scalar matrix. We denote it by $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$, and it can be described as

$$\left\langle a_1, a_2, b \,:\, a_1a_2 = a_2a_1,\ a_1^p = a_2^p = b^q = 1,\ a_1^{\iota(b)} = a_1^{\lambda},\ a_2^{\iota(b)} = a_2^{\lambda} \right\rangle,$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$.

Type 8: These are the non-abelian groups for $q \mid p-1$, $q > 3$, in which a generator of $\mathcal{C}_q$ acts on $\mathcal{C}_p \times \mathcal{C}_p$ as a diagonal, non-scalar matrix with no eigenvalue 1, and determinant different from 1. We denote this type by $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$, and it consists of the groups G_k which can be described as

$$G_k = \left\langle a_1, a_2, b \,:\, a_1a_2 = a_2a_1,\ a_1^p = a_2^p = b^q = 1,\ a_1^{\iota(b)} = a_1^{\lambda},\ a_2^{\iota(b)} = a_2^{\lambda^k} \right\rangle,$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$, and k is an integer modulo q, $k \neq 0, \pm 1$.

Since for each $k \neq 0, \pm 1$ we have that $G_k \simeq G_{k^{-1}}$, the type 8 includes $\frac{q-3}{2}$ isomorphism classes of groups.

We will denote by $\mathcal{K}$ the set of the elements $k \neq 0, \pm 1$ for which $\{G_k : k \in \mathcal{K}\}$ is a set of representatives of the isomorphism classes of groups of type 8.

Type 9: This is the non-abelian group for $q \mid p-1$, $q > 2$, in which a generator of $\mathcal{C}_q$ acts on $\mathcal{C}_p \times \mathcal{C}_p$ as a diagonal, non-scalar matrix with no eigenvalue 1, and determinant 1. We denote it by $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$, and it can be described as

$$\left\langle a_1, a_2, b \,:\, a_1 a_2 = a_2 a_1,\ a_1^p = a_2^p = b^q = 1,\ a_1^{\iota(b)} = a_1^{\lambda},\ a_2^{\iota(b)} = a_2^{\lambda^{-1}} \right\rangle,$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$.

Type 10: This is the non-abelian group group for $q \mid p+1$, $q > 2$, in which a generator of $\mathcal{C}_q$ acts on $\mathcal{C}_p \times \mathcal{C}_p$ as a matrix C with $\det(C) = 1$ and $\mathrm{tr}(C) = \lambda + \lambda^{-1}$, where $\lambda \neq 1$ is a q-th root of unity in a quadratic extension of $\mathbb{F}_p$. We denote it by $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$, and it can be described as

$$\left\langle a_1, a_2, b \,:\, a_1 a_2 = a_2 a_1,\ a_1^p = a_2^p = b^q = 1,\ a_1^{\iota(b)} = a_1^{\lambda+\lambda^{-1}} a_2,\ a_2^{\iota(b)} = a_1^{-1} \right\rangle.$$

Type 11: This is the non-abelian group with centre of order p for $p \mid q-1$, which we denote by $(\mathcal{C}_q \rtimes \mathcal{C}_p) \times \mathcal{C}_p$. It can be described as

$$\left\langle a_1, a_2, b \,:\, a_1 a_2 = a_2 a_1,\ a_1^p = a_2^p = b^q = 1,\ b^{\iota(a_1)} = b^u \right\rangle,$$

where u is an element of order p in $\mathbb{Z}/q\mathbb{Z}$.

TABLE 3. Groups of order $p^2 q$ with elementary abelian Sylow p-subgroups and their automorphisms

Type	Conditions	G	Aut(G)	|Z(G)|	ev
5		$\mathcal{C}_p \times \mathcal{C}_p \times \mathcal{C}_q$	$\mathrm{GL}(2,p) \times \mathcal{C}_{q-1}$	$p^2 q$	$1, 1$
6	$q \mid p-1$	$\mathcal{C}_p \times (\mathcal{C}_p \rtimes \mathcal{C}_q)$	$\mathcal{C}_{p-1} \times \mathrm{Hol}(\mathcal{C}_p)$	p	$1, \lambda$
7	$q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$	$\mathrm{Hol}(\mathcal{C}_p \times \mathcal{C}_p)$	1	λ, λ
8	$3 < q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$	$\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p)$	1	$\lambda, \lambda^k,\ k \neq 0, \pm 1$
9	$2 < q \mid p-1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D1} \mathcal{C}_q$	$(\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p)) \rtimes \mathcal{C}_2$	1	λ, λ^{-1}
10	$2 < q \mid p+1$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$	$(\mathcal{C}_p \times \mathcal{C}_p) \rtimes (\mathcal{C}_{p^2-1} \rtimes \mathcal{C}_2)$	1	no ev
11	$p \mid q-1$	$(\mathcal{C}_q \rtimes \mathcal{C}_p) \times \mathcal{C}_p$	$\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_q)$	p	$-$

Remark 3.1. *The column "ev" determines the eigenvalues of the action of an element of order q on the Sylow p-subgroup.*

4. The main case distinction

In this section we spell out the case distinction we will pursue in the following sections, and collect a few facts that will be useful at several points in the classification.

In what follows we will discuss about duality, so that we consider only the non-abelian groups.

For type 6, we can apply Corollary 2.10 to a non-central subgroup of order p. For type 11, we can apply this to the Sylow q-subgroup C.

For G of the remaining types, namely 7, 8, 9 and 10, denote by A the elementary abelian Sylow p-subgroup of G. We will show in Sections 7 and 11 that for the types 8, 9 and 10 (when $p > 2$) one has

$$\forall a \in A,\ \gamma(a) = \iota(a^{-\sigma}), \tag{4.1}$$

for some $\sigma \in \operatorname{End}(A)$. If G is of type 7 then it is not always the case that $\gamma(A) \leq \operatorname{Inn}(G)$; we will treat the case $\gamma(A) \nleq \operatorname{Inn}(G)$ separately in Section 10. Therefore, for G of types 8, 9, 10, or of type 7 and $\gamma(A) \leq \operatorname{Inn}(G)$, equation (4.1) holds and we can apply Lemma 2.7 with $C = A$, getting equation (2.5).

We have the following case distinction.

4.1. $\sigma, 1 - \sigma$ are not both invertibile. This means that σ has an eigenvalue 0 or 1. If it is 0, then $p \mid |\ker(\gamma)|$. If it is 1, consider the dual gamma function defined as $\tilde{\gamma}(g) = \gamma(g^{-1})\iota(g^{-1})$ (see [7, Proposition 2.22]). Then for $a \in A$, $\tilde{\gamma}(a) = \gamma(a^{-1})\iota(a^{-1}) = \iota(a^{\sigma-1})$, so that $p \mid |\ker(\tilde{\gamma})|$. Therefore, up to switch γ with $\tilde{\gamma}$, we can assume the eigenvalue is 0, so that p divides the order of the kernel of γ.

4.2. $\sigma, 1 - \sigma$ are both invertibile. This means that σ has no eigenvalues 0, 1. Then equation (2.5) yields

$$(\sigma^{-1} - 1)^{-1}\gamma(b)_{\restriction A}(\sigma^{-1} - 1) = \gamma(b)_{\restriction A}\iota(b)_{\restriction A}, \tag{4.2}$$

where $b \neq 1$ is a q-element. Thus $\gamma(b)_{\restriction A}$ and $\gamma(b)_{\restriction A}\iota(b)_{\restriction A}$ are conjugate, and this yields some information about the eigenvalues of $\gamma(b)_{\restriction A}$.

For type 7, if $q > 2$ (4.2) is plainly impossible, as

$$\iota(b) = \begin{bmatrix} \lambda & \\ & \lambda \end{bmatrix},$$

for some $\lambda \neq 1$, λ of order q.

For type 8, the two normal subgroups of order p are characteristic, so $\gamma(b)_{\restriction A}$ and $\iota(b)_{\restriction A}$ commute, as they are simultaneously diagonal. Let

$$\iota(b)_{\restriction A} = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}, \qquad \gamma(b)_{\restriction A} = \begin{bmatrix} \alpha_1 & \\ & \alpha_2 \end{bmatrix},$$

with $\lambda_i \neq 1$. This implies $\alpha_1 = \lambda_2\alpha_2$ and $\alpha_2 = \lambda_1\alpha_1$, so that $\alpha_1 = \lambda_1\lambda_2\alpha_1$ and $\lambda_1\lambda_2 = 1$, against the assumption of type 8.

For type 9, however, this is well possible. This time there is an automorphism of order two exchanging the two eigenspaces, but since $\gamma(b)_{\restriction A}$ has odd order q, it leaves them invariant, so that once more $\gamma(b)_{\restriction A}$ and $\iota(b)_{\restriction A}$ commute, as they are simultaneously diagonal.

In the same notation as for type 8, here we get $\lambda_1 = \lambda$, $\lambda_2 = \lambda^{-1}$, $\alpha_1 = \alpha$ and $\alpha_2 = \lambda\alpha$. We get

$$\sigma^{-1} - 1 = \begin{bmatrix} & s_1 \\ s_2 & \end{bmatrix}$$

(with $s_1s_2 \neq 1$), or

$$\sigma = (1 - s_1s_2)^{-1} \begin{bmatrix} 1 & -s_1 \\ -s_2 & 1 \end{bmatrix}.$$

For type 10, the eigenvalues of $\iota(b)$ are not in the base field, but in a quadratic extension of it. Still, this is similar to type 9.

4.3. Some results on GL(2, p). We collect here some information about $\mathrm{GL}(2, p)$, which will be useful for the groups G of type 5 or 7. We will denote by A and B, the Sylow p-subgroup (which is unique in both cases) and a Sylow q-subgroup of G, respectively.

4.3.1. Sylow p-subgroups. $\mathrm{GL}(2, p)$ has $p + 1$ Sylow p-subgroups and each of them fixes a subgroup of order p of $\mathcal{C}_p \times \mathcal{C}_p$. In the following we will denote by α an element of order p of $\mathrm{GL}(2, p)$.

4.3.2. Elements of order p when $p \parallel |\ker(\gamma)|$. Suppose that G is of type 5 or 7, and let γ be a GF on G such that $\langle a_1 \rangle \leq \ker(\gamma) \neq A$, where $a_1 \in A$, $a_1 \neq 1$. Let $a_2 \in A \setminus \langle a_1 \rangle$, then $\gamma(a_2) = \alpha$ (possibly modulo an automorphism of A which is the identity on A), where $\alpha \in \mathrm{GL}(2, p)$ has order p. Then

$$a_1^{\alpha} a_2 = a_1 \circ a_2 = a_2 \circ a_1 = a_2 a_1, \tag{4.3}$$

so that a_1 is fixed by α. This means that $\ker(\gamma)$ determines $\langle \alpha \rangle$, which is the Sylow p-subgroup of $\mathrm{GL}(2, p)$ fixing $\ker(\gamma)$. With respect to the basis $\{a_1, a_2\}$ we can write

$$\alpha = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix},$$

where $0 \neq d \in \mathbb{Z}/p\mathbb{Z}$.

4.3.3. Sylow q-subgroups. Suppose that $q \mid p-1$ and recall that $|\mathrm{GL}(2, p)| = (p-1)^2 p(p+1)$.

If $q > 2$ a Sylow q-subgroup of $\mathrm{GL}(2, p)$ has order q^{2e}, where $q^e \parallel p - 1$.

Every Sylow q-subgroup of $\mathrm{GL}(2, p)$ is of the form

$$\begin{aligned} Q_{A_1,A_2} &= \{\beta \in \mathrm{GL}(2, p) :\ A_1, A_2 \text{ are eigenspaces of } \beta \text{ with respect} \\ &\qquad\qquad \text{to eigenvalues of order dividing } q^e\} \\ &\cong \mathcal{C}_{q^e} \times \mathcal{C}_{q^e}, \end{aligned}$$

for any choice of a pair $\{A_1, A_2\}$ of distinct one-dimensional subspaces of A. Thus there are $\frac{p(p+1)}{2}$ Sylow q-subgroups.

Moreover, each Sylow q-subgroup of $\mathrm{GL}(2, p)$ has $q^2 - 1$ elements of order q. However, the scalar elements are common to all the Sylow q-subgroups. Hence, $\mathrm{GL}(2, p)$ has

$$(q^2 - q) \cdot \frac{(p+1)p}{2} + q - 1$$

elements of order q.

If $q = 2$, the Sylow 2-subgroups of $\mathrm{GL}(2, p)$ are described in [9]. Note that in this case if ϑ has order 2, then its minimal polynomial divides x^2-1, and therefore its eigenvalues belong to $\{\pm 1\}$. Moreover all the elements with eigenvalues $1, -1$ are conjugate, and such an element, say ϑ, is stabilised by the diagonal matrices, therefore $|\mathrm{Orb}(\vartheta)| = p(p + 1)$. Thus there are $p(p + 1)$ non-scalar elements of order 2, plus the scalar matrix $\mathrm{diag}(-1, -1)$.

4.3.4. Elements of order q when $|\ker(\gamma)| = p$. Suppose that $q \mid p - 1$ and G is of type 5 or 7. Let γ be a GF on G with kernel $\langle a_1 \rangle$, where $a_1 \in A$. Let $a_2 \in A \setminus \langle a_1 \rangle$ and set $\gamma(a_2) = \alpha$, an element of order p in $\mathrm{GL}(2, p)$. Let $b \in G$ be such that $\gamma(b) = \beta$ (possibly modulo $\iota(A)$), where β is an element of order q in the normaliser of $\langle \alpha \rangle$. Then $\alpha^\beta = \alpha^t$ for a certain t, and Subsection 4.3.2 yields that a_1 is fixed by α, so that

$$a_1^{\beta\alpha} = a_1^{\alpha^{t^{-1}}\beta} = a_1^{\beta},$$

namely a_1^β is fixed by α as well. Therefore $a_1^\beta \in \langle a_1 \rangle$, so that $\langle a_1 \rangle$ is an eigenspace for β too.

Let a_3 be another eigenvector for β. Then, since $\det(\alpha)^p = 1$, we can write, with respect to the basis $\langle a_1, a_3 \rangle$,

$$\alpha = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix}, \ \beta = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix},$$

where $0 \neq d \in \mathbb{Z}/p\mathbb{Z}$, λ has order q, and x_1, x_2 are not both 0.

Note that replacing a_3 with a suitable element in $\langle a_3 \rangle$, with respect to that new basis we have

$$\alpha = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Note that if β is a scalar matrix, there are $q-1$ elements β as above. If β is non-scalar, taking into account the choice of $\langle a_3 \rangle$, there are $q(q - 1)p$ possibilities for β.

5. Type 5

Here $G = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes \mathcal{C}_q$, and $\mathrm{Aut}(G) = \mathrm{GL}(2, p) \times \mathcal{C}_{q-1}$.

Let A be the Sylow p-subgroup, and $B = \langle b \rangle$ the Sylow q-subgroup.

In the following we will denote by α an element of order p of $\mathrm{GL}(2, p)$. If $p \mid q - 1$ we will denote by η an element of order p of $\mathcal{C}_{q-1}$: clearly η fixes A point-wise. If $q \mid p - 1$ we will denote by β an element of order q of $\mathrm{GL}(2, p)$.

5.1. Abelian groups. Assume here $(G, \circ)$ abelian. These are in particular the only cases when there are no divisibilities.

Since B is characteristic, by Proposition 2.3 and Theorem 2.2, (iv), $\gamma(b)$ will have order dividing q, so it is an element in $\mathrm{GL}(2, p)$ of order dividing q. Then for $a \in A$ we have

$$a = b^{\ominus 1} \circ a \circ b = b^{-\gamma(b)^{-1}\gamma(a)\gamma(b)} b a^{\gamma(b)} = b^{-\gamma(a)} b a^{\gamma(b)},$$

from which we get that $\gamma(b) = 1$, and also that $\gamma(a)_{\restriction B} = 1$.

Thus $B \leq \ker(\gamma)$. If $\gamma(G) = \{1\}$, then we obtain the right regular representation, which corresponds to one group of type 5. Otherwise $\gamma(G) \neq \{1\}$, and we can only have $\gamma(G) = \gamma(A) = \langle \alpha \rangle$, where $\alpha \in \mathrm{GL}(2, p)$ has order p. Therefore, each GF on G is the lifting of a RGF $\gamma : A \to \mathrm{Aut}(G)$ with $|\gamma(A)| = p$.

Let $1 \neq a_1 \in A$ and let $\ker(\gamma) = \langle a_1 \rangle$ (we have $p + 1$ choices for such a subgroup); the argument in 4.3.2 shows that a_1 is fixed by α, so that $\ker(\gamma)$ determines $\gamma(A)$, and for $a_2 \in A \setminus \langle a_1 \rangle$ we have $\gamma(a_2) = \alpha^i$, for $1 \leq i \leq p - 1$. Note that for each i the unique morphism defined by $\gamma(a_1) = 1$ and $\gamma(a_2) = \alpha^i$ is such that $[A, \gamma(A)] = \ker(\gamma)$, so by Lemma 2.5, these morphisms coincide with the RGF's. Therefore, here we have $(p + 1)(p - 1) = p^2 - 1$ different GF's on G giving groups $(G, \circ)$ of type 5.

As to the conjugacy classes, since $B \leq \ker(\gamma)$ is characteristic, every automorphism φ of G stabilises $\gamma_{|B}$. Moreover, if $\mu \in \mathrm{Aut}(B) \cong \mathcal{C}_{q-1}$, then μ fixes a and centralises $\gamma(a)$ for $a \in A$, so that it stabilises γ.

Now, let $\delta \in \mathrm{Aut}(A) \cong \mathrm{GL}(2, p)$. If δ stabilises γ, then $\gamma^{\delta}(a_1) = 1$, namely $\gamma(a_1^{\delta^{-1}}) = 1$. Therefore δ^{-1} fixes $\langle a_1 \rangle$, and writing $\delta = (\delta_{ij})_{i,j}$ with respect to the basis $\{a_1, a_2\}$, this implies that $\delta_{12} = 0$.

As for a_2, we have

$$\gamma^{\delta}(a_2) = \delta^{-1}\gamma(a_2^{\delta^{-1}})\delta = \delta^{-1}\alpha^{\delta_{22}^{-1}}\delta,$$

and it coincides with $\gamma(a_2)$ precisely when $\delta^{-1}\alpha^{\delta_{22}^{-1}}\delta = \alpha$. Taking α as in Subsubsection 4.3.2, an explicit computation shows that the latter yields $\delta_{11} = \delta_{22}^2$. Therefore, the stabiliser of γ has order $(q - 1)p(p - 1)$, and there is one orbit of length $p^2 - 1$.

In the following we exclude the abelian cases just dealt with.

5.2. $p \mid q - 1$. Here $B \leq \ker(\gamma)$, and the only type of groups we can have here is the type 11, beside the type 5 already considered.

Suppose first $\ker(\gamma) = \langle a_1 \rangle B$ has order pq. Then $\gamma(G)$ has order p, and let a_2 be such that $\gamma(a_2) = \alpha^i\eta^j$, where $0 \leq i < p$, $j \neq 0$ (since we are assuming $(G, \circ)$ is non abelian). The argument in 4.3.2 shows that $a_1^{\alpha} = a_1$, and Lemma A.2 yields that γ is a RGF if and only if it is a morphism. Therefore, the GF's are as many as the choices of $(\langle a_1 \rangle, i, j)$, namely $(p + 1)p(p - 1) = p(p^2 - 1)$, and each of them corresponds to a group $(G, \circ)$ of type 11.

As to the conjugacy classes, again $\mathcal{C}_{q-1}$ stabilises every γ. Moreover, if $\delta \in \mathrm{GL}(2, p)$ stabilises γ, then δ^{-1} fixes $\langle a_1 \rangle$, so that $\delta_{12} = 0$. This time

$$\gamma^{\delta}(a_2) = \delta^{-1}\gamma(a_2^{\delta_{22}^{-1}})\delta = \delta^{-1}\alpha^{i\delta_{22}^{-1}}\delta\eta^{j\delta_{22}^{-1}},$$

where $j \neq 0$. Therefore, δ stabilises γ precisely when $\delta_{12} = 0$, $\delta_{22} = 1$, and δ centralises α^i. If $i = 0$, the latter yields no condition, while it corresponds to take $\delta_{11} = 1$ if $i \neq 0$. So the δ's in the stabiliser are those of the form

$$\delta = \begin{bmatrix} \delta_{11} & 0 \\ \delta_{21} & 1 \end{bmatrix} \text{ if } i = 0, \text{ and } \delta = \begin{bmatrix} 1 & 0 \\ \delta_{21} & 1 \end{bmatrix} \text{ if } i \neq 0.$$

Therefore, if $i = 0$ the stabiliser has order $(q-1)p(p-1)$, and there is one orbit of length p^2-1. If $i \neq 0$, the stabiliser has order $(q-1)p$, and there is one orbit of length $(p^2-1)(p-1)$.

Now suppose $\ker(\gamma) = B$ has order q. Then $\gamma(G) = \gamma(A) = \langle \alpha, \eta \rangle$. Let $a_1, a_2 \in A$ be such that

$$\begin{cases} \gamma(a_1) = \eta \\ \gamma(a_2) = \alpha \end{cases} . \tag{5.1}$$

Since

$$a_1^{\alpha} a_2 = a_1 \circ a_2 = a_2 \circ a_1 = a_2 a_1,$$

a_1 is a fixed point of α, and since α has determinant equal to 1, we can suppose

$$\alpha = \begin{bmatrix} 1 & 0 \\ d & 1 \end{bmatrix},$$

with respect to $\{a_1, a_2\}$, where $1 \leq d \leq p-1$. By Lemma 2.12 and Lemma A.1, each assignment (5.1) defines exactly one GF. Therefore, in this case, we have $p+1$ choices for $\gamma(G)$, and once $\gamma(G)$ has been chosen, there are $p-1$ ways to choose a_1 among the fixed points of α, and p^2-p choices for a_2, which is any element of $A \setminus \langle a_1 \rangle$. So there are $(p^2-1)p(p-1)$ groups of type 11.

As to the conjugacy classes, every automorphism in $\mathcal{C}_{q-1}$ stabilises γ. Since $B = \ker(\gamma)$ is characteristic, by Lemma 2.12, we just consider the action of $\mathrm{GL}(2,p)$ on γ defined on the generators of A.

Let $\delta \in \mathrm{GL}(2,p)$. Then, $\gamma^{\delta}(a_1) = \gamma(a_1)$ if and only if $\gamma(a_1^{\delta^{-1}}) = \gamma(a_1)$, as δ^{-1} centralises η. The latter yields $\gamma(a_1^{\delta^{-1}})_{|A} = 1$, so that $a_1^{\delta^{-1}} \in \langle a_1 \rangle$, namely $\delta_{12} = 0$. Moreover, since $\gamma_{|\langle a_1 \rangle}$ is a morphism, $\gamma(a_1^{\delta^{-1}}) = \eta$ if and only if $\delta_{11} = 1$. Now, since

$$\gamma(a_2^k) = \gamma(a_1)^{-d(\frac{k(k-1)}{2})} \gamma(a_2)^k = \eta^{-d(\frac{k(k-1)}{2})} \alpha^k,$$

we have

$$\gamma^{\delta}(a_2) = \delta^{-1}\gamma(a_2^{\delta^{-1}})\delta = \delta^{-1}\gamma(a_1^{-\delta_{21}\delta_{22}^{-1}} a_2^{\delta_{22}^{-1}})\delta = \eta^{-\delta_{22}^{-1}(\delta_{21}+\frac{d}{2}(\delta_{22}^{-1}-1))}\delta^{-1}\alpha^{\delta_{22}^{-1}}\delta,$$

and the latter coincides with $\gamma(a_2)$ precisely when

$$\begin{cases} \delta^{-1}\alpha^{\delta_{22}^{-1}}\delta = \alpha \\ \delta_{21} = -\frac{d}{2}(\delta_{22}^{-1} - 1). \end{cases}$$

The first condition yields $\delta_{22}^2 = 1$, namely $\delta_{22} = \pm 1$, so that the second yields $\delta_{21} = 0, d$ respectively when $\delta_{22} = 1, -1$. Therefore, the stabiliser has order $2(q-1)$ and we get 2 orbits of length $\frac{1}{2}(p^2-1)p(p-1)$.

5.3. $q \mid p-1$. Here $\gamma(G) \subseteq \mathrm{GL}_2(p)$, so $p \mid |\ker(\gamma)|$ and $\gamma(G)$ acts trivially on B, so that

$$b^{\ominus 1} \circ a \circ b = b^{-\gamma(b)^{-1}\gamma(a)\gamma(b)} a^{\gamma(b)} b = b^{-1} b a^{\gamma(b)} = a^{\gamma(b)}. \tag{5.2}$$

If $pq \mid \ker(\gamma)$, then equation (5.2) becomes

$$b^{\ominus 1} \circ a \circ b = a,$$

so $(G, \circ)$ of type 5 and has already been considered. Thus we just deal with the cases of kernel p^2 and p.

If $\ker(\gamma) = A$ the GF's are exactly the morphisms. Let $\lambda \in \mathbb{Z}/p\mathbb{Z}$ be an element of multiplicative order q. By Subsubsection 4.3.4, with respect to a suitable basis $\{a_1, a_2\}$ of A, we have

$$T = [\gamma(b)] = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix}.$$

Now, since $a^t = a^{\circ t}$ for all t, equation (5.2) yields that the action of b on A in $(G, \circ)$ is precisely $\gamma(b)$. According to the choices of $\gamma(b)$ we easily obtain, besides the abelian cases,

(1) $q-1$ groups of type 7, corresponding to the choices $x_1 = x_2 \neq 0$.
(2) $\frac{(p+1)p}{2} \cdot 2 \cdot (q-1)$ groups of type 6: choose the eigenspaces, and then the eigenvalue different from 1.
(3) if $q > 2$, we get $\frac{(p+1)p}{2} \cdot (q-1)$ groups of type 9.
(4) if $q > 3$ we get $\frac{(p+1)p}{2} \cdot (q-1)(q-3)$ groups of type 8. More precisely, denoting by $Z_\circ$ the action of b on A in $(G, \circ)$, since $Z_\circ$ is similar to $\mathrm{diag}(\mu^{x_1 x_2^{-1}}, \mu)$, where $\mu = \lambda^{x_2}$, they split in $p(p+1)(q-1)$ groups isomorphic to G_s for every $s \in \mathcal{K}$.

Remark 5.1. *In the following we will write $Z_1 \sim Z_2$ to mean that the two square matrices Z_1, Z_2 are similar.*

As to the conjugacy classes, since $A = \ker(\gamma)$ is characteristic, $\gamma_{|A}$ is stabilised by every automorphism φ of G.

As for $\gamma_{|B}$, let $\mu \in \mathcal{C}_{q-1}$, so that $b^{\mu^{-1}} = b^m$ for some m, and let $\delta \in \mathrm{GL}(2, p)$. Then

$$\gamma^{\mu\delta}(b) = \delta^{-1}\gamma(b)^m \delta.$$

Therefore, $\mu\delta$ stabilises γ precisely when T and T^m are conjugate via δ, and in that case they need to have the same eigenvalues, namely $mx_1 = x_1$ and $mx_2 = x_2$ or $mx_1 = x_2$ and $mx_2 = x_1$. Note that if $m = 1$, then δ stabilises γ if and only if it is in the centraliser of T: if T is scalar, then every $\delta \in \mathrm{GL}(2, p)$ stabilises γ, while for a non-scalar matrix T the condition is equivalent to have δ a diagonal matrix with no diagonal elements equal to zero.

Referring to the cases above, we have the following.

(1) T is scalar and $T \sim T^m$ if and only if $m = 1$, so that the stabiliser has order $|\mathrm{GL}(2, p)|$, and there is one orbit of length $q-1$.

(2) T is non-scalar and $m = 1$. In this case the centraliser of T consists of the elements $\delta = \operatorname{diag}(\delta_{11}, \delta_{22})$, with $\delta_{ii} \neq 0$, therefore it has $(p-1)^2$ elements. Thus $|\operatorname{Stab}(\gamma)| = (p-1)^2$, and there is one orbit of length $p(p+1)(q-1)$.

(3) T is non-scalar and $m = \pm 1$. If $m = 1$ then the elements in the stabiliser are the diagonal matrices as above. If $m = -1$ the stabiliser consists of the elements $\mu\delta$, where $b^{\mu^{-1}} = b^{-1}$, and

$$\delta = \begin{bmatrix} 0 & \delta_{12} \\ \delta_{21} & 0 \end{bmatrix},$$

where $\delta_{12} \neq 0 \neq \delta_{21}$. Therefore $|\operatorname{Stab}\gamma| = 2(p-1)^2$, and there is one orbit of length $\frac{1}{2}p(p+1)(q-1)$.

(4) T is non-scalar and $m = 1$, indeed if $mx_1 = x_2$ and $mx_2 = x_1$, then $x_2^{-1}x_1 = m = x_1^{-1}x_2$, namely $x_1 = \pm x_2$ (contradiction). Therefore $|\operatorname{Stab}\gamma| = (p-1)^2$, and for each G_s, $s \in \mathcal{K}$, there is one orbit of length $p(p+1)(q-1)$.

If $\ker(\gamma) = \langle a_1 \rangle$ has order p, then $\gamma(G)$ is a subgroup of order pq of $\mathrm{GL}(2,p)$, so $\gamma(G) = \langle \alpha, \beta \rangle$, where α has order p, $a_1^{\alpha} = a_1$, and β is an element of order q in the normaliser of $\langle \alpha \rangle$ in $\mathrm{GL}(2,p)$. By Subsubsection 4.3.4, we can choose $a_2 \in A$ such that, with respect to the basis $\{a_1, a_2\}$,

$$[\alpha] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ [\beta] = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix},$$

where λ has multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$, and $(x_1, x_2) \neq (0,0)$. Since A is characteristic, a_2 has order p in $(G, \circ)$, so that $\gamma(a_2)$ is an element of order p (as $a_2 \notin \ker(\gamma)$). Therefore $\gamma(a_2) = \alpha^d$, where $1 \le d \le p-1$. Moreover, let $b \in B$ be such that $\gamma(b) = \beta$.

By applying γ to (5.2), we get

$$\gamma(b)^{-1}\gamma(a)\gamma(b) = \gamma(a^{\gamma(b)})$$

which for $a = a_2$, in terms of our notation, can be rewritten as

$$\begin{aligned} \beta^{-1}\alpha^d\beta &= \alpha^{d\lambda^{x_2}} \\ \alpha^{d\lambda^{x_1 - x_2}} &= \alpha^{d\lambda^{x_2}} \end{aligned}$$

which correspond to the condition

$$x_1 \equiv 2x_2 \pmod{q}. \tag{5.3}$$

This condition restricts the choices of β to a set of $(q-1)p$ maps, namely the elements of order q in the normaliser of $\langle \alpha \rangle$ with diagonal $\lambda^{2x_2}, \lambda^{x_2}$. Thus for each choice of $\langle \alpha \rangle$ only one group of order pq can be the image of a GF.

We note that the maps β fulfilling equation (5.3) normalise but do not centralise $\langle \alpha \rangle$, so $\langle \alpha, \beta \rangle$ is not abelian.

The condition (5.3) is also sufficient to have that the map γ, defined as

$$\gamma(a_1^e a_2^f b^g) = \beta^g \alpha^f,$$

is a gamma function, indeed we have

$$\begin{aligned}\gamma((a_1^e a_2^f b^g)^{\gamma(a_1^u a_2^v b^z)} a_1^u a_2^v b^z) &= \gamma((a_1^e a_2^f b^g)^{\beta^z \alpha^v} a_1^u a_2^v b^z)\\ &= \gamma((a_1^* a_2^{f\lambda^{x_2 z}} b^g) a_1^u a_2^v b^z)\\ &= \gamma(a_1^* a_2^{f\lambda^{x_2 z}+v} b^{g+z})\\ &= \beta^{g+z}\alpha^{f\lambda^{x_2 z}+v}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\gamma(a_1^e a_2^f b^g)\gamma(a_1^u a_2^v b^z) &= \beta^g \alpha^f \beta^z \alpha^v\\ &= \beta^{g+z}\alpha^{f\lambda^{(x_1-x_2)z}+v},\end{aligned}$$

so that γ defined as above is a GF if and only if $x_1 \equiv 2x_2 \pmod q$.

Moreover, since we have $p+1$ choices for $\langle \alpha \rangle$, $p-1$ for d, and $p(q-1)$ for β, we obtain $p(p^2-1)(q-1)$ groups $(G, \circ)$.

As for the type of $(G, \circ)$, with respect to the basis $\{a_1, a_2\}$ we have

$$T = [\beta] = \begin{bmatrix} \lambda^{2x_2} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix}.$$

Since $a_1^t = a_1^{\circ t}$ and $a_2^t = a_2^{\circ t}$ modulo $\langle a_1 \rangle$ for all t, denoting by $Z_\circ$ the action of b on A in $(G, \circ)$, we have $Z_\circ \sim T$. Therefore,

- if $q > 3$ all groups $(G, \circ)$ are of type 8, and they are all isomorphic to G_2;
- if $q = 3$ all groups $(G, \circ)$ are of type 9;
- if $q = 2$ we have $x_1 = 0$, $x_2 = 1$, so all groups $(G, \circ)$ are of type 6.

As to the conjugacy classes, let $\varphi \in \mathrm{Aut}(G)$, and write $\varphi = \mu\delta$ as above. Recall that $b^{\mu^{-1}} = b^m$. If φ is in the stabiliser of γ then φ, and hence δ, stabilises $\langle a_1 \rangle$, so $\delta_{12} = 0$. Moreover,

$$\gamma^\varphi(a_2) = \varphi^{-1}\gamma(a_2^{\delta^{-1}})\varphi = \varphi^{-1}\gamma(a_2^{\delta_{22}^{-1}})\varphi = \delta^{-1}\alpha^{\delta_{22}^{-1}}\delta,$$

and $\gamma^\varphi(a_2) = \gamma(a_2)$ if and only if $\delta_{11} = \delta_{22}^2$. Now,

$$\gamma^\varphi(b) = \varphi^{-1}\gamma(b^{\mu^{-1}})\varphi = \varphi^{-1}\gamma(b^m)\varphi = \delta^{-1}T^m\delta,$$

so that, if φ stabilises γ, then T and T^m are conjugate, and they have the same eigenvalues. This implies that either $m = 1$ or $m = 2$ and $q = 3$. If $m = 1$, then every diagonal matrix δ commutes with T. If $q = 3$ and $m = 2$, then the condition $\delta^{-1}T^{-1}\delta = T$ yields $\lambda^{x_2} = \lambda^{-x_2}$, and since $x_2 \neq 0$ this case does not arise. Therefore the stabiliser has order $p-1$, and there is one orbit of length $p(p^2-1)(q-1)$.

5.4. $q \mid p+1$. We have to exclude the cases already considered, so we restrict to $q > 2$ (otherwise q also divides $p-1$) and $(G, \circ)$ non-abelian. Therefore, $(G, \circ)$ can only have type 10.

As in Subsection 5.3, $\gamma(G) \subseteq \mathrm{GL}_2(p)$, so $p \mid |\ker(\gamma)|$. The only possibility is $|\ker(\gamma)| = p^2$ since a group of type 10 has no normal subgroups of order p or pq.

Lemma 2.5 guarantees that in this case all the GF's are morphisms, so to count them we can just count the possibilities for the image of b.

An element $\vartheta \in \mathrm{GL}(2, p)$ of order q has determinant equal to 1, as $q \nmid p-1$, and its eigenvalues λ, λ^{-1} belong to a quadratic extension of $\mathcal{C}_p$. Therefore, every subgroup of $\mathrm{GL}(2, p)$ of order q is conjugate to $\langle \vartheta \rangle$, and in $\mathrm{GL}(2, p)$ there are

$$\frac{|\mathrm{GL}(2, p)|}{|\mathrm{Stab}(\langle \vartheta \rangle)|}$$

subgroups of order q. Now, if ϑ and ϑ^k are conjugate, they have the same eigenvalues, and this yields $k = \pm 1$. For each of these two choices we obtain $p^2 - 1$ elements in the stabiliser, therefore there are

$$\frac{(p^2-1)(p^2-p)}{2(p^2-1)} = \binom{p}{2}$$

subgroups of order q in $\mathrm{GL}(2, p)$.

So we can choose the image of b in such a subgroup in $q-1$ ways, and we get

(1) $\binom{p}{2}(q-1)$ groups of type 10.

As to the conjugacy classes, $A = \ker(\gamma)$ is characteristic, therefore every automorphism φ of G stabilises $\gamma_{|A}$.

Let $b \in B$ be such that $\gamma(b) = \vartheta$, and let $\varphi = \mu\delta \in \mathrm{Aut}(G)$, where $\mu \in \mathcal{C}_{q-1}$, $\delta \in \mathrm{GL}(2, p)$ and m is such that $b^{\mu^{-1}} = b^m$. Then

$$\gamma^{\varphi}(b) = \delta^{-1}\gamma(b^m)\delta,$$

so that φ stabilises γ if and only if ϑ and ϑ^m are conjugate via δ. As above, in this case $m = \pm 1$, and for each of these values of m there are $p^2 - 1$ possibilities for δ.

Therefore, we get one orbit of length $\frac{1}{2}(q-1)p(p-1)$.

We summarise, including the right regular representation.

Proposition 5.2. *Let G be a group of order p^2q, $p > 2$, of type 5. For each isomorphism class of groups (Γ), the number of regular subgroups in* Hol(G) *(RS), and the number (n) and the lengths (l) of the conjugacy classes in* Hol(G) *are listed in the following tables.*

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

(i)

Γ	RS	n	l
5	p^2	1	1
		1	(p^2-1)

(ii) for $p \mid (q-1)$:

Γ	RS	n	l
5	p^2	1	1
		1	(p^2-1)
11	$p^2(p^2-1)$	1	(p^2-1)
		1	$(p-1)(p^2-1)$
		2	$\frac{1}{2}p(p-1)(p^2-1)$

(iii) for $q \mid (p-1)$,

Γ	*Conditions*	RS	n	l
5		p^2	1	1
			1	p^2-1
6		$p(p+1)(q-1)$	1	$p(p+1)(q-1)$
	$q=2$	$p(p^2-1)$	1	$p(p^2-1)$
7		$q-1$	1	$q-1$
8_{G_2}	$q>3$	$p^2(p+1)(q-1)$	1	$p(p+1)(q-1)$
			1	$p(p^2-1)(q-1)$
8_{G_s}, $s \neq 2$	$q>3$	$p(p+1)(q-1)$	1	$p(p+1)(q-1)$
9	$q>2$	$\frac{1}{2}p(p+1)(q-1)$	1	$\frac{1}{2}p(p+1)(q-1)$
	$q=3$	$p(p^2-1)(q-1)$	1	$p(p^2-1)(q-1)$

In the row of 8_{G_s} we mean that for every $s \in \mathcal{K}$, $s \neq 2$, there are $p(p+1)(q-1)$ regular subgroups isomorphic to G_s.

(iv) for $q \mid (p+1)$ and $q > 2$:

Γ	RS	n	l
5	p^2	1	1
		1	(p^2-1)
10	$\frac{1}{2}p(p-1)(q-1)$	1	$\frac{1}{2}p(p-1)(q-1)$

6. Type 6

In this case $q \mid p-1$, and $G = \mathcal{C}_p \times (\mathcal{C}_p \rtimes \mathcal{C}_q)$. The Sylow p-subgroup A is characteristic in G. Write $C = \langle c \rangle$ for the normal subgroup of order p in $\mathcal{C}_p \rtimes \mathcal{C}_q$, and $Z = \langle z \rangle$ for the central factor of order p, so that $A = CZ = \langle c, z \rangle$.

We have

$$\operatorname{Aut}(G) = \mathcal{C}_{p-1} \times \operatorname{Hol}(\mathcal{C}_p).$$

Write $\langle \psi \rangle = \mathcal{C}_{p-1}$ for the central factor in $\operatorname{Aut}(G)$, and let $\operatorname{Hol}(\mathcal{C}_p) = \iota(C) \rtimes \langle \mu \rangle$, where, according to [8],

$$\psi : \begin{cases} z \mapsto z^k \\ c \mapsto c \\ b \mapsto b \end{cases}, \quad \mu : \begin{cases} z \mapsto z \\ c \mapsto c^h \\ b \mapsto b \end{cases}, \tag{6.1}$$

for some $1 \leq h, k \leq p-1$, and where $b \in G$ is chosen in such a way that $\iota(b) \in \langle \mu \rangle$.

$\mathrm{Aut}(G)$ contains a unique Sylow p-subgroup of order p, namely $\iota(C)$. C is characteristic in G, so that it is a subgroup of $(G, \circ)$ too. Therefore $\gamma(C)$ is a subgroup of $\mathrm{Aut}(G)$, and necessarily it is contained in $\iota(C)$. We can apply Proposition 2.9 and assume that $C \le \ker(\gamma)$. Then, by Corollary 2.10, to count all the GFs we simply double the numbers obtained.

In the following, it is useful to keep in mind that

$$\begin{cases} b^{\ominus 1} \circ c \circ b = c^{\gamma(b)\iota(b)} \\ b^{\ominus 1} \circ z \circ b = (b^{-\gamma(b)^{-1}\gamma(z)\gamma(b)} b) z^{\gamma(b)}. \end{cases}$$

6.1. The case $A \le \ker(\gamma)$. Suppose $\ker(\gamma) = A$, as the case $\ker(\gamma) = G$ yields the right regular representation. So $\gamma(G)$ has order q.

The action of $\gamma(G)$ of order q on the set of the Sylow q-subgroups of G has at least one fixed point, namely there is at least one $\gamma(G)$-invariant Sylow q-subgroup B of G. Therefore, by Proposition 2.6, the GF's on G are induced by the RGF's on B, and each γ is obtained r times, where r is the number of $\gamma(G)$-invariant Sylow q-subgroups of G.

Note moreover that $[B, \gamma(B)] = 1$, as B and $\gamma(B)$ have order q ($\gamma(B) = \gamma(G)$, see Corollary 2.4), so that by Lemma 2.5 the RGF's on B are precisely the morphisms $B \to \mathrm{Aut}(G)$.

Let β be the element of order q in the central factor $\mathcal{C}_{p-1}$ of $\mathrm{Aut}(G)$, such that $z^\beta = z^\lambda$, where λ is the eigenvalue of C under the action of b, namely $c^b = c^\lambda$.

Here $c^{\circ t} = c^t$ and $z^{\circ t} = z^t$ for all t. Let $Z_\circ$ be the action of b on A in $(G, \circ)$. We will write $Z_\circ$ with respect to the basis $\{c, z\}$ of $(A, \circ)$.

(1) If $\gamma(b) = \beta^i$, for some $0 < i < q$, then $Z_\circ = \mathrm{diag}(\lambda, \lambda^i)$. Here the choice of B is immaterial, and we get
  (a) 1 group of type 7 when $i = 1$;
  (b) 1 group of type 9 when $i = q - 1$ and $q > 2$;
  (c) $q - 3$ groups of type 8, when $q > 3$. They split in 2 groups isomorphic to G_s for every $s \in \mathcal{K}$.
(2) If $\gamma(b) = \iota(b)^j$, for some $0 < j < q$, then $Z_\circ = \mathrm{diag}(\lambda^{j+1}, 1)$ and we get
  (a) p groups of type 5 when $j = q - 1$, for the possible choices of B;
  (b) $p(q - 2)$ groups of type 6 when $0 < j < q - 1$, for the possible choices of B.
(3) If $\gamma(b) = \beta^i \iota(b)^j$, for some $0 < i, j < q$, then $Z_\circ = \mathrm{diag}(\lambda^{1+j}, \lambda^i)$ and we get
  (a) $p(q - 1)$ groups of type 6, when $j = q - 1$;
  (b) $p(q - 2)$ of type 7, when $i = j + 1 \neq 0$;
  (c) $p(q - 2)$ of type 9, when $-i = j + 1 \neq 0$ and $q > 2$;
  (d) $p((q - 1)^2 - 3q + 5) = p(q - 2)(q - 3)$ groups of type 8 in the remaining cases; they occur only for $q > 3$. They split in $2p(q - 2)$ groups isomorphic to G_s, for every $s \in \mathcal{K}$.

As to the conjugacy classes, since $A = \ker(\gamma)$ is characteristic, to find the automorphisms which stabilise γ, we can look at the action of $\mathrm{Aut}(G)$ on $\gamma_{|B}$.

The central factor $\langle \psi \rangle$ of $\mathrm{Aut}(G)$ and $\langle \mu \rangle$ are in the stabiliser of γ, as they fix b and centralise $\gamma(b)$. As for $\iota(C)$, for $\gamma(b) = \beta^i \iota(b)^j$, we have

$$\gamma^{\iota(c^m)}(b) = \iota(c^{-m})\gamma(b)\iota(c^m) = \beta^i \iota(b^j c^{m(1-\lambda^j)}),$$

so that $\iota(c^m)$ stabilises γ if and only if $m = 0$ or $j = 0$.

Therefore, if γ is a GF defined by $\gamma(b) = \beta^i \iota(b)^j$, $j \neq 0$, the stabiliser has order $(p-1)^2$, and the orbits have length p. Otherwise $\gamma(b) = \beta^i$ and every automorphism stabilises γ, so that the orbits have length 1. More precisely we obtain

(1) p groups of type 5 which form one class of length p;
(2) $p(q-2) + p(q-1) = p(2q-3)$ groups of type 6 which split in $2q-3$ classes of length p;
(3) $p(q-2)+1$ groups of type 7, which split in $q-2$ classes of length p and one class of length one (the last one is for $j = 0$).
(4) if $q > 3$, $2p(q-2)+2$ groups for each isomorphism class G_s of groups of type 8, which split in $2(q-2)$ classes of length p, and 2 classes of length one (these are for $j = 0$).
(5) if $q > 2$, $p(q-2)+1$ groups of type 9, which split in $q-2$ classes of length p, and one class of length one (this is for $j = 0$).

6.2. The case $C \leq \ker(\gamma) \neq A$. Suppose now $C \leq \ker(\gamma) \neq A$, so that we will have $\gamma(z) = \iota(c)^e$, for some $e \neq 0$. If $\gamma(b)$ is a (possibly trivial) q-element in $\gamma(G)$, then b is a q-element in G, and we will have

$$\gamma(b) \in \langle \beta, \iota(bc^m) \rangle$$

for some m.

Recall that λ is the eigenvalue of C under the action of b. If $\gamma(b) = \beta^t$, for some t, then $\gamma(G)$ is abelian, so that $(G, \circ)$ is of type 5 or 6. However,

$$b^{\ominus 1} \circ c \circ b = c^{\gamma(b)\iota(b)} = c^{\lambda} = c^{\circ \lambda} \neq c,$$

so that $(G, \circ)$ is not abelian, and thus of type 6.

We also have

$$b^{\ominus 1} \circ z \circ b = b^{-\gamma(b)^{-1}\gamma(z)\gamma(b)} z^{\gamma(b)} b \equiv_{\bmod C} z^{\lambda^t} = z^{\circ \lambda^t},$$

so that $t = 0$, as $(G, \circ)$ has to be of type 6. Therefore the kernel has order pq, $\gamma(G) = \gamma(Z)$ and $[Z, \gamma(Z)] = 1$, so that by Proposition 2.6 and Lemma 2.5 the GF's on G are precisely the morphisms $Z \to \mathrm{Aut}(G)$, which are as many as the choices for e, namely $p-1$.

If $\gamma(b) = \beta^t \iota(bc^m)^l$ for some $l \neq 0$ and t, replacing b with bc^m we see that we can take $m = 0$.

We have

$$b^{\ominus 1} \circ c \circ b = c^{\gamma(b)\iota(b)} = c^{\lambda^{l+1}} = c^{\circ \lambda^{l+1}}.$$

Then

$$b^{\ominus 1} \circ z \circ b = b^{-\gamma(b)^{-1}\gamma(z)\gamma(b)} z^{\gamma(b)} b \equiv_{\bmod C} z^{\lambda^t} = z^{\circ \lambda^t}.$$

However
$$\gamma(b^{\ominus 1}\circ z\circ b) = \gamma(b)^{-1}\gamma(z)\gamma(b) = \iota(c)^{e\lambda^l} = \gamma(z)^{\lambda^l}.$$
It follows that $t = l$.

The latter is also a sufficient condition in order to have that the map γ defined by
$$\gamma(b^m c^k z^n) = \beta^{mt}\iota(b^{ml}c^{ne})$$
satisfies the GFE. Indeed
$$\begin{aligned}\gamma(b^m c^k z^n)\gamma(b^u c^v z^w) &= \beta^{mt}\iota(b^{ml}c^{ne})\beta^{ut}\iota(b^{ul}c^{we})\\ &= \beta^{(m+u)t}\iota(b^{(m+u)l}c^{(n\lambda^{ul}+w)e}),\end{aligned}$$

$$\begin{aligned}\gamma((b^m c^k z^n)^{\gamma(b^u c^v z^w)}b^u c^v z^w) &= \gamma((b^m c^k z^n)^{\beta^{ut}\iota(b^{ul}c^{we})}b^u c^v z^w)\\ &= \gamma((b^m c^* z^{n\lambda^{ut}})b^u c^v z^w)\\ &= \gamma((b^{m+u}c^* z^{n\lambda^{ut}+w})\\ &= \beta^{(m+u)t}\iota(b^{(m+u)l}c^{(n\lambda^{ut}+w)e}),\end{aligned}$$

and they are equal if and only if $l = t$.

As for $(G, \circ)$ we have that $Z_\circ \sim \mathrm{diag}(\lambda^{t+1}, \lambda^t)$.

(1) For $t = -1$ we get $p(p-1)$ groups of type 6, with p choices for B and $p-1$ choices for e.
(2) For $q > 2$ and $t = (q-1)/2$ we have $\lambda^{t+1}\lambda^t = \lambda^{2t+1} = 1$, so $p(p-1)$ groups of type 9.
(3) For $q > 3$ for each of the remaining $q-3$ values of t, we get $p(p-1)$ groups of type 8, so $(q-3)p(p-1)$ in total. They split in $2p(p-1)$ groups isomorphic to G_s, for every $s \in \mathcal{K}$.

As to the conjugacy classes, write $\varphi = \psi\iota(c^m)\mu$ for an automorphism of G, with ψ and μ as in (6.1). Here $C \leq \ker(\gamma)$ is characteristic, so that by Lemma 2.12, we can look at the action of φ on γ defined on the generators z, b.

Write $\mu^{-1}\iota(c^m)\mu = \iota(c^m)^r$ for the commutation rule in $\mathrm{Hol}(\mathcal{C}_p)$, where $1 \leq r \leq p-1$. Then
$$\gamma^\varphi(z) = \varphi^{-1}\gamma(z^{\psi^{-1}})\varphi = \mu^{-1}\iota(c^{ek^{-1}})\mu = \iota(c^{ek^{-1}})^r,$$
so that $\gamma^\varphi(z) = \gamma(z)$ if and only if $k = r$. Moreover, for $\gamma(b) = \beta^t\iota(b)^t$,
$$\begin{aligned}\gamma^\varphi(b) &= \varphi^{-1}\gamma(c^{m(1-\lambda^{-1})}b)\varphi\\ &= \mu^{-1}\iota(c^{-m})\gamma(b)\iota(c^m)\mu\\ &= \beta^t\mu^{-1}\iota(c^{-m})\iota(b^t)\iota(c^m)\mu\\ &= \beta^t\mu^{-1}\iota(b^t c^{m(1-\lambda^t)})\mu\\ &= \beta^t\iota(b^t)\iota(c^{m(1-\lambda^t)})^r,\end{aligned}$$
so that $\gamma^\varphi(b) = \gamma(b)$ if and only if $t = 0$ or $m = 0$.

Therefore, if $t = 0$, namely when $\ker(\gamma)$ has size pq, the stabiliser has order $p(p-1)$, and there is one orbit of length $p-1$. Otherwise, if $t \neq 0$, namely $\ker(\gamma)$ has size p, then the stabiliser has order $p-1$, and there are $q-1$ orbits of length $p(p-1)$.

We summarise, including the right regular representations, and doubling the numbers just obtained.

Proposition 6.1. *Let G be a group of order p^2q, $p > 2$, of type 6. For each isomorphism class of groups (Γ), the number of regular subgroups in* $\mathrm{Hol}(G)$ *(RS), and the number (n) and the lengths (l) of the conjugacy classes in* $\mathrm{Hol}(G)$ *are listed in the following table.*

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

Γ	*Conditions*	*RS*	n	l
5		$2p$	2	p
6		$2p(p+2q-3)$	2	1
			$2(2q-3)$	p
			2	$p-1$
			2	$p(p-1)$
7		$2(p(q-2)+1)$	2	1
			$2(q-2)$	p
8_{G_s}	$q > 3$	$4(1+p(p+q-3))$	4	1
			$4(q-2)$	p
			4	$p(p-1)$
9	$q > 2$	$2(1+p(p+q-3))$	2	1
			$2(q-2)$	p
			2	$p(p-1)$

In the row of 8_{G_s} we mean that for every $s \in \mathcal{K}$ there are $4(1+p(p+q-3))$ *regular subgroups isomorphic to G_s.*

7. Prologue to Sections 8, 9 and 10

In this section we collect some arguments which are common to the study of the groups of types 7, 8 and 9.

In these groups the Sylow p-subgroup A of G is characteristic. With respect to a suitable basis a_1, a_2, the action of a generator b of a Sylow q-subgroup B on A can be represented by the matrix

$$Z = Z_k = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^k \end{bmatrix},$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$ and $k \neq 0$ is an integer modulo q. If $k = 1$ the type is 7, if $k = -1$ the type is 9, and if $k \neq 0, \pm 1$ the type is 8.

Remark 7.1. *Note that if we choose as a basis a_2, a_1, then the action of the generator $b^{k^{-1}}$ of B on A is represented by the matrix $Z_{k^{-1}}$.*

For types 8 and 9, each gamma function γ is such that

$$\text{there is } \sigma \in \operatorname{End}(A) \text{ such that } \gamma(a) = \iota(a^{-\sigma}), \text{ for all } a \in A \tag{7.1}$$

(see Sections 8, 9). This is not always the case for groups of type 7 (see Section 10). Therefore, in this section we will work under the following:

Assumption 7.2.
(G, γ) is a pair, where G is a group, and γ is a GF on G, such that

- G is a group of type 8 or 9, and γ is arbitrary, or
- G is a group of type 7 and γ on G satisfies $\gamma(A) \leq \operatorname{Inn}(G)$.

Under this assumption we have $\gamma(A) \leq \operatorname{Inn}(G)$. This implies $\gamma(A) \leq \iota(A)$, since $\gamma(a)$, for $a \in A$, has order dividing p^2, and thus has to be a conjugation by an element of G. Lemma 2.7 shows that the condition $\gamma(A) \leq \iota(A)$ implies (7.1).

We will prove that for a pair (G, γ) as in Assumption 7.2, the group G admits an invariant Sylow q-subgroup B, so that γ can be obtained as a *lifting* of a RGF on B or as a *gluing* of a RGF on B and the map $a \mapsto \iota(a^{-\sigma})$, for some $\sigma \in \operatorname{End}(A)$. It follows that γ will be of the form (2.7).

To count each GF exactly once we will also need to determine the exact number of invariant Sylow q-subgroups in each case.

A proof of these facts will require a detailed analysis, that will be carried out in several steps in the next subsections. We will then complete the classification for the three types in Sections 8, 9, 10.

7.1. Invariant Sylow q-subgroups of G. When $q \mid |\ker(\gamma)|$, there is a Sylow q-subgroup contained in $\ker(\gamma)$, and this is clearly invariant.

Consider thus the case $q \nmid |\ker(\gamma)|$, so that $\gamma(G)$ contains an element of order q. In this subsection we give a characterization of the invariant Sylow q-subgroups of G, that we will then use to count them for the various types.

Under Assumption 7.2, an element of order q of $\operatorname{Aut}(G)$ has the form $\iota(a_*)\beta$, where $a_* \in A$, and β is an element of order q. As we will explain in Subsections 8.2.1, 9.2 and 10.2, an element β of order q fixes every $b \in B$. Moreover, with respect to the chosen basis, β acts as

$$T = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{kx_2} \end{bmatrix}, \tag{7.2}$$

where λ and k are as above, and x_1 and x_2 are not both zero (this will be detailed in Sections 8, 9 and 10).

Note that if an element of order q in $\operatorname{Aut}(G)$ belongs to $\gamma(G)$, then necessarily it will be the image of an element $b \in G$ of order q. In fact, the elements of a group G of type 7, 8 or 9 can have order 1, p or q. The elements whose order divides p are those of the Sylow p-subgroup A, which is characteristic, so that combining Proposition 2.3 and Theorem 2.2, (iv) for $a \in A$, $\gamma(a)$ has order dividing p.

So let $b \in G$ be an element of order q such that

$$\gamma(b) = \iota(a_*)\beta. \tag{7.3}$$

A Sylow q-subgroup of G is of the form $\langle b^x \rangle$, for $x \in A$, and it is invariant if and only if $(b^x)^{\gamma(b^x)} \in \langle b^x \rangle$. Now,

$$\gamma(b^x) = \gamma(x^{-1+Z^{-1}}b) = \iota(x^{(1-Z^{-1})T^{-1}\sigma}a_*)\beta,$$

so that

$$\begin{aligned}(b^x)^{\gamma(b^x)} &= (x^{-1+Z^{-1}}b)^{\iota(x^{(1-Z^{-1})T^{-1}\sigma}a_*)\beta} \\ &= (x^{-(1-Z^{-1})(1+T^{-1}\sigma(1-Z^{-1}))}a_*^{-(1-Z^{-1})}b)^{\beta} \\ &= x^{-(1-Z^{-1})(1+T^{-1}\sigma(1-Z^{-1}))T}a_*^{-(1-Z^{-1})T}b.\end{aligned}$$

Since $(b^x)^j = x^{-1+Z^{-j}}b^j$ for all j, the last expression is in $\langle b^x \rangle$ if and only if

$$x^{-(1-Z^{-1})(1+T^{-1}\sigma(1-Z^{-1}))T}a_*^{-(1-Z^{-1})T}b = x^{-(1-Z^{-1})}b,$$

which, writing $M = 1 - (1 + T^{-1}\sigma(1 - Z^{-1}))T$, can be rewritten as

$$x^{(1-Z^{-1})M} = a_*^{(1-Z^{-1})T}. \tag{7.4}$$

The number of solutions x of the system (7.4) is the number of invariant Sylow q-subgroups: a solution x corresponds to the invariant Sylow q-subgroup $B = \langle b^x \rangle$.

When the kernel of γ has size p^2, the existence of an invariant Sylow q-subgroup can be easily shown by noticing that the action of the group $\gamma(G)$, of size q, on the set of Sylow q-subgroups, that has cardinality p^2, admits at least one fixed point. This means that the system (7.4) is always solvable.

When the kernel of γ has size p or 1 the system (7.4) can be unsolvable for some a_*. However, we will show in 7.3.1 and in 8.7 that under Assumption 7.2 the following are equivalent: for a given a_*,

(1) equation (7.4) admits a solution x, and
(2) the assignments given in (7.1) and (7.3) can be extended to a GF on G.

In fact, if the assignments in (7.1) and (7.3) can be extended to a GF, then

$$\gamma(b^m) = \gamma((b^{m-1})^{\gamma(b)^{-1}})\gamma(b),$$

and an inductive argument shows that

$$\gamma(b^m) = \iota(a_*^{-A_m\sigma+1+T^{-1}+\cdots+T^{-(m-1)}})\beta^m, \tag{7.5}$$

where

$$A_m = \sum_{i=1}^{m-1}(1 - Z^{-i})T^{-i}.$$

Since $\gamma(b^q) = 1$ and the center of G is trivial, (7.5) yields

$$a_*^{A_q\sigma} = a_*^{1+T^{-1}+\cdots+T^{-(q-1)}}. \tag{7.6}$$

In Subsections 7.3 and 8.7 we will see that the elements a_* satisfying (7.6) are exactly those for which the system (7.4) admits solutions. The case $|\ker(\gamma)| = 1$ is specific to the type 9 and it will be considered in Section 8.

7.2. The case $|\ker(\gamma)| = p^2$. As we already noticed, the action on G of the group $\gamma(G)$ of order q fixes at least one of the p^2 Sylow q-subgroups of G, say $B = \langle b \rangle$. Therefore, by Proposition 2.6, each γ on G is the lifting of at least one RGF defined on such a Sylow q-subgroup B. To count each GF exactly once, we need to compute the number of invariant Sylow q-subgroups.

In this case, the element $\gamma(b)$ of order q acts trivially on B, so that $a_* = 1$ and $[B, \gamma(B)] = \{1\}$; by Lemma 2.5, the RGF's on B are thus precisely the morphisms.

Let $\gamma(b)_{|A} = \beta$, where β is as in (7.2).

Equation (7.4) yields

$$x^{(1-Z^{-1})M} = 1,$$

where

$$M = 1 - T = \begin{bmatrix} 1-\lambda^{x_1} & 0 \\ 0 & 1-\lambda^{kx_2} \end{bmatrix}.$$

Since $\det(1 - Z^{-1}) \neq 0$, we obtain that

(1) there is a unique solution, namely a unique invariant Sylow q-subgroup, when both $x_1, x_2 \neq 0$;
(2) there are p solutions, that is, p invariant Sylow q-subgroups, when either $x_1 = 0$ or $x_2 = 0$.

The action of b on A with respect to the operation $\circ$ is given by

$$b^{\ominus 1} \circ a \circ b = b^{-\gamma(b)^{-1}\gamma(a)\gamma(b)} a^{\gamma(b)} b = a^{\gamma(b)\iota(b)},$$

and denoting by $Z_\circ$ its associated matrix, we have

$$Z_\circ \sim \begin{bmatrix} \lambda^{1+x_1} & 0 \\ 0 & \lambda^{k+kx_2} \end{bmatrix}.$$

The enumeration of the groups $(G, \circ)$ depends on the type of the groups G, that is, on the parameter k. We will treat the various types separately in Subsections 8.5, 9.4 and 10.4.

7.3. The case $|\ker(\gamma)| = p$. Here $|\gamma(G)| = pq$. Write $\gamma(b) = \iota(a_*)\beta$, where b has order q, and a_* and β are as in Subsection 7.1. Equation (2.5) yields

$$\sigma T(\sigma - 1) = (\sigma - 1)TZ\sigma. \tag{7.7}$$

The condition $|\ker(\gamma)| = p$ means that $|\ker(\sigma)| = p$, so let $\ker(\sigma) = \langle v \rangle$. Using (7.7) we obtain $v^{-TZ\sigma} = 1$, therefore $v^{-TZ} \in \langle v \rangle$, namely, v is an eigenvector for TZ. If TZ is not scalar, then its eigenspaces are $\langle a_1 \rangle$ and $\langle a_2 \rangle$, so either $v \in \langle a_1 \rangle$ or $v \in \langle a_2 \rangle$. When TZ is scalar, v can be any non-zero element of A.

We proceed by distinguishing three cases, namely when $\ker(\sigma) = \langle a_1 \rangle$, when $\ker(\sigma) = \langle a_2 \rangle$, and lastly when $\ker(\sigma)$ is generated by $v = a_1^x a_2^y$, where $x, y \neq 0$.

Case A: $\ker(\sigma) = \langle a_1 \rangle$. Here $a_1^\sigma = 1$, and $a_2^\sigma = a_1^\mu a_2^\nu$ for some μ, ν not both 0. Evaluating equation (7.7) in a_2, we get

$$\begin{cases} \mu(\nu - \lambda^{x_1 - x_2 k}) = \mu(\nu - 1)\lambda^k \\ (\nu - 1)\nu = (\nu - 1)\nu\lambda^k. \end{cases} \tag{7.8}$$

Since $\lambda^k \neq 1$, the second equation gives that either $\nu = 0$ or $\nu = 1$. If $\nu = 0$, then $\mu \neq 0$ and $x_1 - x_2 k = k$. If $\nu = 1$, then either $\mu = 0$ or $\mu \neq 0$ and $x_1 = x_2 k$.

Case A^*: $\ker(\sigma) = \langle a_2 \rangle$. This case can be reduced to case A according to Remark 7.1, obtaining a system as in (7.8) with $\lambda^{k^{-1}}$ replacing λ^k.

When $k = k^{-1}$, that is for the groups G of type 7 and 9, the conditions are the same, so we will double the results we will obtain in case A.

For the groups G of type 8, $k^{-1} \neq k$, therefore we will sum the results we will obtain in case A for k with the same results for k^{-1}.

Case B: $\ker(\sigma) = \left\langle a_1^x a_2^y \right\rangle$, where $x, y \neq 0$. Here TZ is scalar, namely $x_1 = x_2 k + k - 1$. We can replace the generator $v = a_1^x a_2^y$ of the kernel by $v^{x^{-1}} = a_1 a_2^{yx^{-1}}$, and thus assume $v = a_1 a_2^z$, for some $z \neq 0$. Rescaling a_2, we can also assume $z = 1$, keeping in mind that this covers $p - 1$ cases here. Therefore, here σ is defined as

$$a_1^\sigma = a_1^{-\mu} a_2^{-\nu},\ a_2^\sigma = a_1^\mu a_2^\nu,$$

for some μ, ν not both 0. Evaluating equation (7.7) on a_2, we get

$$\begin{cases} \mu(-\mu\lambda^{-1} - \lambda^{-1} + \nu\lambda^{-k}) = \mu(-\mu + \nu - 1) \\ \nu(-\mu\lambda^{-1} - \lambda^{-k} + \nu\lambda^{-k}) = \nu(-\mu + \nu - 1). \end{cases} \tag{7.9}$$

If $\nu = 0$ then $\mu \neq 0$ and we get $-(\mu + 1)\lambda^{-1} = -(\mu + 1)$, that is, $\mu = -1$, so $A^\sigma = \langle a_1 \rangle$. If $\nu \neq 0$ and $\mu = 0$, we get $(\nu - 1)\lambda^{-k} = \nu - 1$, that is, $\nu = 1$ and $A^\sigma = \langle a_2 \rangle$. Lastly, if both $\nu, \mu \neq 0$, then if $k = 1$ we get $\mu + 1 - \nu = 0$, and if $k \neq 1$ the system has no solution.

If $k = 1$, namely G is of type 7, then the condition $\mu + 1 - \nu = 0$ includes also the cases above in which $\mu = -1$ and $\nu = 0$, or $\mu = 0$ and $\nu = 1$.

If $k \neq 1$, namely G is of type 8 or 9, the last case does not happen. Moreover we reduce the case $(\mu, \nu) = (0, -1)$ to the case $(\mu, \nu) = (1, 0)$ according to Remark 7.1, obtaining a system as in (7.9) with $\lambda^{k^{-1}}$ replacing λ^k.

As for case A^*, for the groups G of type 9 we will double the results we will obtain in case $(\mu, \nu) = (1, 0)$, and for the groups G of type 8 we will sum the results we will obtain in case $(\mu, \nu) = (1, 0)$ for k with the same results for k^{-1}.

Under Assumption 7.2, for a group $G \simeq G_k$, we sum up our analysis as follows:

- Case A: $\ker(\sigma) = \langle a_1 \rangle$.
  (A1) $\nu = 0, \mu \neq 0, x_1 - x_2 k = k$;
  (A2) $\nu = 1, \mu \neq 0, x_1 = x_2 k$;
  (A3) $\nu = 1, \mu = 0$.

- Case A*: $\ker(\sigma) = \langle a_2 \rangle$.
  It is equivalent to considering the case A for $G \simeq G_{k^{-1}}$, as explained above. Therefore, replacing λ^k with $\lambda^{k^{-1}}$ in (A1), (A2) and (A3), here we obtain the subcases (A1*), (A2*) and (A3*).
- Case B: $\ker(\sigma) = \langle a_1 a_2 \rangle$.
  (1) If $k \neq 1$ (namely G is of type 8 or 9):
   (B1) $\nu = 0, \mu = -1, x_1 = x_2 k + k - 1$;
   (B1*) $\nu = 1, \mu = 0, x_1 = x_2 k + k - 1$. As explained above, this case is equivalent to considering the case (B1) for $G \simeq G_{k^{-1}}$.
  (2) If $k = 1$ (namely G is of type 7):
   (B2) $\mu + 1 - \nu = 0, x_1 = x_2$.

7.3.1. Invariant Sylow q-subgroups. We are now ready to prove, under Assumption 7.2 and when $|\ker(\gamma)| = p$, that the groups always have at least one invariant Sylow q-subgroup and to determine their number in terms of our assignments $\gamma(a) = \iota(a^{-\sigma})$ and $\gamma(b) = \iota(a_*)\beta$.

Subsection 7.1 yields that, for $x \in A$, the Sylow q-subgroup $\langle b^x \rangle$ is invariant if and only if x is a solution of (7.4):

$$x^{(1-Z^{-1})M} = a_*^{(1-Z^{-1})T},$$

where $M = 1 - (1 + T^{-1}\sigma(1 - Z^{-1}))T$, and $\det(1 - Z^{-1}) \neq 0$.

If $\det(M) \neq 0$, then the system (7.4) admits a unique solution for each a_*, or equivalently, for each choice of $\gamma(b)$.

If $\det(M) = 0$, then for some a_*'s the system (7.4) admits p or p^2 solutions, and for others it has no solution. However, we show that in the latter case the values for $\gamma(a), \gamma(b)$ do not extend to a full GF on G.

In fact, if γ is a GF on G such that $\gamma(b) = \iota(a_*)\beta$, then γ satisfies (7.5) and (7.6). We will show that if the system (7.4) has no solution for some a_*, then a_* does not satisfy the condition (7.6), and thus the assignments $\gamma(a) = \iota(a^{-\sigma})$ and $\gamma(b) = \iota(a_*)\beta$ cannot be extended to a GF.

Write $a_* = a_1^x a_2^y$. We distinguish several cases.

Case A: Here $\ker(\sigma) = \langle a_1 \rangle$, and

$$M = \begin{bmatrix} 1 - \lambda^{x_1} & 0 \\ -\mu\lambda^{x_1 - x_2 k}(1 - \lambda^{-1}) & 1 - \lambda^{x_2 k} - \nu(1 - \lambda^{-k}) \end{bmatrix}.$$

According to the division into subcases, we have

A1: $\det(M) = (1 - \lambda^{x_1})(1 - \lambda^{x_1 - k})$, so that there is a unique invariant Sylow q-subgroup when $x_1 \neq 0, k$.
If $x_1 = 0$, then the system (7.4) admits (a number of p) solutions if and only if $a_* = a_1^{-\mu y} a_2^y$. In this case there are p invariant Sylow q-subgroups. Moreover, if $a_* = a_1^x a_2^y$, the condition (7.6) yields that $x = -\mu y$. Therefore, by the discussion above, the case in which there are no invariant Sylow q-subgroups does not arise.

If $x_1 = k$, then (7.4) admits (a number of p) solutions if and only if $a_* = a_1^x$, and in this case there are p invariant Sylow q-subgroups. (7.6) yields that $y = 0$, therefore the case in which there are no invariant Sylow q-subgroups does not arise.

A2: $\det(M) = (1 - \lambda^{x_1})(\lambda^{-k}(1 - \lambda^{x_1+k}))$, so that there is a unique invariant Sylow q-subgroup when $x_1 \neq 0, -k$. Note that here necessarily $x_1 \neq 0$, otherwise we would also have $x_2 = 0$, namely $\beta = 1$.

If $x_1 = -k$, then the system (7.4) admits (a number of p) solutions if and only if $a_* = a_1^x$, and in this case there are p invariant Sylow q-subgroups. Also here the a_*'s for which (7.4) has no solutions are precisely those for which (7.6) is not satisfied, in fact here (7.6) yields $y = 0$.

A3: $\det(M) = (1 - \lambda^{x_1})(\lambda^{-k}(1 - \lambda^{x_2 k+k}))$, so that there is a unique solution when $x_1 \neq 0$ and $x_2 \neq -1$.

If either $x_1 = 0$ and $x_2 \neq -1$, or $x_1 \neq 0$ and $x_2 = -1$, then $(1 - Z^{-1})M$ has rank 1, and the system (7.4) admits p solutions if and only if $a_* = a_2^y$ in the first case, and $a_* = a_1^x$ in the second case. Once again, the a_*'s for which (7.4) has no solutions are precisely those for which (7.6) is not satisfied, in fact here (7.6) yields $x = 0$ in the case $x_1 = 0$ and $x_2 \neq -1$, and $y = 0$ in the case $x_1 \neq 0$ and $x_2 = -1$.

If $x_1 = 0$ and $x_2 = -1$, then $(1 - Z^{-1})M$ has rank 0, and the system (7.4) admits p^2 solutions if and only if $a_*^{(1-Z^{-1})T} = 1$, namely when $a_* = 1$. Moreover, the condition (7.6) yields $x, y = 0$, so that the case in which the system has no solution does not arise.

Case B: Here $\ker(\sigma) = \langle\, a_1 a_2 \,\rangle$ and (7.4) yields

$$M = \begin{bmatrix} 1 - \lambda^{x_1} + \mu(1 - \lambda^{-1}) & \lambda^{-x_1+kx_2}\nu(1 - \lambda^{-k}) \\ -\lambda^{x_1-kx_2}\mu(1 - \lambda^{-1}) & 1 - \lambda^{kx_2} - \nu(1 - \lambda^{-k}) \end{bmatrix}.$$

According to the division into subcases, we have

B1: $\det(M) = \lambda^{-1}(1 - \lambda^{x_1+1})(1 - \lambda^{x_1+1-k})$, and there exists a unique invariant Sylow q-subgroup when $x_1 \neq -1, k - 1$.

If $x_1 = -1, k-1$, then there are p invariant Sylow q-subgroups when $a_* = a_1^x a_2^x$ in the case $x_1 = -1$, and $a_* = a_1^x$ in the case $x_1 = k - 1$. The other cases, namely those for which there are no invariant Sylow q-subgroups, do not arise, in fact the condition (7.6) yields precisely $x = y$ in the case $x_1 = -1$, and $y = 0$ in the case $x_1 = k - 1$.

B2: $\det(M) = (\lambda^{x_1} - 1)\lambda^{-1}(\lambda^{x_1+1} - 1)$, and there exists a unique invariant Sylow q-subgroup when $x_1 \neq 0, -1$. Note that here necessarily $x_1 \neq 0$, otherwise we would also have $x_2 = 0$.

If $x_1 = -1$, then there are p invariant Sylow q-subgroups when $a_* = a_1^x a_2^x$. The other cases, namely those for which there are no invariant

Sylow q-subgroups, do not arise, in fact the condition (7.6) yields precisely $x = y$.

By the discussion above, we get the following.

Proposition 7.3. *Under Assumption* 7.2*, if G is a group and γ is a GF on G with $|\ker(\gamma)| = p$, then the number of invariant Sylow q-subgroups is*

(A1) 1 *when $x_1 \neq 0, k$ and p otherwise.*
(A2) 1 *when $x_1 \neq -k$ and p otherwise.*
(A3) 1 *when $x_1 \neq 0$ and $x_2 \neq -1$, p^2 when $x_1 = 0$ and $x_2 = -1$, and p otherwise.*
(A1*) 1 *when $x_2 \neq 0, k^{-1}$ and p otherwise.*
(A2*) 1 *when $x_2 \neq -k^{-1}$ and p otherwise.*
(A3*) 1 *when $x_2 \neq 0$ and $x_1 \neq -1$, p^2 when $x_2 = 0$ and $x_1 = -1$, and p otherwise.*
(B1) 1 *when $x_1 \neq -1, k-1$ and p otherwise.*
(B1*) 1 *when $x_2 \neq -1, -1+k^{-1}$ and p otherwise.*
(B2) 1 *when $x_1 \neq -1$ and p otherwise.*

7.3.2. Enumerating the GF's. We have shown that there is always a Sylow q-subgroup B which is invariant under $\gamma(B)$, and Proposition 7.3 yields the exact number of such invariant Sylow q-subgroups. Since (7.1) is also satisfied, we have that γ is of the form (2.7).

To enumerate the GF's we can, according to Proposition 2.8, count the possible couples (γ_A, γ_B) with the properties above, taking into account that every such choice defines a unique γ, and that a given γ built in this way is obtained s times, where s is the number of invariant Sylow q-subgroups of G. Thus to obtain the number of distinct GF's on G we count the choices for (γ_A, γ_B) as above, and then divide this number by s.

Let $Z_\circ$ denote the action of b on A in $(G, \circ)$. We have

$$\begin{aligned} b^{\ominus 1} \circ a \circ b &= (b^{\gamma(b)^{-1}\gamma(a)\gamma(b)})^{-1} a^{\gamma(b)} b \\ &= (b^{\iota(a^{-\sigma})\beta})^{-1} a^{\gamma(b)} b \\ &= ((a^{-\sigma(-1+Z^{-1})} b)^{\beta})^{-1} a^{\gamma(b)} b \\ &= (a^{-\sigma(-1+Z^{-1})T} b)^{-1} a^{T} b \\ &= b^{-1} a^{\sigma(-1+Z^{-1})T+T} b \\ &= a^{(\sigma(1-Z)+Z)T}, \end{aligned}$$

and since $a^{\circ t} = a^t$ for all t, with respect to the basis $\{a_1, a_2\}$ of $(A, \circ)$ we have

$$Z_\circ = (\sigma(1-Z) + Z)T. \tag{7.10}$$

Case A. Here $\ker(\sigma) = \langle\, a_1 \,\rangle$ and equality (7.10) yields

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} & 0 \\ \mu(1-\lambda)\lambda^{x_1} & \lambda^{x_2 k}(\nu(1-\lambda^k) + \lambda^k) \end{bmatrix}.$$

(A1) We have $p-1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

(A2) We have $p-1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

(A3) We have 1 choice for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_2 k} \end{bmatrix}.$$

Case B. Here $\ker(\sigma) = \langle a_1 a_2 \rangle$, $x_1 = x_2 k + k - 1$, and equality (7.10) yields

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} - \lambda^{x_1}\mu(1-\lambda) & -\lambda^{k x_2}\nu(1-\lambda^k) \\ \lambda^{x_1}\mu(1-\lambda) & \lambda^{k x_2 + k} + \lambda^{k x_2}\nu(1-\lambda^k) \end{bmatrix}.$$

(B1) Here $k \neq 1$. We have $p-1$ choices for σ, and

$$Z_\circ = \begin{bmatrix} \lambda^{x_1} & 0 \\ -\lambda^{x_1}(1-\lambda) & \lambda^{x_2 k + k} \end{bmatrix} \sim \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_1+1} \end{bmatrix}.$$

(B2) Here $k = 1$, $x_1 = x_2$, $\mu + 1 = \nu$, and we have $p(p-1)$ choices for

$$\sigma = \begin{bmatrix} -\mu & -\mu - 1 \\ \mu & \mu + 1 \end{bmatrix}.$$

We have

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} - \lambda^{x_1}\mu(1-\lambda) & -\lambda^{x_1}(\mu+1)(1-\lambda) \\ \lambda^{x_1}\mu(1-\lambda) & \lambda^{x_1+1} + \lambda^{x_1}(\mu+1)(1-\lambda) \end{bmatrix},$$

and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

7.3.3. Conjugacy classes. Here we exhibit a general scheme to compute the conjugacy classes for groups which satisfy the assumptions of this section in the case $|\ker(\gamma)| = p$, when the automorphisms are of the form $\varphi = \iota(x)\delta$, where $x \in A$ and $\delta_{|A} \in \mathrm{GL}(2, p)$. We will show in Subsections 8.6, 9.5 and 10.5 that this scheme can be applied to the groups of types 9, 8 and to the groups G of type 7, when $\gamma(A) \leq \mathrm{Inn}(G)$.

Suppose thus that an automorphism of G has the form $\varphi = \iota(x)\delta$, where $x \in A$ and $\delta \in \mathrm{GL}(2, p)$. We have

$$\begin{aligned} \gamma^\varphi(a) &= \varphi^{-1}\gamma(a^{\delta^{-1}})\varphi \\ &= \delta^{-1}\iota(a^{-\delta^{-1}\sigma})\delta \\ &= \iota(a^{-\delta^{-1}\sigma\delta}), \end{aligned}$$

and

$$\begin{aligned}\gamma^{\varphi}(b) &= \varphi^{-1}\gamma(x^{1-Z^{-1}}b)\varphi\\ &= \varphi^{-1}\iota(x^{-(1-Z^{-1})T^{-1}\sigma})\beta\varphi\\ &= \delta^{-1}\iota(x^{-1+T^{-1}-(1-Z^{-1})T^{-1}\sigma})\beta\delta\\ &= \iota(x^{(-1+T^{-1}-(1-Z^{-1})T^{-1}\sigma)\delta})\delta^{-1}\beta\delta.\end{aligned}$$

Write $H = -1 + T^{-1} - (1 - Z^{-1})T^{-1}\sigma$. Setting $\gamma^{\varphi}(a) = \iota(a^{-\sigma})$ and $\gamma^{\varphi}(b) = \beta$, we obtain that φ stabilises γ if and only if the conditions

$$[\sigma, \delta] = 1, \tag{7.11}$$

$$[\beta, \delta] = 1, \tag{7.12}$$

$$x^{H\delta} = 1, \tag{7.13}$$

hold.

We now distinguish two cases, namely when δ is diagonal and when δ is not necessarily diagonal.

7.3.4. δ is a diagonal matrix. Suppose first that δ is diagonal, namely $\delta = \operatorname{diag}(\delta_{11}, \delta_{22})$; as we will see in Subsections 8.6 and 9.5, this will be the case for the groups G of type 9 and 8, therefore we do not consider here the case (B2).

In this case equation (7.12) is satisfied for every δ. In both cases (A) and (B1), the condition (7.11) yields $\mu\delta_{22}^{-1}\delta_{11} = \mu$, so that $\mu = 0$ or δ is scalar. Note that $\mu = 0$ only in the case (A3), thus we consider any δ in this case, and δ scalar in the cases (A1), (A2) and (B1).

In the case (A1) we have

$$H = \begin{bmatrix} -1 + \lambda^{-x_1} & 0\\ -\mu(1-\lambda^{-k})\lambda^{k+x_1} & -1 + \lambda^{k+x_1}\end{bmatrix},$$

so that (7.11) has one solution if $x_1 \neq 0, k$, and p solutions if $x_1 = 0, k$.

In the case (A2),

$$H = \begin{bmatrix} -1 + \lambda^{-x_1} & 0\\ -\mu(1-\lambda^{-k})\lambda^{-x_1} & -1 + \lambda^{-k-x_1}\end{bmatrix},$$

and (7.11) has one solution if $x_1 \neq -k$, and p solutions if $x_1 = -k$.

In the case (A3),

$$H = \begin{bmatrix} -1 + \lambda^{-x_1} & 0\\ 0 & -1 + \lambda^{-k-kx_2}\end{bmatrix},$$

and (7.11) has one solution if $x_1 \neq 0$ and $x_2 \neq -1$, p^2 solutions if $x_1 = 0$ and $x_2 = -1$, and p solutions otherwise.

In the case (B1),

$$H = \begin{bmatrix} -1 + \lambda^{-x_1-1} & 0\\ (1-\lambda^{-k})\lambda^{k-1-x_1} & -1 + \lambda^{k-1-x_1}\end{bmatrix},$$

and (7.11) has one solution if $x_1 \neq -1, k-1$, and p solutions otherwise.

7.3.5. δ is a (not necessarily diagonal) matrix. Suppose now that δ is an arbitrary matrix. As we will see in Subsection 10.5 this will be the case for the groups G of type 7, therefore we do not consider here the case (B1).

If we are in case (A1), then (7.11) yields $\delta_{12} = 0$ and $\delta_{11} = \delta_{22}$, and (7.12) yields $\delta_{21} = 0$. Therefore δ is scalar, and with the same computations above (taking $k = 1$) we obtain that H has rank 2 if $x_1 \neq 0, 1$, and 1 otherwise.

If we are in case (A2), then (7.11) yields $\delta_{12} = 0$ and $\delta_{21} = \mu(\delta_{22} - \delta_{11})$. Since in the case (A2) T is scalar, equation (7.12) is satisfied for every δ. Moreover H has rank 2 if $x_1 \neq -1$, and 1 otherwise.

If we are in case (A3), then (7.11) yields $\delta_{12}, \delta_{21} = 0$, namely δ is diagonal. In particular (7.12) is satisfied. H has rank 2 if $x_1 \neq 0$ and $x_2 \neq -1$, 0 if $x_1 = 0$ and $x_2 = -1$, and 1 otherwise.

If we are in case (B2), then (7.11) yields

$$\begin{cases} \mu\delta_{12} = -(\mu+1)\delta_{21} \\ \mu(\delta_{12} - \delta_{11}) = \mu(\delta_{21} - \delta_{22}) \\ (\mu+1)(\delta_{12} - \delta_{11}) = (\mu+1)(\delta_{21} - \delta_{22}). \end{cases}$$

If $\mu = 0$ then $\delta_{21} = 0$ and $\delta_{12} = \delta_{11} - \delta_{22}$; if $\mu = -1$, then $\delta_{12} = 0$ and $\delta_{21} = \delta_{22} - \delta_{21}$; if $\mu \neq 0, -1$, then $\delta_{12} = -\frac{\mu+1}{\mu}\delta_{21}$ and $\frac{2\mu+1}{\mu}\delta_{21} = \delta_{22} - \delta_{11}$. Therefore, in all cases we have one choice for the elements δ_{12}, δ_{21}, and $(p-1)^2$ choices for δ_{11}, δ_{22}. Since T is scalar here, (7.12) is always satisfied. Moreover

$$H = \begin{bmatrix} -1 + \lambda^{-x_1} + \mu(1-\lambda^{-1})\lambda^{-x_1} & (\mu+1)(1-\lambda^{-1})\lambda^{-x_1} \\ -\mu(1-\lambda^{-1})\lambda^{-x_1} & -1 + \lambda^{-x_1} + (\mu+1)(1-\lambda^{-1})\lambda^{-x_1} \end{bmatrix}$$

has determinant $(1-\lambda^{-x_1})(1-\lambda^{-x_1-1})$. Since $x_1 \neq 0$, we obtain that H has rank 2 if $x_1 \neq -1$, and 1 otherwise.

8. Type 9

Here $q \mid p-1$, where $q > 2$, and $G = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D_1} \mathcal{C}_q$. The Sylow p-subgroup $A = \langle\, a_1, a_2 \,\rangle$ of G is characteristic, and if $a_1, a_2 \in A$ are in the eigenspaces of the action of a generator b of a Sylow q-subgroup B on A, then this action can be represented by a non-scalar diagonal matrix Z, with no eigenvalues 1 and $\det(Z) = 1$.

For all of this section, we consider $A = \langle\, a_1, a_2 \,\rangle$, where a_1, a_2 are eigenvectors for $\iota(b)$. With respect to that basis, we have

$$Z = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix},$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$.

The divisibility condition on p and q implies that $(G, \circ)$ can be of type 5, 6, 7, 8 and 9.

According to Subsections 4.1 and 4.3 of [8], we have

$$\mathrm{Aut}(G) = (\mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_p)) \rtimes \mathcal{C}_2.$$

The Sylow p-subgroup of $\mathrm{Aut}(G)$ has order p^2 and is characteristic, so, since G has trivial center, all of its elements are conjugation by elements of A.

If γ is a GF on G, then $\gamma_{|A} : A \to \mathrm{Inn}(G) \leq \mathrm{Aut}(G)$ is a RGF, as A is characteristic in G. Moreover, Lemma 2.5 yields that $\gamma_{|A}$ is a morphism, as $\iota(A)$ acts trivially on the abelian group A. Therefore, for each gamma function γ there exists $\sigma \in \mathrm{End}(A)$ such that

$$\gamma(a) = \iota(a^{-\sigma}) \tag{8.1}$$

for each $a \in A$.

8.1. Duality. Since every γ on G satisfies equation (8.1), we can apply Lemma 2.7 with $C = A$, and this yields equation (2.5).

Now, by the discussion in Subsections 4.1 and 4.2, if σ and $1 - \sigma$ are not both invertible, then $p \mid |\ker(\gamma)|$ or $p \mid |\ker(\tilde{\gamma})|$, namely σ has 0 or 1 as an eigenvalue. Otherwise σ and $1 - \sigma$ are both invertible, and there are actually σ with no eigenvalues 0 and 1, and this corresponds to the existence of γ such that $p \nmid |\ker(\gamma)|, |\ker(\tilde{\gamma})|$.

Except for the case when both γ and $\tilde{\gamma}$ have kernel of size not divisible by p, we will use duality to swich to a more convenient kernel.

8.2. Outline. We will use Proposition 2.6 to deal with the kernels of size q, pq, and p^2. As for the kernels of size p and 1, we will appeal to Proposition 2.8. To do this, we will show that each γ on G with kernel of size p or 1 always admits at least one invariant Sylow q-subgroup B.

In order to do that, in Subsubsection 8.2.1 we describe the elements of $\mathrm{Aut}(G)$ of order q.

8.2.1. Description of the elements of order q of $\mathrm{Aut}(G)$. The Sylow q-subgroups of $\mathrm{Aut}(G)$ are of the form $\mathcal{C}_{q^e} \times \mathcal{C}_{q^e}$, for $q^e || p-1$, and they can be described as the Sylow q-subgroups of the centraliser $C_{\mathrm{Aut}(G)}(\langle \iota(b) \rangle)$ of $\langle \iota(b) \rangle$, generated in $\mathrm{Aut}(G)$, where $\langle b \rangle$ varies among the Sylow q-subgroups of G. Since they are abelian, each of them contains exactly one subgroup of type $\iota(\langle b \rangle)$, and this establishes a one-to-one correspondence between the Sylow q-subgroups of G and the Sylow q-subgroups of $\mathrm{Aut}(G)$.

We note also that for $a \in A$ one has

$$C_{\mathrm{Aut}(G)}(\langle \iota(b) \rangle)^{\iota(a)} = C_{\mathrm{Aut}(G)}(\langle \iota(b^a) \rangle).$$

For $b \in G \setminus A$, recalling that $\iota(b)$ acts on A as $\mathrm{diag}(\lambda, \lambda^{-1})$, we write

$$\begin{aligned} \beta_1 :\ & a_1 \mapsto a_1^{\lambda} & \beta_2 :\ & a_1 \mapsto a_1 \\ & a_2 \mapsto a_2 & & a_2 \mapsto a_2^{\lambda^{-1}} \\ & b \mapsto b & & b \mapsto b \end{aligned} \tag{8.2}$$

so that $\iota(b) = \beta_1\beta_2$, and $\langle \beta_1, \beta_2 \rangle$ is the elementary abelian subgroup of order q^2 of a Sylow q-subgroup of $C_{\mathrm{Aut}(G)}(\langle \iota(b) \rangle)$.

Now, if $\beta \in \mathrm{Aut}(G)$ is an element of order q, then it belongs to the centraliser of $\langle \iota(b) \rangle$, where $\langle b \rangle$ is a Sylow q-subgroup of G. Therefore, if β_1, β_2 are as above, then $\beta \in \langle \beta_1, \beta_2 \rangle$, namely $\beta = \beta_1^{x_1}\beta_2^{x_2}$, where $0 \le x_1, x_2 < q$ not both zero.

Let us start with the enumeration of the GF's on G. We proceed case by case, according to the size of the kernel.

As usual, if $|\ker(\gamma)| = p^2q$, then γ corresponds to the right regular representation, so that we will assume $\gamma \ne 1$.

8.3. The case $|\ker(\gamma)| = q$. Let $B = \ker(\gamma)$. Here $(G, \circ)$ is necessarily of type 5, as it is the only type having a normal subgroup of order q.

By Proposition 2.6, since A is characteristic, each GF on G is the lifting of a RGF on A, and, conversely, a RGF on A lifts to G if and only if B is invariant under $\{\gamma(a)\iota(a) \mid a \in A\}$.

For each $a \in A$, $\gamma(a) = \iota(a^{-\sigma})$, where $\sigma \in \mathrm{GL}(2, p)$, so that $\gamma(a)\iota(a) = \iota(a^{1-\sigma})$. Taking into account that each Sylow q-subgroup of G is self-normalising, we obtain that γ lifts to G if and only if $\sigma = 1$, namely when

$$\gamma(a) = \iota(a^{-1}).$$

Since this map is a morphism and $[A, \gamma(A)] = \{1\}$, by Lemma 2.5 γ is actually a RGF. Therefore, for each of the p^2 choices for a Sylow q-subgroup, there is a unique RGF on A which lifts to G, and we obtain p^2 groups.

Note that for all the γ's in this case $p \mid |\ker(\tilde{\gamma})|$.

As to the conjugacy classes, if γ has kernel B, then, for $x \in A$, $\gamma^{\iota(x)}$ has kernel $B^{\iota(x)}$, as for $b \in \ker(\gamma)$,

$$\gamma^{\iota(x)}(b^{\iota(x)}) = \iota(x^{-1})\gamma(b)\iota(x) = 1.$$

Since $\iota(A)$ conjugates transitively the p^2 Sylow q-subgroups of G, the orbits contain at least p^2 elements. Since there are p^2 GF, there is a unique orbit of length p^2.

8.4. The case $|\ker(\gamma)| = pq$. Here $K = \ker(\gamma)$ is a subgroup of G isomorphic to $\mathcal{C}_p \rtimes \mathcal{C}_q$, therefore we will obtain $(G, \circ)$ of type 6, as it is the only type having a non abelian normal subgroup of order pq.

We can choose K in $2p$ ways, indeed for each of the p^2 choices for a Sylow q-subgroup B, the subgroups of order p that are B-invariant are the 1-dimensional invariant subspaces of the action of B. Therefore, there are 2 of such subgroups. Moreover, since $\mathcal{C}_p \rtimes \mathcal{C}_q$ has p subgroups of order q, exactly p choices for B give the same group.

Let $K = \langle a_1, b \rangle$, and let $a_2 \in A$ be such that $A = \langle a_1, a_2 \rangle$. The cyclic complement $\langle a_2 \rangle$ of K in G can be chosen in p ways, and since $\gamma(G) \le \iota(A)$, each of these choices yields a $\gamma(G)$-invariant subgroup.

Therefore, by Proposition 2.6, each γ is the lifting of a RGF defined on any of the complements of order p. So, we fix $\langle a_2 \rangle$ and we consider the RGF's $\gamma' : \langle a_2 \rangle \to \mathrm{Aut}(G)$, taking into account that the choice of the complement is immaterial. Again appealing to Proposition 2.6, the RGF's γ' which can be lifted to G are those for which K is invariant under $\{\gamma'(x)\iota(x) : x \in \langle a_2 \rangle\}$, namely the maps defined as

$$\gamma'(a_2) = \iota(a_1^j a_2^{-1}),$$

for some j, $0 \leq j \leq p-1$. Moreover, since $[\langle a_2 \rangle, \gamma(\langle a_2 \rangle)] = \{1\}$, by Lemma 2.5 the RGF's correspond to the morphisms. Therefore, since there are p choices for j and $2p$ for K, the number of distinct gamma functions is $2p^2$.

Notice that, for every γ as above, $p \mid |\ker(\tilde{\gamma})|$.

As to the conjugacy classes, let $\varphi \in \mathrm{Aut}(G)$. According to [8], φ has the form $\iota(x)\delta\psi$, where $x \in A$, $\delta_{|B} = 1$ and, with respect to the fixed basis, $\delta_{|A} = (\delta_{ij}) \in \mathrm{GL}(2, p)$ is diagonal. ψ is defined as $b^\psi = b^r$ and $a^\psi = a^S$, where either $r = 1$ and $S = 1$, or $r = -1$ and

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We have that $\gamma(a_1^{\varphi^{-1}}) = \gamma(a_1^{\psi\delta^{-1}})$, and $\gamma^\varphi(a_1) = 1$ if and only if $a_1^{\varphi^{-1}} \in \ker(\gamma) \cap A = \langle a_1 \rangle$, therefore $\psi = 1$. Moreover,

$$\gamma^\varphi(b) = \varphi^{-1}\gamma(b^{\iota(x^{-1})})\varphi = \varphi^{-1}\gamma(x^{1-Z^{-1}})\varphi, \tag{8.3}$$

so it is equal to $\gamma(b) = 1$ when $x \in \langle a_1 \rangle$. Now, writing $a = a_1^j a_2^{-1}$, we have

$$\gamma^\varphi(a_2) = \varphi^{-1}\gamma(a_2^{\delta^{-1}})\varphi = \varphi^{-1}\gamma(a_2^{\delta_{22}^{-1}})\varphi = \iota(a^{\delta_{22}^{-1}})^\delta, \tag{8.4}$$

so that φ stabilises γ if and only if $\iota(a^{\delta_{22}^{-1}})^\delta = \iota(a)$, and this yields the condition $j(\delta_{11} - \delta_{22}) = 0$.

So, if $j = 0$ the last condition is always satisfied, and if $j \neq 0$ the δ's in the stabiliser are the scalar matrices. Therefore we get one orbit of length $2p$ and one orbit of length $2p(p-1)$.

8.5. The case $|\ker(\gamma)| = p^2$. The action of $\gamma(G)$ of order q on G fixes at least one of the p^2 Sylow q-subgroups of G, say $B = \langle b \rangle$.

Now, as B is $\gamma(G)$-invariant, $\gamma(b)_{|B} = 1$, and let $\gamma(b)_{|A} = \beta$. The discussion in Subsubsection 8.2.1 yields that $\beta = \beta_1^{x_1}\beta_2^{x_2}$, where x_1, x_2 are not both zero, so that, with respect to the basis $\{a_1, a_2\}$, we can represent β as the matrix

$$T = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{-x_2} \end{bmatrix},$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$.

Therefore, since we are under the assumptions of Subsection 7.2, we obtain that

(1) there is a unique invariant Sylow q-subgroup when both $x_1, x_2 \neq 0$;
(2) there are p invariant Sylow q-subgroups when either $x_1 = 0$ or $x_2 = 0$.

Moreover, taking $k = -1$ in Subsection 7.2, we find that the action of b on A with respect to the operation $\circ$ has associated matrix

$$Z_\circ \sim \begin{bmatrix} \lambda^{1+x_1} & 0 \\ 0 & \lambda^{-1-x_2} \end{bmatrix}.$$

Therefore we obtain the following groups $(G, \circ)$.

Type 5: if $x_1 = x_2 = -1$, and there are p^2 groups.

Type 6: if either $x_1 = -1$ and $x_2 \neq -1$, or $x_1 \neq -1$ and $x_2 = -1$. In both the cases there is a unique invariant Sylow q-subgroup, except if either $x_2 = 0$ or $x_1 = 0$, when there are p invariant Sylow q-subgroups. Therefore there are $2p^2(q-2)$ groups when we are in the case (1), plus other $2p$ groups for the case (2).

Type 7: if $x_1 + 1 = -(x_2 + 1) \neq 0$. There are $p^2(q-3)$ groups for the case (1), plus $2p$ groups for the case (2).

Type 8: if $Z_\circ$ is a non scalar matrix with no eigenvalues 1, and determinant different from 1.

In case (1) this corresponds to the conditions $x_2 \neq 0, -1$ and the four conditions $x_1 \neq 0, -1, -x_2, -x_2 - 2$, which are independent if and only if in addition $x_2 \neq -2$. Therefore, for $x_2 \neq 0, -1, -2$ we obtain $p^2(q-4)(q-3)$, groups. For $x_2 = -2$ the four conditions on x_1 reduce to three conditions, and we obtain further $p^2(q-3)$, groups.

In case (2), suppose $x_1 = 0$. Then there are three independent conditions on x_2. Doubling for the case $x_2 = 0$, we obtain $2p(q-3)$.

Summing up, we have just obtained $p^2(q-3)^2 + 2p(q-3)$ groups of type 8; looking at the eigenvalues of $Z_\circ$, we easily obtain that they are $2p^2(q-3) + 4p$ groups isomorphic to G_s, for every $s \in \mathcal{K}$.

Type 9: if $Z_\circ$ is a non-scalar matrix with no eigenvalue 1 and determinant 1, namely $x_1 \neq -1, -x_2 - 2$, $x_2 \neq -1$, and $x_1 - x_2 = 0$. The case (2) can not happen, otherwise $\beta = 1$. In case (1) we have $x_2 \neq 0, -1$, therefore there are $p^2(q-2)$ groups.

As to the conjugacy classes, since the kernel A is characteristic, we have that $\gamma^\varphi(a) = \gamma(a)$, for every $\varphi \in \mathrm{Aut}(G)$.

In the notation of Subsection 8.4 , write $\varphi = \iota(x)\delta\psi$. Since $b^{\varphi^{-1}} = b^{\psi\iota(x^{-1})} \equiv b^r \bmod \ker(\gamma)$, we have

$$\gamma^\varphi(b) = \varphi^{-1}\gamma(b^r)\varphi = \psi\delta^{-1}T^r\iota(x^{1-T^r})\delta\psi = \psi\delta^{-1}T^r\delta\iota(x^{(1-T^r)\delta})\psi.$$

Therefore, φ stabilises γ if and only if

$$\begin{cases} x^{(1-T^r)\delta} = 1 \\ \delta^{-1}T^r\delta = \psi T\psi. \end{cases}$$

The first condition yields $x = 1$ or, if $x = a_1^u a_2^v$, either $x_1 = 0$ and $v = 0$, or $x_2 = 0$ and $u = 0$. If $\psi = 1$ the second condition is always satisfied. If $\psi \neq 1$, since $\delta^{-1}T^{-1}\delta = T^{-1}$ and ψ acts on T by conjugation exchanging the eigenvalues, the second condition yields $x_1 = x_2$.

We obtain the following.

(1) For $(G,\circ)$ of type 5 the stabiliser has order $2(p-1)^2$, so that there is one orbit of length p^2.
(2) For $(G,\circ)$ of type 6 the stabiliser has order $p(p-1)^2$ when either $x_1 = 0$ or $x_2 = 0$, and $(p-1)^2$ when $x_1, x_2 \neq 0$. Therefore, there is one orbit of length $2p$ together with $q-2$ orbits of length $2p^2$.
(3) For $(G,\circ)$ of type 7 the stabiliser has order $(p-1)^2$ when $x_1, x_2 \neq 0$, and $p(p-1)^2$ otherwise. Therefore there are $\frac{q-3}{2}$ orbits of length $2p^2$, and one orbit of length $2p$.
(4) For $(G,\circ)$ of type 8, if $x_1, x_2 \neq 0$ then the stabiliser has order $(p-1)^2$; otherwise either $x_1 = 0$ or $x_2 = 0$, and the stabiliser has order $p(p-1)^2$. Therefore, if $(G,\circ) \simeq G_s$, for every $s \in \mathcal{K}$ we obtain $q-3$ orbits of length $2p^2$ and two orbits of length $2p$.
(5) For $(G,\circ)$ of type 9, $x_1, x_2 \neq 0$ so that the stabiliser has order $(p-1)^2$, and there are $q-2$ orbits of length p^2.

8.6. The case $|\ker(\gamma)| = p$. To count the GF's of this case we will use Proposition 2.8.

Here $|\gamma(G)| = pq$. The discussion in Subsubsection 8.2.1 yields that $\gamma(G) = \langle \iota(a_0), \beta \rangle$, for some $1 \neq a_0 \in A$ with $A^\sigma = \langle \iota(a_0) \rangle$, and $\beta \neq 1$. We can assume $\gamma(b) = \iota(a_0^j)\beta$ for some j, where $\beta = \beta_1^{x_1}\beta_2^{x_2}$. With respect to the basis $\{a_1, a_2\}$, where $\langle a_1 \rangle$ and $\langle a_2 \rangle$ are the eigenspaces of $\iota(b)$, the matrix associated to β is $\operatorname{diag}(\lambda^{x_1}, \lambda^{-x_2})$, where x_1 and x_2 are not both zero.

We write $\gamma(b) = \iota(a_*)\beta$, and with respect to $\{a_1, a_2\}$, we can represent $\beta_{|A}$ as

$$T = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{-x_2} \end{bmatrix},$$

where x_1, x_2 are not both zero.

Following Subsection 7.3, and recalling that for G of type 9 $k = -1$, here we find the following cases:

- Case A: $\ker(\sigma) = \langle a_1 \rangle$.
  - (A1) $\nu = 0, \mu \neq 0, x_1 + x_2 = -1$;
  - (A2) $\nu = 1, \mu \neq 0, x_1 = -x_2$;
  - (A3) $\nu = 1, \mu = 0$.
- Case B: $\ker(\sigma) = \langle a_1 a_2 \rangle$.
  - (B1) $\nu = 0, \mu = -1, x_1 = -x_2 - 2$;

As explained in Subsection 7.3, the results in the cases (A1*), (A2*), (A3*) and (B1*) can be obtained by doubling the results we will obtain in the cases (A1), (A2), (A3) and (B1).

Notice that p divides both $|\ker(\gamma)|$ and $|\ker(\tilde{\gamma})|$ if and only if σ has both 0 and 1 as eigenvalues, that is, in all the cases above except (A1), where, since σ has only 0 as eigenvalue, $p \mid |\ker(\gamma)|$ but $p \nmid |\ker(\tilde{\gamma})|$.

8.6.1. Invariant Sylow q-subgroups. Taking $k = -1$ in Subsubsection 7.3.1, we find the following.

Proposition 8.1. *If G is of type 9 and γ is a GF on G with $|\ker(\gamma)| = p$, the number of invariant Sylow q-subgroups is*

(A1) 1 *when* $x_1 \neq 0, -1$ *and* p *otherwise.*

(A2) 1 *when* $x_1 \neq 1$ *and* p *otherwise.*

(A3) 1 *when* $x_1 \neq 0$ *and* $x_2 \neq -1$, p^2 *when* $x_1 = 0$ *and* $x_2 = -1$, *and* p *otherwise.*

(B1) 1 *when* $x_1 \neq -1, -2$ *and* p *otherwise.*

8.6.2. Computations. By Subsubsection 7.3.2 the action $Z_\circ$ of b on A in $(G, \circ)$ is given by

$$Z_\circ = (\sigma(1 - Z) + Z)T,$$

and we obtain the following.

Case A. Here $\ker(\sigma) = \langle a_1 \rangle$ and equality (7.10) yields

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} & 0 \\ \mu(1-\lambda)\lambda^{x_1} & \lambda^{-x_2}(\nu(1-\lambda^{-1}) + \lambda^{-1}) \end{bmatrix}.$$

(A1) We have $p - 1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: if $x_1 = 0$ or $x_1 = -1$. If $x_1 = 0$ for each of the $(p-1)$ choices for σ we have p^2/p choices for B giving different GF's, so $p(p-1)$ groups. If $x_1 = -1$, then there are $p(p-1)$ groups.

Type 7: does not arise.

Type 8: if $x_1 \neq 0, -1, (q-1)/2$, and these are always three independent conditions. Since $x_1 \neq -1$, we get $p^2(p-1)(q-3)$ groups. They split in $2p^2(p-1)$ groups isomorphic to G_s for every $s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. Since $x_1 \neq -1$, we get $p^2(p-1)$ groups.

(A2) We have $p - 1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: when $x_1 = -1$. Since $x_1 \neq 1$, there are $p^2(p-1)$ groups.

Type 7: does not arise.

Type 8: if $x_1 \neq 0, -1, (q-1)/2$, and these are always three independent conditions. Here x_1 can be equal to 1 and so there are $p(p-1) + p^2(p-1)(q-4)$ groups. They split in $p(p-1) + p^2(p-1)$ groups isomorphic to G_2 and $2p^2(p-1)$ groups isomorphic to G_s for every $s \neq 2, s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. Here $x_1 = 1$ if and only if $q = 3$, so that there are $p^2(p-1)$ groups if $q > 3$ and $p(p-1)$ if $q = 3$.

(A3) We have 1 choice for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{-x_2} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: if $1 + x_1 = -x_2 = 0$. Since $x_1 \neq 0$ and $x_2 \neq -1$, there are p^2 groups.

Type 6: if either $x_1 = -1$ and $x_2 \neq 0$ or $x_1 \neq -1$ and $x_2 = 0$. In the first case, there are p groups when $x_2 = -1$, otherwise, for $x_2 \neq -1$, there are $p^2(q-2)$ groups. In the second case, since $x_2 = 0$, we have to take $x_1 \neq 0$ and there are $p^2(q-2)$ groups.

Type 7: when $-x_2 = 1 + x_1 \neq 0$. If $x_1 = 0$ and $x_2 = -1$, then there is one group. The cases $x_1 \neq 0$, $x_2 = -1$, and $x_1 = 0$, $x_2 \neq -1$ can not happen, while if $x_1 \neq 0$ and $x_2 \neq -1$ there are $p^2(q-2)$ groups.

Type 8: when $x_1 \neq -1, -x_2 - 1, x_2 - 1, x_2 \neq 0$. The case $x_1 = 0$ and $x_2 = -1$ can not happen. If $x_1 \neq 0$ and $x_2 = -1$, the four conditions on x_1 are actually three conditions, and there are $p(q-3)$ groups. If $x_1 = 0$ and $x_2 \neq -1$ we get further $p(q-3)$ groups.
Suppose now $x_1 \neq 0$, $x_2 \neq -1$.
There are always four independent conditions on x_1 except when $x_2 = 1$, where the conditions become three. There are $p^2((q-4)(q-3) + (q-3)) = p^2(q-3)^2$ groups.
Therefore we have just obtained $2p(q-3) + p^2(q-3)^2$ groups, which split in $4p + 2p^2(q-3)$ groups isomorphic to G_s for every $s \in \mathcal{K}$.

Type 9: if $x_1 \neq -1$, $x_2 \neq 0$ and $1 + x_1 - x_2 = 0$. For $x_2 = -1$ and $x_1 = -2 \neq 0$ there are p groups. Similarly, for $x_2 \neq -1$ and $x_1 = 0$ and there are p groups. For $x_2 \neq -1$ and $x_1 = x_2 - 1 \neq 0$, namely $x_2 \neq 0, \pm 1$, we get $p^2(q-3)$ groups.

Case B. Here $\ker(\sigma) = \langle a_1 a_2 \rangle$, $x_1 = -x_2 - 2$.

(B1) We have $p - 1$ choices for σ, and equality (7.10) yields

$$Z_\circ = \begin{bmatrix} \lambda^{x_1} & 0 \\ -\lambda^{x_1}(1-\lambda) & \lambda^{-x_2-1} \end{bmatrix} \sim \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_1+1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: when $x_1 = 0$ or $x_1 = -1$. In the first case $x_1 \neq -2, -1$ and there are $p^2(p-1)$ groups, while in the second case there are $p(p-1)$ groups.

Type 7: does not arise.

Type 8: when $x_1 \neq 0, -1, (q-1)/2$. If $x_1 = -2$ there are $p(p-1)$ groups. Suppose now $x_1 \neq -2$; the four conditions on x_1 are independent and there are $(p-1)(q-4)p^2$ groups. Therefore there are $p^2(p-1)(q-4)+p(p-1)$ groups, which split in $p(p-1)+p^2(p-1)$ groups isomorphic to G_2, and $2p^2(p-1)$ groups isomorphic to G_s for every $s \neq 2, s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$ ($x_1 \neq 0$ and $x_1 \neq -1$). There are $(p-1)p^2$ groups.

8.6.3. Conjugacy classes. As to the conjugacy classes, let $\varphi = \iota(x)\delta\psi$, where $x \in A$, $\delta \in \mathrm{GL}(2, p)$ is diagonal, and $\psi \in \mathcal{C}_2$. Suppose $\psi \neq 1$. Then

$$\gamma^{\varphi}(a_1) = \varphi^{-1}\gamma(a_1^{S\delta^{-1}})\varphi = \varphi^{-1}\iota(a_2^{-\delta_{22}^{-1}\sigma})\varphi = \varphi^{-1}\iota(a_1^{-\mu\delta_{22}^{-1}} a_2^{-\nu\delta_{22}^{-1}})\varphi.$$

In the case (A) we have that $\gamma(a_1) = 1$, and since $\iota(a_1^{-\mu\delta_{22}^{-1}} a_2^{-\nu\delta_{22}^{-1}}) \neq 1$, then $\gamma^{\varphi}(a_1) \neq \gamma(a_1)$. In the case (B1), $\gamma(a_1) = \iota(a_1^{-\sigma}) = \iota(a_1)$, and since here $\nu = 0$ and $\mu = -1$,

$$\gamma^{\varphi}(a_1) = \psi^{-1}\delta^{-1}\iota(a_1^{\delta_{22}^{-1}})\delta\psi = \psi^{-1}\iota(a_1^{\delta_{22}^{-1}\delta_{11}})\psi = \iota(a_2^{\delta_{22}^{-1}\delta_{11}}) \neq \gamma(a_1).$$

Therefore $\psi = 1$, and consider $\varphi = \iota(x)\delta$. Now taking $k = -1$ and δ diagonal in Subsubsection 7.3.3, we obtain the following.

In the case (A1), equation (7.11) has one solution if $x_1 \neq 0, -1$, and p solutions if $x_1 = 0, -1$, therefore the orbits have length $2p(p-1)$ if $x_1 = 0, -1$, and $2p^2(p-1)$ when $x_1 \neq 0, -1$.

In the case (A2), equation (7.11) has one solution if $x_1 \neq 1$, and p solutions if $x_1 = 1$, therefore the orbits have length $2p(p-1)$ when $x_1 = 1$, and $2p^2(p-1)$ when $x_1 \neq 1$.

In the case (A3), equation (7.11) has one solution if $x_1 \neq 0$ and $x_2 \neq -1$, p^2 solutions if $x_1 = 0$ and $x_2 = -1$, and p solutions otherwise. Therefore, the orbits have length $2p^2$ if $x_1 \neq 0$ and $x_2 \neq -1$, 1 if $x_1 = 0$ and $x_2 = -1$, and $2p$ otherwise.

In the case (B1), equation (7.11) has one solution if $x_1 \neq -1, -2$, and p solutions otherwise. Therefore, the orbits have length $2p^2(p-1)$ if $x_1 \neq -1, -2$, and $2p(p-1)$ otherwise.

Therefore, here the orbits have length:

1. in the case (A1), $2p(p-1)$ if $x_1 = 0, -1$, and $2p^2(p-1)$ otherwise;
2. in the case (A2), $2p(p-1)$ if $x_1 = -1$, and $2p^2(p-1)$ otherwise;
3. in the case (A3), $2p^2$ if $x_1 \neq 0$ and $x_2 \neq -1$, 2 if $x_1 = 0$ and $x_2 = -1$, and $2p$ otherwise;
4. in the case (B1), $2p(p-1)$ if $x_1 = -1, -2$, and $2p^2(p-1)$ otherwise.

Recap 8.2. *For G of type 9 and γ a GF on G with kernel of size p, we list for each isomorphism class of groups ($(G, \circ)$), the number (n) and the lengths (l) of the conjugacy classes in* $\mathrm{Hol}(G)$*.*

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

$T(n)$ denotes the total number of conjugacy classes.

$(G,\circ)$	*Conditions*	n	l	$T(n)$
5		1	$2p^2$	1
6		1	$2p$	$2(q+1)$
		$2(q-2)$	$2p^2$	
		3	$2p(p-1)$	
		2	$2p^2(p-1)$	
7		1	2	$q-1$
		$q-2$	$2p^2$	
8_{G_2}	$q>3$	4	$2p$	$2(q+2)$
		$2(q-3)$	$2p^2$	
		2	$2p(p-1)$	
		4	$2p^2(p-1)$	
8_{G_k}, $k \neq 2$	$q>3$	4	$2p$	$2(q+2)$
		$2(q-3)$	$2p^2$	
		6	$2p^2(p-1)$	
9	$q>2$	2	$2p$	$q+2$
		$q-3$	$2p^2$	
	$q=3$	1	$2p(p-1)$	
		2	$2p^2(p-1)$	
	$q>3$	3	$2p^2(p-1)$	

In the row of 8_{G_k} we mean that for every $k \in \mathcal{K}$, $k \neq 2$ there are n classes of length l of regular subgroups isomorphic to G_k.

8.7. The case $|\ker(\gamma)| = 1$. The GF's of this case can be divided into subclasses according to the size of $\ker(\tilde{\gamma})$. Those for which $|\ker(\tilde{\gamma})| \neq 1$ can be recovered via duality from the previous computations applied to $\tilde{\gamma}$. For the others, for which $|\ker(\tilde{\gamma})| = 1$, we will use Proposition 2.8.

We recall that $\tilde{\gamma}(x) = \gamma(x^{-1})\iota(x^{-1})$ for all $x \in G$, so $|\ker(\tilde{\gamma})| \neq 1$ means that there exists $x_0 \in G$, $x_0 \neq 1$, such that

$$\gamma(x_0) = \iota(x_0^{-1}) \tag{8.5}$$

whereas the condition $|\ker(\gamma)| = 1$ corresponds to

$$\tilde{\gamma}(x) \neq \iota(x^{-1}), \text{ for each } x \in G, x \neq 1. \tag{8.6}$$

Clearly, when $|\ker(\tilde{\gamma})| = p^2q$, $\gamma = \tilde{\tilde{\gamma}}$ corresponds to the left regular representation, and this gives one group of the same type as G.

In the remaining cases for which $q \mid |\ker(\tilde{\gamma})|$, the condition (8.6) is not fulfilled, so none of the corresponding γ's has trivial kernel.

Consider now the GF's γ for which $p \mid |\ker(\tilde{\gamma})|$ (and $q \nmid |\ker(\tilde{\gamma})|$). Here $\gamma(a) = \iota(a^{-\sigma})$, where σ has 1, but not 0, as eigenvalue (because $p \mid |\ker(\tilde{\gamma})|$ and γ is injective). Therefore, for each $a \in A$, we have that $\tilde{\gamma}(a) = \gamma(a^{-1})\iota(a^{-1}) = \iota(a^{\sigma-1})$.

Suppose that $\sigma = 1$. Then $|\ker(\tilde{\gamma})| = p^2$ and $\gamma(a) = \iota(a^{-1})$. Therefore, $p^2 \mid |\gamma(G)|$, and $\ker(\gamma)$ can have size 1 or q. We have $|\ker(\gamma)| = 1$ if and only if (8.6) is satisfied, and by Subsection 8.5, $\tilde{\gamma}(b) = \iota(b^{-1})$ if and only if $x_1 = x_2 = -1$. Therefore, the p^2 GF's $\tilde{\gamma}$ corresponding to $(G,\circ)$ of type 5 are such that the corresponding γ have kernel of size q, and all the others $\tilde{\gamma}$ correspond to γ with kernel of size 1.

Suppose now that $\sigma \neq 1$. Since σ has 1 as eigenvalue, then $\left|\ker(\tilde{\gamma})\right| = p$, and if $a_0 \in A$ generates $\ker(\tilde{\gamma})$, then $\gamma(a_0) = \iota(a_0^{-1})$, namely $p \mid \left|\gamma(G)\right|$. Moreover, since 0 is not an eigenvalue for σ, $p \nmid \left|\ker(\gamma)\right|$. Therefore, again, $\ker(\gamma)$ can have size 1 or q. By Subsection 8.6, the $\tilde{\gamma}$'s such that $p \mid \left|\ker(\tilde{\gamma})\right|$ and $p \nmid \left|\ker(\gamma)\right|$ are those of the cases (A1) and (A1*). Moreover, for every $\tilde{\gamma}$ belonging to these cases the condition (8.6) is satisfied, namely the corresponding γ are injective.

We are left with the case when $\left|\ker(\gamma)\right| = \left|\ker(\tilde{\gamma})\right| = 1$. In the following, we suppose that σ has no eigenvalues 0 or 1.

Here $\gamma(G) = \langle\, \iota(a_1), \iota(a_2), \beta \,\rangle$, where $\beta \neq 1$. As in Subsubsection 7.1, if b is an element of order q fixed by β, we can assume $\gamma(b) = \iota(a_*)\beta$ for some $a_* \in A^\sigma$, and $\beta = \beta_1^{x_1}\beta_2^{x_2}$. As usual, denote by T the matrix of $\gamma(b)_{|A}$ with respect to the basis $\{a_1, a_2\}$. The discussion in Subsection 8.2 yields equation (4.2), which in our notation here is

$$(\sigma^{-1} - 1)^{-1} T (\sigma^{-1} - 1) = TZ. \tag{8.7}$$

Now T and TZ, being conjugate, have the same eigenvalues, so that $\lambda^{-x_2} = \lambda^{x_1+1}$. Therefore $T = \operatorname{diag}(\lambda^{x_1}, \lambda^{1+x_1})$, and $\sigma^{-1} - 1$ exchanges the two eigenspaces, so

$$\sigma^{-1} - 1 = \begin{bmatrix} 0 & s_1 \\ s_2 & 0 \end{bmatrix} \tag{8.8}$$

with the conditions $s_1, s_2 \neq 0$ (due to our assumptions on the eigenvalues of σ) and $s_1 s_2 \neq 1$.

8.7.1. Invariant Sylow *q*-subgroups. We will show that also in this case there always exists at least one invariant Sylow q-subgroup. By the discussion above $\gamma(b) = \iota(a_*)\beta_1^{x_1}\beta_2^{-(x_1+1)}$, for some $a_* \in A^\sigma$ and $0 \leq x_1 < q$.

By Subsection 7.1 there exists an invariant Sylow q-subgroup, $\langle\, b^x \,\rangle$ where $x \in A$, if and only if the equation (7.4), namely

$$x^{(1-Z^{-1})M} = a_*^{(1-Z^{-1})T}$$

where $M = 1 - (1 + T^{-1}\sigma(1 - Z^{-1}))T$, has a solution in x.

Here

$$M = \begin{bmatrix} 1 - \lambda^{x_1} - \frac{(1-\lambda^{-1})}{1-s_1s_2} & \frac{s_1\lambda(1-\lambda)}{1-s_1s_2} \\ \frac{s_2\lambda^{-1}(1-\lambda^{-1})}{1-s_1s_2} & 1 - \lambda^{x_1+1} - \frac{(1-\lambda)}{1-s_1s_2} \end{bmatrix},$$

and since $\det(1 - Z^{-1}) \neq 0$ and $\det(M) = (1 - \lambda^{x_1})(1 - \lambda^{x_1+1})$, we have the following.

(1) If $x_1 \neq 0, -1$, then M has rank 2 and the system (7.4) admits a unique solution.

(2) If $x_1 = 0$, then, writing $a_* = a_1^x a_2^y$, the system (7.4) admits solutions if and only if $y = -s_1 x$. Moreover, in that case there are p solutions. If $y \neq -s_1 x$, then there are no GF on G extending the assignment $\gamma(b) = \iota(a_*)\beta$, as the condition (7.6) is not satisfied.

(3) If $x_1 = -1$, then (7.4) admits p solutions if and only if $a_* = a_1^{-s_2y}a_2^y$. Reasoning as above, if $x \neq -s_2y$ there are no GF on G extending the assignment $\gamma(b) = \iota(a_*)\beta$.

8.7.2. Computations. Since in this case all gamma functions fulfil (2.7), we can count them using Proposition 2.8, as follows.

- Choose $\sigma \in \mathrm{GL}(2,p)$ without eigenvalues 1, and a RGF $\gamma : B \to \mathrm{Aut}(G)$ such that σ and γ satisfy (2.5) (q choices for γ corresponding to $\gamma(b) = \beta_1^{x_1}\beta_2^{-(x_1+1)}$ and $(p-1)(p-2)$ choices for σ as in equation (8.8)).
- By Proposition 2.8 each such assignment defines a unique function γ, and the GF's obtained in this way are distinct for $x_1 \neq 0, -1$ (namely when $\gamma(G)$ is centerless) and each of them is obtained p times when $x_1 = 0$ or -1 (namely when $Z(\gamma(G))$ is non trivial).

Now, let γ be a GF obtained for a choice of σ, B, x_1. Since γ is injective, $(G, \circ)$ is isomorphic to $\gamma(G)$. We have

$$\gamma(b)^{-1}\gamma(a)\gamma(b) = \gamma(b)^{-1}\iota(a^{-\sigma})\gamma(b) = \iota(a^{-\sigma T}) = \gamma(a^{\sigma T\sigma^{-1}}),$$

from which we obtain,

$$b^{\ominus 1}\circ a\circ b = a^{\sigma T\sigma^{-1}}.$$

Since $a^{\circ t} = a^t$ for all t, the action of $\iota(b)$ on A in $(G, \circ)$ is

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_1+1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.
Type 6: when $x_1 = 0$ or $x_1 = -1$ and there are $2p(p-1)(p-2)$ groups.
Type 7: does not arise.
Type 8: when $x_1 \neq 0, -1, (q-1)/2$ and there are $p^2(p-1)(p-2)(q-3)$ groups. They are $2p^2(p-1)(p-2)$ groups isomorphic to G_s for every $s \in \mathcal{K}$.
Type 9: when $x_1 = (q-1)/2$ and there are $p^2(p-1)(p-2)$ groups.

8.7.3. Conjugacy classes. As to the conjugacy classes, in the notation of Subsection 8.4 , let $\varphi = \iota(x)\delta\psi \in \mathrm{Aut}(G)$. We have

$$\gamma^\varphi(a) = \varphi^{-1}\gamma(a^{S\delta^{-1}})\varphi = \psi^{-1}\delta^{-1}\iota(a^{-S\delta^{-1}\sigma})\delta\psi = \iota(a^{-S\delta^{-1}\sigma\delta S}),$$

so that $\gamma^\varphi(a) = \gamma(a)$ if and only if $\sigma^{-1}\delta S\sigma = \delta S$. The last condition yields $\delta_{22} = \delta_{11}$ if $S = 1$, and $\delta_{22} = \frac{s_2}{s_1}\delta_{11}$ if $S \neq 1$.

Now,

$$\gamma^\varphi(b) = \varphi^{-1}\gamma(b^{\varphi^{-1}})\varphi = \varphi^{-1}\gamma(x^{1-Z^{-r}}b^r)\varphi;$$

suppose first $\psi = 1$. Using Proposition 2.8, we obtain

$$\begin{aligned}\gamma^\varphi(b) &= \varphi^{-1}\gamma(x^{1-Z^{-1}}b)\varphi \\ &= \varphi^{-1}\iota(x^{-(1-Z^{-1})T^{-1}\sigma})\beta\varphi\end{aligned}$$

$$\begin{aligned} &= \delta^{-1}\iota(x^{-1+T^{-1}-(1-Z^{-1})T^{-1}\sigma})\beta\delta \\ &= \iota(x^{(-1+T^{-1}-(1-Z^{-1})T^{-1}\sigma)\delta})\beta, \end{aligned}$$

so that $\gamma^{\varphi}(b) = \gamma(b)$ if and only if the system $x^{H_1} = 1$, where $H_1 := -1 + T^{-1} - (1-Z^{-1})T^{-1}\sigma$, admits a solution. Since $\det(H_1) = (1-\lambda^{-x_1})(1-\lambda^{-x_1-1})$, there is one solution if $x_1 \neq 0, -1$, and p solutions otherwise.

Suppose now that $\psi \neq 1$. Since $\gamma(b^{-1}) = \beta^{-1}$, we have

$$\begin{aligned} \gamma^{\varphi}(b) &= \varphi^{-1}\gamma(x^{1-Z}b^{-1})\varphi \\ &= \varphi^{-1}\iota(x^{-(1-Z)T\sigma})\gamma(b^{-1})\varphi \\ &= \varphi^{-1}\iota(x^{-(1-Z)T\sigma})\beta^{-1}\varphi \\ &= \psi\delta^{-1}\iota(x^{-1+T-(1-Z)T\sigma})\beta^{-1}\delta\psi \\ &= \psi\iota(x^{(-1+T-(1-Z)T\sigma)\delta})\beta^{-1}\psi. \end{aligned}$$

If $H_2 := -1 + T - (1-Z)T\sigma$, we have that $\gamma^{\varphi}(b) = \gamma(b)$ if and only if

$$\iota(x^{H_2\delta})\beta^{-1} = \psi\beta\psi,$$

namely if and only if

$$\begin{cases} x^{H_2} = 1 \\ T^{-1} = STS. \end{cases}$$

Since $\det(H_2) = (1-\lambda^{x_1})(1-\lambda^{x_1+1})$, the system $x^{H_2} = 1$ has one solution if $x_1 \neq 0, -1$ and p solutions otherwise, while the condition $T^{-1} = STS$ is satisfied if and only if $x_1 = \frac{q-1}{2}$.

We obtain the following.

(1) if $x_1 = 0, -1$, then the stabiliser has order $p(p-1)$. Here there are $2p(p-1)(p-2)$ groups $(G, \circ)$ of type 6, so that there are $p-2$ orbits of length $2p(p-1)$.
(2) if $x_1 = \frac{q-1}{2}$, then $(G, \circ)$ is of type 9, and the stabiliser has order $2(p-1)$. Since there are $p^2(p-1)(p-2)$ groups, they split in $p-2$ orbits of length $p^2(p-1)$.
(3) if $x_1 \neq 0, -1, \frac{q-1}{2}$, then $(G, \circ)$ is of type 8, and the stabiliser has order $p-1$. Since for every $s \in \mathcal{K}$ there are $2p^2(p-1)(p-2)$ groups isomorphic to G_s, they split in $p-2$ classes for every $s \in \mathcal{K}$.

8.8. Results.

Proposition 8.3. *Let G be a group of order p^2q, $p, q > 2$, of type 9. For each isomorphism class of groups (Γ), the number of regular subgroups in* $\mathrm{Hol}(G)$ *(RS), and the number (n) and the lengths (l) of the conjugacy classes in* $\mathrm{Hol}(G)$ *are listed in the following table.*

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

$T(n)$ denotes the total number of conjugacy classes.

Γ	*Conditions*	*RS*	n	l	$T(n)$
5		$4p^2$	2	p^2	3
			1	$2p^2$	
6		$2p^2(4q+3p-7)$	4	$2p$	$4q+p+2$
			$4(q-2)$	$2p^2$	
			$p+4$	$2p(p-1)$	
			2	$2p^2(p-1)$	
7		$2+4p+2p^2(2q-5)$	1	2	$2(q-1)$
			2	$2p$	
			$2q-5$	$2p^2$	
8_{G_2}	$q>3$	$2p(p^3+3p^2-14p+4pq-6)$	8	$2p$	$4q+p+2$
			$4(q-3)$	$2p^2$	
			2	$2p(p-1)$	
			$p+4$	$2p^2(p-1)$	
8_{G_s}, $s \neq 2$	$q>3$	$2p(p^3+5p^2-18p+4pq+8)$	8	$2p$	$4q+p+2$
			$4(q-3)$	$2p^2$	
			$p+6$	$2p^2(p-1)$	
9	$q>3$	$2+4p+p^2(p^2+5p+4q-16)$	2	1	$3q+p-1$
			$2(q-2)$	p^2	
			2	$2p$	
			$q-3$	$2p^2$	
			$p-2$	$p^2(p-1)$	
			4	$2p^2(p-1)$	
	$q=3$	$2+2p+p^3(p+3)$	2	1	$8+p$
			2	p^2	
			2	$2p$	
			$p-2$	$p^2(p-1)$	
			1	$2p(p-1)$	
			3	$2p^2(p-1)$	

In the row of 8_{G_s} we mean that for every $s \in \mathcal{K}$, $s \neq 2$, there are $2p(p^3+5p^2-18p+4pq+8)$ *regular subgroups isomorphic to G_s.*

9. Type 8

This case can be handled in a very similar way to the case in which G is type 9, so in the following we will often refer to the previous Section, highlighting only the points that require a different treatment.

Here $q \mid p-1$, $q > 3$, and G is isomorphic to one of the groups $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_{D0} \mathcal{C}_q$. The Sylow p-subgroup $A = \langle a_1, a_2 \rangle$ of G is characteristic, and if $a_1, a_2 \in A$ are in the eigenspaces of the action of a generator b of a Sylow q-subgroup B on

A, then this action can be represented by a non-scalar diagonal matrix Z, with no eigenvalues 1 and $\det(Z) \neq 1$.

For all of this section, we consider $A = \langle a_1, a_2 \rangle$, where a_1, a_2 are eigenvectors for $\iota(b)$. With respect to that basis, we have

$$Z = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^k \end{bmatrix},$$

where λ has order q, and $k \neq 0, \pm 1$.

Recall that the type 8 includes $\frac{q-3}{2}$ different isomorphism classes of groups, and that $\{G_k : k \in \mathcal{K}\}$ denotes a set of representatives of the isomorphism classes (see Section 3).

According to Subsections 4.1 and 4.3 of [8], we have

$$\operatorname{Aut}(G) = \operatorname{Hol}(\mathcal{C}_p) \times \operatorname{Hol}(\mathcal{C}_p);$$

as in the case of the groups of type 9, all the elements of the Sylow p-subgroup of $\operatorname{Aut}(G)$ are conjugation by elements of A, and for each gamma function γ there exists $\sigma \in \operatorname{End}(A)$ such that (7.1), namely $\gamma(a) = \iota(a^{-\sigma})$, is satisfied for each $a \in A$ (see Section 8).

9.1. Duality. For G of type 8 the discussion in Subsections 4.1 and 4.2 yields that σ has 0 or 1 as an eigenvalue, and this corresponds to have $p \mid |\ker(\gamma)|$ or $p \mid |\ker(\tilde{\gamma})|$.

Therefore, we can assume that $p \mid |\ker(\gamma)|$ (equivalently σ has 0 as an eigenvalue), and once we have counted the gamma functions with this property, we will double the number of those for which moreover $p \nmid |\ker(\tilde{\gamma})|$ (we will double only those GF for which 1 is not an eigenvalue of σ).

9.2. Description of the elements of order q of Aut(G). The discussion in Subsubsection 8.2.1 yields that, if $b \in G \setminus A$, recalling that $\iota(b)$ acts on A as $\operatorname{diag}(\lambda, \lambda^k)$, we can write

$$\begin{aligned} \beta_1 : a_1 &\mapsto a_1^{\lambda} & \qquad \beta_2 : a_1 &\mapsto a_1 \\ a_2 &\mapsto a_2 & a_2 &\mapsto a_2^{\lambda^k} \\ b &\mapsto b & b &\mapsto b \end{aligned} \tag{9.1}$$

so that $\iota(b) = \beta_1\beta_2$, and if $\beta \in \operatorname{Aut}(G)$ is an element of order q, then $\beta \in \langle \beta_1, \beta_2 \rangle$, namely $\beta = \beta_1^{x_1}\beta_2^{x_2}$, where $0 \leq x_1, x_2 < q$ are not both zero.

Let us start with the enumeration of the GF's on G. As usual, if $|\ker(\gamma)| = p^2 q$, then γ corresponds to the right regular representation, so that we will assume $\gamma \neq 1$.

Suppose moreover that $G \simeq G_k$, for a certain $k \in \mathcal{K}$.

9.3. The case $|\ker(\gamma)| = pq$. Reasoning as in Subsection 8.4, we obtain $2p^2$ gamma functions corresponding to groups $(G, \circ)$ of type 6.

Moreover, for every γ here, $p \mid |\ker(\tilde{\gamma})|$.

As to the conjugacy classes, this time an automorphism φ of G has the form $\varphi = \iota(x)\delta$, where $x \in A$ and δ is such that $\delta_{|B} = 1$, $\delta_{|A} = (\delta_{ij}) \in \mathrm{GL}(2, p)$ diagonal with respect to the fixed basis.

With the same computations of Subsection 8.4 we obtain two orbits of length p and two orbits of length $p(p-1)$.

9.4. The case $|\ker(\gamma)| = p^2$. Reasoning as in Subsection 8.5, we obtain that each γ on G is the lifting of at least one RGF defined on an invariant Sylow q-subgroup B, and the RGF's on B are precisely the morphisms. We have $\gamma(b)_{|B} = 1$, and let $\gamma(b)_{|A} = \beta$; then the discussion in Subsubsection 9.2 yields that $\beta = \beta_1^{x_1}\beta_2^{x_2}$, where x_1, x_2 not both zero, so that, with respect to the basis $\{a_1, a_2\}$, we can represent β as the matrix

$$T = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{kx_2} \end{bmatrix},$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$, x_1, x_2 are not both zero, and $k \in \mathcal{K}$ is such that $G \simeq G_k$.

To know the exact number of the invariant Sylow q-subgroups we appeal to the discussion in Subsubsection 7.1; here equation (7.4) yields $x^{(1-Z^{-1})M} = 1$, where

$$M = 1 - T = \begin{bmatrix} 1-\lambda^{x_1} & 0 \\ 0 & 1-\lambda^{kx_2} \end{bmatrix},$$

and we obtain that

(1) if both $x_1, x_2 \neq 0$, there is a unique invariant Sylow q-subgroup;
(2) if either $x_1 = 0$ or $x_2 = 0$, there are p invariant Sylow q-subgroups.

Denoting as usual by $Z_\circ$ the associated matrix of the action of b on A with respect to the operation $\circ$, here we have

$$Z_\circ \sim \begin{bmatrix} \lambda^{1+x_1} & 0 \\ 0 & \lambda^{k+kx_2} \end{bmatrix},$$

and we obtain precisely the same number of groups $(G, \circ)$ of type 5, 6, and 7 as in Subsection 8.5. As for the type 8 and 9, we have the following.

Type 8. In case (1) the type 8 corresponds to the conditions $x_2 \neq 0, -1$ and $x_1 \neq 0, -1, -kx_2 - k - 1, kx_2 + k - 1$. The conditions on x_1 are independent if and only if in addition $x_2 \neq k^{-1} - 1, -k^{-1} - 1$. When these four conditions on x_1 are dependent, they reduce to three.

We obtain $p^2(q-4)^2$ groups if $x_2 \neq -k^{-1} - 1, k^{-1} - 1$, plus further $p^2(q-3)$ groups if $x_2 = -k^{-1} - 1, k^{-1} - 1$.

In case (2), suppose $x_1 = 0$. Then, there are four independent conditions on x_2. Doubling for the case $x_2 = 0$, we obtain $2p(q-4)$ groups.

Now, looking at the eigenvalues of $Z_\circ$ as x_1 and x_2 vary, and taking into account the conditions on x_1 and x_2, one can see that the $2p^2(q-3)$ groups split in $4p^2$ groups isomorphic to G_s for every $s \in \mathcal{K}$, and the $p^2(q-4)^2 + 2p(q-4)$ groups split in $2p^2(q-5) + 4p$ groups isomorphic to G_s for every $s \neq k$, and $2p^2(q-5) + p^2 + 2p$ groups isomorphic to G_k. Therefore, in total, we have

- $2p^2(q-3) + 4p$ groups isomorphic to G_s for every $s \neq k$;
- $p^2(2q-5) + 2p$ groups isomorphic to G_k.

Type 9. $(G, \circ)$ is of type 9 when $x_1 \neq -1, kx_2 + k - 1, x_2 \neq -1$, and $x_1 + 1 + kx_2 + k = 0$. In case (1) $x_2 \neq 0, -1$ and also $x_2 \neq -k^{-1} - 1$ (otherwise we would have $x_1 = 0$). Since the last one is a further condition, we obtain $p^2(q-3)$ groups. The case (2) yields $2p$ groups.

Summing up, there are $2p + p^2(q-3)$ groups.

As to the conjugacy classes, with the same computations as in Subsection 8.5 (imposing $\psi = 1$), we obtain the following.

(1) For $(G, \circ)$ of type 5 there is one orbit of length p^2;
(2) For $(G, \circ)$ of type 6 we obtain 2 orbits of length p and $2(q-2)$ orbits of length p^2;
(3) For $(G, \circ)$ of type 7 there are $q-3$ orbits of length p^2 and 2 orbits of length p.
(4) For $(G, \circ)$ of type 8, $(G, \circ) \simeq G_s$, then for every $s \neq k$, $s \in \mathcal{K}$, we obtain $2(q-3)$ orbits of length p^2 and 4 orbits of length p; otherwise $s = k$, and we get $2q-5$ orbits of length p^2 and 2 orbits of length p.
(5) For $(G, \circ)$ of type 9 there are $q-3$ orbits of length p^2 and 2 orbits of length p.

9.5. The case $|\ker(\gamma)| = p$. As in Subsection 8.6, we have $\gamma(G) = \langle \iota(a_0), \beta \rangle$, for some $1 \neq a_0 \in A$ with $A^\sigma = \langle \iota(a_0) \rangle$, and $\beta \neq 1$. We can assume $\gamma(b) = \iota(a_0^j)\beta$ for some j, where $\beta = \beta_1^{x_1}\beta_2^{x_2}$ (by Subsection 9.2). With respect to the fixed basis we can represent $\beta_{|A}$ as

$$T = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{kx_2} \end{bmatrix}.$$

Also here $\ker(\gamma) = \ker(\sigma) = \langle v \rangle$ and from equation (2.5) we obtain (7.7), and we distinguish in the three cases $v \in \langle a_1 \rangle$, $v \in \langle a_2 \rangle$, and $v = a_1^x a_2^y$.

Following Subsection 7.3, and recalling that $k \neq \pm 1, 0$, here we find the following cases.

- Case A: $\ker(\sigma) = \langle a_1 \rangle$.
  (A1) $\nu = 0, \mu \neq 0, x_1 - x_2 k = k$;
  (A2) $\nu = 1, \mu \neq 0, x_1 = x_2 k$;
  (A3) $\nu = 1, \mu = 0$.
- Case B: $\ker(\sigma) = \langle a_1 a_2 \rangle$.
  (B1) $\nu = 0, \mu = -1, x_1 = x_2 k + k - 1$;

As explained in Subsection 7.3, the cases (A1*), (A2*), (A3*) and (B1*) can be recovered by the cases (A1), (A2), (A3) and (B1) considering k^{-1} instead of k.

Notice that p divides both $|\ker(\gamma)|$ and $|\ker(\tilde{\gamma})|$ if and only if σ has both 0 and 1 as eigenvalues, that is, in all the cases above except (A1) and (A1*), where, since σ has only 0 as eigenvalue, $p \mid |\ker(\gamma)|$ but $p \nmid |\ker(\tilde{\gamma})|$.

9.5.1. Invariant Sylow q-subgroups. Taking $k \neq \pm 1, 0$ in Subsubsection 7.3.1, we find the following.

Proposition 9.1. *If G is of type 8 and $|\ker(\gamma)| = p$, then the number of invariant Sylow q-subgroups is*

(A1) 1 *when* $x_1 \neq 0, k$ *and* p *otherwise.*
(A2) 1 *when* $x_1 \neq -k$ *and* p *otherwise.*
(A3) 1 *when* $x_1 \neq 0$ *and* $x_2 \neq -1$, p^2 *when* $x_1 = 0$ *and* $x_2 = -1$, *and* p *otherwise.*
(B1) 1 *when* $x_1 \neq -1, k-1$ *and* p *otherwise.*
(B1*) 1 *when* $x_2 \neq -1, -1+k^{-1}$ *and* p *otherwise.*

9.5.2. Computations. By Subsubsection 7.3.2 the action $Z_\circ$ of b on A in $(G, \circ)$ is given by

$$Z_\circ = (\sigma(1-Z) + Z)T,$$

and we obtain the following.

Case A. Here

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} & 0 \\ \mu(1-\lambda)\lambda^{x_1} & \lambda^{x_2 k}(\nu(1-\lambda^k)+\lambda^k) \end{bmatrix}.$$

(A1) We have $p-1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: if $x_1 = 0$ or $x_1 = -1$. If $x_1 = 0$ we obtain $p(p-1)$ groups; if $x_1 = -1$ there are $p^2(p-1)$ groups.

Type 7: does not arise.

Type 8: if $x_1 \neq 0, -1, (q-1)/2$. Suppose first that $k = (q-1)/2$; then $x_1 \neq k$ and there are $p^2(p-1)(q-3)$ groups. Otherwise $k \neq (q-1)/2$ and there are $p(p-1) + p^2(p-1)(q-4)$ groups. Therefore, if $k = \frac{q-1}{2}$, there are $2p^2(p-1)$ groups isomorphic to G_s for every $s \in \mathcal{K}$, and if $k \neq \frac{q-1}{2}$, then there are $p(p-1)+p^2(p-1)$ groups isomorphic to $G_{1+k^{-1}}$ (obtained for $x_1 = k, -(k+1)$) and $2p^2(p-1)$ groups isomorphic to G_s for every $s \neq 1+k^{-1}, s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. When $k = (q-1)/2$ then $x_1 = k$ and there are $p(p-1)$ groups. Otherwise $k \neq (q-1)/2$ so that $x_1 \neq k$ and there are $p^2(p-1)$ groups.

(A2) We have $p-1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: when $x_1 = -1$. Here $x_1 \neq -k$ and there are $p^2(p-1)$ groups.

Type 7: does not arise.

Type 8: if $x_1 \neq 0, -1, (q-1)/2$. When $k = 1/2$ then $x_1 \neq -k$ and there are $(p-1)(q-3)p^2$ groups, which split $2p^2(p-1)$ groups isomorphic to G_s for every $s \in \mathcal{K}$. If $k \neq 1/2$ then there are $(p-1)p + p^2(p-1)(q-4)$ groups, which split in $p^2(p-1) + p(p-1)$ groups isomorphic to $G_{1-k^{-1}}$, and $2p^2(p-1)$ groups isomorphic to G_s for every $s \neq 1-k^{-1}, s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. If $k = 1/2$ then $x_1 = -k$ and there are $p(p-1)$ groups. Otherwise $k \neq 1/2$, so $x_1 \neq -k$ and there are $p^2(p-1)$ groups.

(A3) We have 1 choice for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_2 k} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: if $1 + x_1 = x_2 k = 0$. There are p^2 groups.

Type 6: if either $x_1 = -1$ and $x_2 \neq 0$ or $x_1 \neq -1$ and $x_2 = 0$. In the first case, there are p groups when $x_2 = -1$, and $p^2(q-2)$ otherwise. In the second case, since $x_2 = 0$, we have to take $x_1 \neq 0$ and there there $p^2(q-2)$ groups.

Type 7: when $x_2 k = 1 + x_1 \neq 0$. In both the cases $x_1 \neq 0$, $x_2 = -1$, and $x_1 = 0$, $x_2 \neq -1$ there are $2p$ groups. If $x_1 \neq 0$ and $x_2 \neq -1$ there are $p^2(q-3)$ groups.

Type 8: when $x_1 \neq -1, x_2 k - 1, -x_2 k - 1$, $x_2 \neq 0$. If $x_1 = 0$ and $x_2 = -1$, then there is one group. If $x_1 \neq 0$ and $x_2 = -1$, the four conditions on x_1 are independent, and so there are $p(q-4)$ groups. If $x_1 = 0$ and $x_2 \neq -1$ we get further $p(q-4)$ groups. Suppose now $x_1 \neq 0$, $x_2 \neq -1$; there are always four independent conditions on x_1, except when $x_2 = \pm k^{-1}$, where the conditions become three. Thus there are $p^2(q-4)^2 + 2p^2(q-3)$ groups.

The $2p^2(q-3)$ groups split in $4p^2$ groups isomorphic to G_s for every $s \in \mathcal{K}$, and the $1+2p(q-4)+p^2(q-4)^2$ groups split in $4p+2p^2(q-5)$ groups isomorphic to G_s for every $s \neq -k$, and $1 + 2p + p^2 + 2p^2(q-5)$ groups isomorphic to G_{-k}. Therefore, in total, there are $1 + 2p + p^2(2q-5)$ groups isomorphic to G_{-k}, and $4p + 2p^2(q-3)$ groups isomorphic to G_s for every $s \neq -k, s \in \mathcal{K}$.

Type 9: if $x_1 \neq -1$, $x_2 \neq 0$ and $1 + x_1 + x_2k = 0$. For $x_2 = -1$ and $x_1 = k - 1 \neq 0$ there are p groups. Similarly, for $x_2 \neq -1$ and $x_1 = 0$ and there are p groups. For $x_2 \neq -1$ and $x_1 = -x_2k - 1 \neq 0$, namely $x_2 \neq 0, -1, k^{-1}$, we get $p^2(q-3)$ groups.

Case B. Here $x_1 = x_2k + k - 1$.

(B1) We have $p - 1$ choices for σ, and

$$Z_\circ = \begin{bmatrix} \lambda^{x_1} & 0 \\ -\lambda^{x_1}(1-\lambda) & \lambda^{x_2k+k} \end{bmatrix} \sim \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_1+1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: when $x_1 = 0$ or $x_1 = -1$. In the first case $x_1 \neq k-1, -1$ and there are $p^2(p-1)$ groups, while in the second case there are $p(p-1)$ groups.

Type 7: does not arise.

Type 8: when $x_1 \neq 0, -1, -1/2$. If $x_1 = k - 1$ (and thus $k \neq 1/2$) there are $p(p-1)$ groups. Suppose now $x_1 \neq k-1$; when $k \neq 1/2$ (in particular when G is of type 9) the four conditions on x_1 are independent and there are $(p-1)(q-4)p^2$ groups. Otherwise $k = 1/2$ and the four conditions are actually three, thus there are $p^2(p-1)(q-3)$ groups. Therefore there are $p^2(p-1)(q-3)$ groups if $k = 1/2$, and they split in $2p^2(p-1)$ groups isomorphic to G_s for every $s \in \mathcal{K}$. If $k \neq 1/2$ there are $p^2(p-1)(q-4)+p(p-1)$ groups, which split in $p(p-1)+p^2(p-1)$ groups isomorphic to $G_{1-k^{-1}}$, and $2p^2(p-1)$ groups isomorphic to G_s for every $s \neq 1-k^{-1}, s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. We have that $x_1 = k - 1$ when $k = 1/2$ and in this case there are $(p-1)p$ groups. Otherwise $k \neq 1/2$ and there are $(p-1)p^2$ groups.

As for the conjugacy classes, here an automorphism of G has the form $\varphi = \iota(x)\delta$, where $x \in A$ and δ is a diagonal matrix. Therefore, we can refer to Subsubsection 7.3.3 for the computation of the conjugacy classes.

Summing up all the results obtained for the kernel of size p, we have the following.

Recap 9.2. *For G of type 8, $G \simeq G_k$, and γ a GF on G with kernel of size p, we list for each isomorphism class of groups ($(G, \circ)$), the number (n) and the lengths (l) of the conjugacy classes in* $\mathrm{Hol}(G)$*.*

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

$T(n)$ denotes the total number of conjugacy classes.

Cases	*Conditions*	$(G,\circ)$	n	l	$T(n)$
		5	2	p^2	2
		6	2	p	$4(q+1)$
			$4(q-2)$	p^2	
			4	$p(p-1)$	
			6	$p^2(p-1)$	
		7	4	p	$2(q-1)$
			$2(q-3)$	p^2	
(A3)+(A3)*		$8_{G_{-k}}$	2	1	$4(q-1)$
			4	p	
			$2(2q-5)$	p^2	
		$8_{G_s}, s \neq -k$	8	p	$4(q-1)$
			$4(q-3)$	p^2	
(A1)+(A1)*	*either k or k^{-1} is equal to $q-2$*	8_{G_2}	1	$p(p-1)$	4
			3	$p^2(p-1)$	
		$8_{G_s}, s \neq 2$	4	$p^2(p-1)$	4
	$k, k^{-1} \neq q-2$	$8_{G_s}, s \neq 1+k, 1+k^{-1}$	4	$p^2(p-1)$	4
	$k, k^{-1} \neq q-2$ *and* $G_{1+k} \simeq G_{1+k^{-1}}$	$8_{G_{1+k}}$	2	$p(p-1)$	4
			2	$p^2(p-1)$	
	$k, k^{-1} \neq q-2$ *and* $G_{1+k} \not\simeq G_{1+k^{-1}}$	$8_{G_{1+k}}, 8_{G_{1+k^{-1}}}$	1	$p(p-1)$	4
			3	$p^2(p-1)$	
(A2)+(A2) + (B1)+(B1*)*	*either k or k^{-1} is equal to 2*	8_{G_2}	2	$p(p-1)$	8
			6	$p^2(p-1)$	
		$8_{G_s}, s \neq 2$	8	$p^2(p-1)$	8
	$k, k^{-1} \neq 2$	$8_{G_s}, s \neq 1-k, 1-k^{-1}$	8	$p^2(p-1)$	8
	$k, k^{-1} \neq 2$ *and* $G_{1-k} \simeq G_{1-k^{-1}}$	$8_{G_{1-k}}$	4	$p(p-1)$	8
			4	$p^2(p-1)$	
	$k, k^{-1} \neq q-2$ *and* $G_{1-k} \not\simeq G_{1-k^{-1}}$	$8_{G_{1-k}}, 8_{G_{1-k^{-1}}}$	2	$p(p-1)$	8
			6	$p^2(p-1)$	
(A3)+(A3)*		9	4	p	$2(q-1)$
			$2(q-3)$	p^2	
(A1)+(A1)*	*either k or k^{-1} is equal to $q-2$*	9	1	$p(p-1)$	2
			1	$p^2(p-1)$	
	$k, k^{-1} \neq q-2$	9	2	$p^2(p-1)$	2
(A2)+(A2) + (B1)+(B1*)*	*either k or k^{-1} is equal to 2*	9	2	$p(p-1)$	4
			2	$p^2(p-1)$	
	$k, k^{-1} \neq 2$	9	4	$p^2(p-1)$	4

In the rows of 8_{G_s} we mean that for every $s \in \mathcal{K}$ there are n classes of length l of regular subgroups isomorphic to G_s.

9.6. Results.

Proposition 9.3. *Let G be a group of order p^2q, $p > 2$, $q > 3$, of type 8, so that G is isomorphic to G_k, where k, an integer modulo q, $k \neq 0, 1, -1$, determines the isomorphism class of G. For each isomorphism class of groups (Γ), the number of regular subgroups in $\mathrm{Hol}(G)$ (RS), and the number (n) and the lengths (l) of the conjugacy classes in $\mathrm{Hol}(G)$ are listed in the following table.*

$T(n)$ denotes the total number of conjugacy classes.

Γ	*Conditions*	*RS*	n	l	$T(n)$
5		$4p^2$	4	p^2	4
6		$8p^2(q+p-2)$	8	p	$8(q+1)$
			$8(q-2)$	p^2	
			8	$p(p-1)$	
			8	$p^2(p-1)$	

7		$8p + 4p^2(q-3)$	8	p	$4(q-1)$
			$4(q-3)$	p^2	
8	$G \not\simeq G_{\pm 2}$, $q > 5$	*Table* 5			$8(q+1)$
	$G \simeq G_{\pm 2}$, $q > 3$	*Table* 6			
9	$G \not\simeq G_{\pm 2}$, $q > 5$	$4p(2 + p(q + 2p - 5))$	8	p	$4(q+1)$
			$4(q-3)$	p^2	
			8	$p^2(p-1)$	
	$G \simeq G_{\pm 2}$, $q > 5$	$2p(3 + p(2q + 3p - 8))$	8	p	$4(q+1)$
			$4(q-3)$	p^2	
			2	$p(p-1)$	
			6	$p^2(p-1)$	
	$G \simeq G_{\pm 2}$, $q = 5$	$8p(1 + p + 2p(p^2 - 1))$	8	p	24
			8	p^2	
			4	$p(p-1)$	
			4	$p^2(p-1)$	

Table 5: Γ, G of type 8, $G \simeq G_k \not\simeq G_{\pm 2}$

Γ	*RS*	n	l
if either k or k^{-1} is a solution of $x^2 - x - 1 = 0$:			
G_k, G_{1-k}	$2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$	2	1
		12	p
		$2(4q-11)$	p^2
		2	$p(p-1)$
		14	$p^2(p-1)$
G_{1+k}	$4(3p + 2p^2q - 8p^2 + 3p^3)$	16	p
		$8(q-3)$	p^2
		4	$p(p-1)$
		12	$p^2(p-1)$
$\forall s \in \mathcal{K} : G_s \not\simeq G_k, G_{1\pm k}$	$8(2p + p^2q - 5p^2 + 2p^3)$	16	p
		$8(q-3)$	p^2
		16	$p^2(p-1)$
if k and k^{-1} are the solutions of $x^2 + x + 1 = 0$:			
G_k	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$	2	1
		12	p
		$2(4q-11)$	p^2
		16	$p^2(p-1)$
G_{1+k}	$2(1 + 4p + 4p^2q - 15p^2 + 6p^3)$	2	1
		12	p
		$2(4q-11)$	p^2
		4	$p(p-1)$
		16	$p^2(p-1)$
$G_{1-k}, G_{1-k^{-1}}$	$2(7p + 4p^2q - 18p^2 + 7p^3)$	16	p
		$8(q-3)$	p^2
		2	$p(p-1)$
		14	$p^2(p-1)$
$\forall s \in \mathcal{K}$: $G_s \not\simeq G_k, G_{1\pm k}, G_{1-k^{-1}}$	$8(2p + p^2q - 5p^2 + 2p^3)$	16	p
		$8(q-3)$	p^2
		16	$p^2(p-1)$
if k and k^{-1} are the solutions of $x^2 - x + 1 = 0$:			

G_{-k}	$2(1+6p+4p^2q-19p^2+8p^3)$	2	1
		12	p
		$2(4q-11)$	p^2
		16	$p^2(p-1)$
G_{1-k}	$2(1+4p+4p^2q-15p^2+6p^3)$	2	1
		12	p
		$2(4q-11)$	p^2
		4	$p(p-1)$
		16	$p^2(p-1)$
$G_{1+k}, G_{1+k^{-1}}$	$2(7p+4p^2q-18p^2+7p^3)$	16	p
		$8(q-3)$	p^2
		2	$p(p-1)$
		14	$p^2(p-1)$
$\forall s \in \mathcal{K}$: $G_s \not\simeq G_k, G_{1\pm k}, G_{1+k^{-1}}$	$8(2p+p^2q-5p^2+2p^3)$	16	p
		$8(q-3)$	p^2
		16	$p^2(p-1)$
if k and k^{-1} are the solutions of $x^2+1=0$:			
G_k	$4(1+2p+2p^2q-9p^2+4p^3)$	4	1
		8	p
		$4(2q-5)$	p^2
		16	$p^2(p-1)$
G_{1+k}, G_{1-k}	$4(3p+2p^2q-8p^2+3p^3)$	16	p
		$8(q-3)$	p^2
		4	$p(p-1)$
		12	$p^2(p-1)$
$\forall s \in \mathcal{K}$: $G_s \not\simeq G_k, G_{1\pm k}$	$8(2p+p^2q-5p^2+2p^3)$	16	p
		$8(q-3)$	p^2
		16	$p^2(p-1)$
if $k^2 \neq \pm k \pm 1, -1$:			
G_k, G_{-k}	$2(1+6p+4p^2q-19p^2+8p^3)$	2	1
		12	p
		$2(4q-11)$	p^2
		16	$p^2(p-1)$
$G_{1\pm k}, G_{1\pm k^{-1}}$	$2(7p+4p^2q-18p^2+7p^3)$	16	p
		$8(q-3)$	p^2
		2	$p(p-1)$
		14	$p^2(p-1)$
$\forall s \in \mathcal{K}$: $G_s \not\simeq G_{\pm k}, G_{1\pm k}, G_{1\pm k^{-1}}$	$8(2p+p^2q-5p^2+2p^3)$	16	p
		$8(q-3)$	p^2
		16	$p^2(p-1)$

When in a row there is more than one isomorphism class for Γ, we mean that there are RS regular subgroups for each of these. In particular, in the rows of G_s we mean that that there are RS regular subgroups for each $s \in \mathcal{K}$.

Note that in each of the cases in Table 5, and for each isomorphism class of Γ, the total number of conjugacy classes is $T(n) = 8(q+1)$.

Table 6: Γ, G of type 8, $G \simeq G_2$

Γ	*Conditions*	*RS*	n	l
G_2	$q = 5$	$4(1 + p + 3p^2(p + 1))$	4	1
			8	p
			20	p^2
			4	$p(p-1)$
			12	$p^2(p-1)$
G_2	$q > 5$	$2(1 + 5p + 4p^2q - 17p^2 + 7p^3)$	2	1
			12	p
			$2(4q - 11)$	p^2
			2	$p(p-1)$
			14	$p^2(p-1)$
G_3	$q = 7$	$2(1 + 4p + 13p^2 + 6p^3)$	2	1
			12	p
			34	p^2
			4	$p(p-1)$
			12	$p^2(p-1)$
G_{-2}	$q > 7$	$2(1 + 6p + 4p^2q - 19p^2 + 8p^3)$	2	1
			12	p
			$2(4q - 11)$	p^2
			16	$p^2(p-1)$
$G_3, G_{\frac{3}{2}}$	$q > 7$	$2(7p + 4p^2q - 18p^2 + 7p^3)$	16	p
			$8(q - 3)$	p^2
			2	$p(p-1)$
			14	$p^2(p-1)$
$\forall s \in \mathcal{K}$: $G_s \not\simeq G_2, G_{q-2}, G_3, G_{(q+3)/2}$	$q > 7$	$8(2p + p^2q - 5p^2 + 2p^3)$	16	p
			$8(q - 3)$	p^2
			16	$p^2(p-1)$

In each of the cases in Table 6*, and for each isomorphism class of* Γ*, the total number of conjugacy classes is* $T(n) = 8(q + 1)$*.*

Proof. For the types 5, 6, 7 and 9, the number of $(G, \circ)$ is obtained just summing up the results in Subsections 9.3, 9.4, and 9.5, and doubling those such that $p \nmid |\ker(\tilde{\gamma})|$ (see the discussion in Subsection 9.1), namely when $|\ker(\gamma)| = p^2$ and the cases (A1), (A1*) when $|\ker(\gamma)| = p$.

If $(G, \circ)$ is of type 8 and $q = 5$, then there is only one isomorphism class of groups of type 8, so that also in this case we obtain the number of $(G, \circ) \simeq G_2$ simply summing up the results in the previous sections, and doubling for the cases $|\ker(\gamma)| = p^2q$, $|\ker(\gamma)| = p^2$ and the cases (A1), (A1*) when $|\ker(\gamma)| = p$.

Suppose now $(G, \circ)$ of type 8 and $q > 5$. To obtain the total number of $(G, \circ)$ for every isomorphism class of groups of type 8, we have to distinguish some cases.

Suppose first $k = \pm 2$, then by Subsections 9.4 and 9.5 the number of $(G, \circ)$ of type 8 depends on the isomorphis classes of the groups $G_2, G_3, G_{\frac{3}{2}}$ and G_{-2}. Since two groups of type 8, say G_{k_1} and G_{k_2}, are isomorphic if and only if $k_1 = k_2$ or $k_1k_2 = 1$, in this case we have that the $G_3 \simeq G_{\frac{3}{2}} \simeq G_{-2} \not\simeq G_2$ if $q = 7$, and that $G_2, G_3, G_{\frac{3}{2}}$ and G_{-2} represent different isomorphism classes if $q > 7$.

Suppose now $k \neq \pm 2$. By Subsections 9.4 and 9.5 the number of $(G, \circ)$ of type 8 depends on the isomorphis classes of the groups $G_k, G_{1+k^{-1}}, G_{1+k}, G_{1-k^{-1}}, G_{1-k}$ and G_{-k}.

Suppose 5 is a quadratic residue modulo q; then $G_k \simeq G_{1+k^{-1}}, G_{1+k} \simeq G_{1-k^{-1}}$ and $G_{1-k} \simeq G_{-k}$ if and only if k is a solution of $x^2 - x - 1 = 0$. Moreover $G_k \simeq G_{1+k}$, $G_{1+k^{-1}} \simeq G_{1-k}$ and $G_{1-k^{-1}} \simeq G_{-k}$ if and only if k is a solution of $x^2 + x - 1 = 0$. Note also that if k is a solution of $x^2 - x - 1 = 0$, then k^{-1} is a solution of $x^2 + x - 1 = 0$. Therefore, if $G \simeq G_k$ and either k or k^{-1} is a solution of $x^2 - x - 1 = 0$, then the groups above are in three different isomorphism classes, namely $G_k \simeq G_{1+k^{-1}}$, $G_{1+k} \simeq G_{1-k^{-1}}$, and $G_{1-k} \simeq G_{-k}$.

Suppose $q - 3$ is a quadratic residue modulo q; then $G_{1+k^{-1}} \simeq G_{1+k} \simeq G_{-k}$ if and only if k is a solution of $x^2+x+1 = 0$. In that case k^{-1} is the other solution, and the groups above are in four different isomorphism classes. Similarly, if k is a solution of $x^2-x+1 = 0$ there are four different isomorphism classes. Note moreover that if α_1, α_2 are the solutions of $x^2 + x + 1 = 0$, then the solutions of $x^2 - x + 1 = 0$ are $-\alpha_1, -\alpha_2$. Therefore, the last case can be obtained from the previous one by changing k in $-k$.

Lastly suppose $q-4$ is a quadratic residue modulo q; then $G_k \simeq G_{-k}$, $G_{1+k} \simeq G_{1-k^{-1}}$ and $G_{1-k} \simeq G_{1+k^{-1}}$ if and only if k is a solution of $x^2 + 1 = 0$. Also here the other solution is k^{-1}.

Now, note that either k is a solution of exactly one of the above equations, or k is not a solution for any of them. In the last case the groups above form 6 different isomorphism classes.

In compliance with these facts, summing up the results of the previous subsections and doubling for the cases $|\ker(\gamma)| = p^2 q$, $|\ker(\gamma)| = p^2$ and the cases (A1), (A1*) when $|\ker(\gamma)| = p$, we obtain *(a)-(e)* in *4* and *(a)-(c)* in *5*. □

10. Type 7

Here $q \mid p-1$ and G is isomorphic to a group $(\mathcal{C}_p \times \mathcal{C}_p) \rtimes_S \mathcal{C}_q$. The Sylow p-subgroup $A = \langle a_1, a_2 \rangle$ of G is characteristic, and if a_1, a_2 are in the eigenspaces of the action of a generator b of a Sylow q-subgroup B on A, then this action can be represented by a scalar matrix Z with no eigenvalues 1. Therefore, if a_1, a_2 are eigenvectors for $\iota(b)$, then with respect to that basis, we have

$$Z = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}.$$

The divisibility condition on p and q implies that $(G, \circ)$ can be of type 5, 6, 7, 8 and 9.

According to Subsections 4.1 and 4.2 of [8], we have

$$\mathrm{Aut}(G) = \mathrm{Hol}(\mathcal{C}_p \times \mathcal{C}_p).$$

Different from the types 8 and 9, in this case if γ is a GF on a group G of type 7, then $\gamma(A)$ is not necessarily contained in $\mathrm{Inn}(G)$, as here a Sylow p-subgroup of $\mathrm{Aut}(G)$ is of the form $\iota(A) \rtimes \mathcal{P}$, where $\mathcal{P}$ is a Sylow p-subgroup of $\mathrm{GL}(2, p)$.

In the following, we will distinguish two cases, namely when $\gamma(A) \le \mathrm{Inn}(G)$ and when $\gamma(A) \nleq \mathrm{Inn}(G)$. In the first case, if γ is a GF on G, then $\gamma_{|A} : A \to \mathrm{Inn}(G) \le \mathrm{Aut}(G)$ is a RGF, as A is characteristic in G. Moreover, Lemma 2.5 yields that $\gamma_{|A}$ is a morphism, as $\iota(A)$ acts trivially on the abelian group A. Therefore, for each gamma function γ there exists $\sigma \in \mathrm{End}(A)$ satisfying equation (7.1), namely

$$\gamma(a) = \iota(a^{-\sigma})$$

for each $a \in A$.

The case $\gamma(A) \le \mathrm{Inn}(G)$ can be handled in a very similar way to the cases in which G is type 8 or 9, therefore in the following we will often refer to Sections 7, 8 and 9. The case $\gamma(A) \nleq \mathrm{Inn}(G)$ instead will require a separate treatment.

10.1. Duality. Suppose first that $\gamma(A) \le \mathrm{Inn}(G)$, so that every γ on G satisfies equation (7.1). We can apply Lemma 2.7 with $C = A$, and this yields equation (2.5). By the discussion in Subsections 4.1 and 4.2, if σ and $1 - \sigma$ are not both invertible, then $p \mid |\ker(\gamma)|$ or $p \mid |\ker(\tilde{\gamma})|$, namely σ has 0 or 1 as an eigenvalue. Otherwise σ and $1 - \sigma$ are both invertible, but this happens only when $q = 2$.

Suppose now that $\gamma(A) \nleq \mathrm{Inn}(G)$. We show that, appealing to duality, we can always suppose that $p \mid |\ker(\gamma)|$.

If $\gamma(A)$ has order p, then $p \mid |\ker(\gamma)|$. Moreover $\gamma(A) = \langle\, \iota(c)\alpha \,\rangle$, for some $c \in A$ and α in $\mathrm{GL}(2, p)$ of order p, therefore, by the discussion in Subsection 4.3.2, the kernel is the fixed point space of α.

Now suppose $|\gamma(A)| = p^2$. We show that there exists a subgroup C of order p which satisfies the hypotheses of Proposition 2.9.

Let $\gamma(A) = \langle\, \iota(c), \iota(d)\alpha \,\rangle$, for some $c, d \in A$, and $\alpha \in \mathrm{GL}(2, p)$ of order p. Since $1 = [\iota(c), \iota(d)\alpha] = \iota([c, \alpha])$, we have that α fixes c. Let $x_1, x_2 \in A$ be such that $\gamma(x_1) = \iota(c)$, and $\gamma(x_2) = \iota(d)\alpha$. Then

$$x_1^{\alpha} x_2 = x_1 \circ x_2 = x_2 \circ x_1 = x_2 x_1,$$

so that $x_1 \in \langle\, c \,\rangle$. It follows that $\gamma(c) = \iota(c^{-k})$ for some k. The subgroup $C = \langle\, c \,\rangle$ is $\gamma(G)$-invariant, as if $b \in G$ has order q, then $\gamma(A) \cap \iota(A)$ is normalised by $\langle\, \gamma(b) \,\rangle$, so that $\gamma(b)$ leaves C invariant. Since C is also normal in G, Proposition 2.9 yields that $\gamma(c) = \iota(c^{-k})$ with $k = 0, 1$, namely either $C \le \ker(\gamma)$ or $C \le \ker(\tilde{\gamma})$. Now by Corollary 2.10 we can assume $C \le \ker(\gamma)$.

Therefore, when $\gamma(A) \le \mathrm{Inn}(G)$ and $q > 2$ we can assume that $p \mid |\ker(\gamma)|$ (equivalentely σ has 0 as an eigenvalue), and once we have counted the gamma functions with this property, we will double the number of those for which moreover $p \nmid |\ker(\tilde{\gamma})|$ (we will double only those GF for which 1 is not an eigenvalue of σ). If $\gamma(A) \le \mathrm{Inn}(G)$ and $q = 2$ then there are actually σ with no eigenvalues 0 and 1, and this corresponds to the existence of γ such that $p \nmid |\ker(\gamma)|, |\ker(\tilde{\gamma})|$. Here, except for the case when both γ and $\tilde{\gamma}$ have kernel of size not divisible by p, we will use duality to swich to a more convenient kernel. Otherwise, when $\gamma(A) \nleq \mathrm{Inn}(G)$, we can assume that $p \mid |\ker(\gamma)|$, and then we will double the numbers we will obtain.

10.2. Description of the elements of order q of $\mathrm{Aut}(G)$. An element of order q in $\mathrm{Aut}(G)$ is of the form $\iota(a_*)\beta$, where $a_* \in A$, and $\beta \in \mathrm{GL}(2, p)$ of order q.

We now show that the number of Sylow q-subgroups of $\mathrm{Aut}(G)$ is p^2 times the number of Sylow q-subgroups of $\mathrm{GL}(2, p)$ (see Subsection 4.3.3).

The image in $\mathrm{GL}(2, p)$ of a Sylow q-subgroup of $\mathrm{Aut}(G)$ is a Sylow q-subgroup of $\mathrm{GL}(2, p)$. Conversely, if Q is a Sylow q-subgroup of $\mathrm{GL}(2, p)$, the Sylow q-subgroups of $\mathrm{Aut}(G)$ that have Q as an image are precisely the ones contained in QA. Let X be the intersection of A with the normaliser of Q in QA, so that $[X, Q] \le Q$. Since A is normal, and $X \le A$, we also have $[X, Q] \le [A, Q] \le A$. Since $A \cap Q = \{1\}$, we have that X is centralised by Q. Since $q \mid p - 1$, in $\mathrm{GL}(2, p)$ there is a group Y of order q of scalar matrices. Since Y is central in $\mathrm{GL}(2, p)$, it is contained in every Sylow q-subgroup of $\mathrm{GL}(2, p)$, hence in Q. It follows that $X = \{1\}$, so that Q is self-normalising in QA, and thus there are $|QA : Q| = |A| = p^2$ Sylow q-subgroups in QA.

Let us start with the enumeration of the GF's on G. We proceed case by case, according to the size of the kernel.

As usual, if $|\ker(\gamma)| = p^2 q$, then γ corresponds to the right regular representation, so that we will assume $\gamma \ne 1$.

10.3. The case $|\ker(\gamma)| = pq$. Here $K = \ker(\gamma)$ is a subgroup of G isomorphic to $\mathcal{C}_p \rtimes \mathcal{C}_q$, therefore we will obtain $(G, \circ)$ of type 6, as it is the only type having a non abelian normal subgroup of order pq.

We can choose K in $p(p + 1)$ ways, indeed for each of the p^2 choices for a Sylow q-subgroup B, the subgroups of order p that are B-invariant are the 1-dimensional invariant subspaces of the action of B. Since the action of a Sylow q-subgroup on A here is scalar, all of the $p + 1$ subgroups of G of order p are B-invariant. Moreover, since $\mathcal{C}_p \rtimes \mathcal{C}_q$ has p subgroups of order q, exactly p choices for B give the same group.

Let $K = \langle a_1, b \rangle$, and let $a_2 \in A$ be such that $A = \langle a_1, a_2 \rangle$.

Suppose first that $\gamma(A) \le \mathrm{Inn}(G)$. Reasoning as in Subsection 8.4, here we obtain $p^2(p + 1)$ gamma functions, corresponding to groups $(G, \circ)$ of type 6. Moreover, for every γ here, $p \mid |\ker(\tilde{\gamma})|$.

As to the conjugacy classes, this time an automorphism φ of G has the form $\varphi = \iota(x)\delta$, where $x \in A$ and δ is such that $\delta_{|B} = 1$, $\delta_{|A} = (\delta_{ij}) \in \mathrm{GL}(2, p)$. With computations similar to those of Subsection 8.4 we obtain

- $\gamma^{\varphi}(a_1) = 1$ if and only if $a_1^{\delta^{-1}} \in \ker(\gamma) \cap A = \langle a_1 \rangle$, namely $\delta_{12} = 0$;
- taking into account (8.3), we have that $\gamma^{\varphi}(b) = 1$ if and only if $x \in \langle a_1 \rangle$;
- taking into account (8.4) and writing $a = a_1^j a_2^{-1}$, we have that $\gamma^{\varphi}(a_2) = \gamma(a_2)$ if and only if $\iota(a^{\delta_{22}^{-1}})^{\delta} = \iota(a)$, namely $j(\delta_{11} - \delta_{22}) = \delta_{21}$.

The latter condition yields δ_{21} as a function of the diagonal elements, so that the stabiliser has order $p(p - 1)^2$, and we obtain one orbit of length $p^2(p + 1)$.

Suppose now that $\gamma(A) \not\leq \mathrm{Inn}(G)$. In this case there are no $\gamma(G)$-invariant complements of K, therefore let us consider $G = KA$. Proposition 2.6 yields that every GF on G is the lifting of a RGF $\gamma' : A \to \mathrm{Aut}(G)$ with $\gamma(G) = \gamma'(A)$, and such that K is invariant under $\{\gamma'(x)\iota(x) : x \in A\}$. Conversely, every RGF γ' such that $\gamma'(\langle a_1 \rangle) = 1$, and which makes K invariant under $\{\gamma'(x)\iota(x) : x \in A\}$, can be lifted to G. Now we show that a such map is a morphism, and it is defined by

$$\gamma'(a_2) = \alpha\iota(a_1^j a_2^{-1}),$$

for some $0 \leq j \leq p-1$, and $\alpha \in \mathrm{GL}(2,p)$ of order p.

Indeed, since $\gamma'(A) = \gamma'(\langle a_2 \rangle)$ has order p, $\gamma'(a_2) = \alpha\iota(a)$ for some $a \in A$, and $\alpha \in \mathrm{GL}(2,p)$ of order p. By Subsubsection 4.3.2, $\langle a_1 \rangle$ is the space of the fixed points of α, so that $a_1^\alpha = a_1$. Moreover, we can write $a_2^\alpha = a_1^d a_2$, for some $1 \leq d \leq p-1$, and by Lemma A.2 in the Appendix, the RGF's are morphisms.

Now, if K is invariant under $\gamma'(x)\iota(x)$ for every $x \in A$, then $\gamma'(a_2)\iota(a_2) = \alpha\iota(aa_2)$ leaves K invariant, so that $aa_2 \in \langle a_1 \rangle$, namely $a = a_1^j a_2^{-1}$ for some j, $0 \leq j \leq p-1$. Conversely, choosing $a = a_1^j a_2^{-1}$ then $\gamma'(a_2)\iota(a_2) = \alpha\iota(a_1^j)$, and since γ' is a morphism, K is invariant under $\gamma'(x)\iota(x)$ for every $x \in \langle a_2 \rangle$, and so for every $x \in A$.

Since there are $p(p+1)$ choices for K, $p-1$ choices for α and p choices for $\iota(a_1^j a_2^{-1})$, we obtain $p^2(p^2-1)$ groups.

As to the conjugacy classes, let $\varphi = \iota(x)\delta \in \mathrm{Aut}(G)$. As above, the conditions $\gamma^\varphi(a_1) = \gamma(a_1)$ and $\gamma^\varphi(b) = \gamma(b)$ yield $\delta_{12} = 0$ and $x \in \langle a_1 \rangle$. Moreover here

$$\begin{aligned}
\gamma^\varphi(a_2) &= \varphi^{-1}\gamma(a_2^{\delta^{-1}})\varphi \\
&= \delta^{-1}\alpha^{\delta_{22}^{-1}}\iota(a^{\alpha^{\delta_{22}^{-1}-1}+\cdots+\alpha+1})\delta \\
&= \delta^{-1}\alpha^{\delta_{22}^{-1}}\delta\iota(a^{\alpha^{\delta_{22}^{-1}-1}+\cdots+\alpha+1})^\delta,
\end{aligned}$$

so that φ stabilises γ if and only if both $\delta^{-1}\alpha^{\delta_{22}^{-1}}\delta = \alpha$ (namely $\delta_{22}^2 = \delta_{11}$) and

$$\iota(a^{\alpha^{\delta_{22}^{-1}-1}+\cdots+\alpha+1})^\delta = \iota(a),$$

namely $\delta_{21} = (j + \frac{d}{2})\delta_{22}(\delta_{22}-1)$. Therefore the stabiliser has order $p(p-1)$, and there is one orbit of length $p^2(p^2-1)$.

10.4. The case $|\ker(\gamma)| = p^2$. Reasoning as in Subsection 8.5, we obtain that each γ on G is the lifting of at least one RGF defined on an invariant Sylow q-subgroup B, and the RGF's on B are precisely the morphisms. We have $\gamma(b)_{|B} = 1$, and let $\gamma(b)_{|A} = \beta$; then the discussion in Subsubsection 10.2 yields that β is a matrix of order q in $\mathrm{GL}(2,p)$, and, with respect to a suitable basis of A, we can represent β as the diagonal matrix

$$T = \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix},$$

where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$ and x_1, x_2 are not both zero (see Subsubsection 4.3.3). We assume that β is diagonal with respect to $\{a_1, a_2\}$, taking into account that if β is non-scalar, then there are $\frac{1}{2}p(p+1)$ choices for a pair $\{A_1, A_2\}$ of distinct one-dimensional subspaces of A.

To know the exact number of the invariant Sylow q-subgroups we appeal to the discussion in Subsubsection 7.1; here equation (7.4) yields $x^{(1-Z^{-1})M} = 1$, where

$$M = 1 - T = \begin{bmatrix} 1-\lambda^{x_1} & 0 \\ 0 & 1-\lambda^{x_2} \end{bmatrix},$$

and we obtain that

(1) if both $x_1, x_2 \neq 0$, there is a unique invariant Sylow q-subgroup;
(2) if either $x_1 = 0$ or $x_2 = 0$, there are p invariant Sylow q-subgroups.

Denoting as usual by $Z_\circ$ the associated matrix of the action of b on A with respect to the operation $\circ$, here we have

$$Z_\circ \sim \begin{bmatrix} \lambda^{1+x_1} & 0 \\ 0 & \lambda^{1+x_2} \end{bmatrix}.$$

We obtain the followings groups $(G, \circ)$.

Type 5: if $x_1 = x_2 = -1$, therefore p^2 groups.

Type 6: if either $x_1 = -1$ and $x_2 \neq -1$, or $x_1 \neq -1$ and $x_2 = -1$. In both cases there is a unique invariant Sylow q-subgroup, except if either $x_2 = 0$ or $x_1 = 0$, when there are p invariant Sylow q-subgroups. Therefore, there are $p^3(p+1)(q-2) + p^2(p+1)$ groups.

Type 7: if $x_1 + 1 = x_2 + 1 \neq 0$, namely $x_1 = x_2 \neq -1$. Since we are in case (1), we obtain $p^2(q-2)$ groups.

Type 8: if $Z_\circ$ is a non scalar matrix with no eigenvalues 1, and determinant different from 1.

In case (1) this corresponds to the conditions $x_2 \neq 0, -1$ and the four conditions $x_1 \neq 0, -1, -x_2 - 2, x_2$, which are independent if and only if in addition $x_2 \neq 0, -2$. When these four conditions are dependent, they reduce to three independent condition on x_1. If $x_2 \neq 0, -1, -2$ we have four independent conditions on x_1, and therefore we obtain $\frac{1}{2}p^3(p+1)(q-4)(q-3)$ groups. For $x_2 = -2$ there are three conditions, and we obtain further $\frac{1}{2}p^3(p+1)(q-3)$ groups.

In case (2), if $x_1 = 0$ there are three independent conditions on x_2. Doubling for the case $x_2 = 0$, we obtain $p^2(p+1)(q-3)$ groups.

Summing up, we have just obtained $\frac{1}{2}p^3(p+1)(q-3)^2 + p^2(p+1)(q-3)$ groups of type 8; looking at the eigenvalues of $Z_\circ$, we easily obtain that they are $2p^2(p+1) + p^3(p+1)(q-3)$ groups isomorphic to G_s, for every $s \in \mathcal{K}$;

Type 9: if $Z_\circ$ is a non-scalar matrix with no eigenvalue 1 and determinant 1, namely $x_1 \neq -1, x_2$, $x_2 \neq -1$, and $x_1 + x_2 + 2 = 0$. In case (1)

$x_2 \neq 0, -1$ and also $x_2 \neq -2$, otherwise we would have $x_1 = 0$; since the latter is a new condition there are $\frac{1}{2}p^3(p+1)(q-3)$ groups. The case (2) yields $p^2(p+1)$.

Summing up, there are $p^2(p+1) + \frac{1}{2}p^3(p+1)(q-3)$ groups.

As to the conjugacy classes, in the notation of Subsection 10.3, let $\varphi = \iota(x)\delta \in \mathrm{Aut}(G)$. With computations similar to those in Subsection 8.5 (taking $\psi = 1$), here we obtain that φ stabilises γ if and only if

$$\begin{cases} x^{(1-T)\delta} = 1 \\ \delta^{-1}T\delta = T. \end{cases}$$

The first condition yields $x = 1$ or, if $x = a_1^u a_2^v$, either $x_1 = 0$ and $v = 0$, or $x_2 = 0$ and $u = 0$. From the second condition we obtain that δ is any matrix when T is scalar, and δ is diagonal when T is non-scalar.

We obtain the following.

(1) For $(G, \circ)$ of type 5 , $x_1 = x_2 = -1$, so that the stabiliser has order $|\mathrm{GL}(2,p)|$, and there is one orbit of length p^2.
(2) For $(G, \circ)$ of type 6, $x_1 \neq x_2$. The stabiliser has order $p(p-1)^2$ when either $x_1 = 0$ or $x_2 = 0$, and $(p-1)^2$ when $x_1, x_2 \neq 0$. Therefore, we obtain one orbit of length $p^2(p+1)$ and $q-2$ orbits of length $p^3(p+1)$.
(3) For $(G, \circ)$ of type 7, $x_1 = x_2 \neq -1$. The stabiliser has order $|\mathrm{GL}(2,p)|$ and there are $q-2$ orbits of length p^2.
(4) For $(G, \circ)$ of type 8 $x_1 \neq x_2$, so that if $x_1, x_2 \neq 0$ the stabiliser has order $(p-1)^2$; otherwise either $x_1 = 0$ or $x_2 = 0$, and the stabiliser has order $p(p-1)^2$. Therefore, if $(G, \circ) \simeq G_s$, for every $s \in \mathcal{K}$ we obtain $q-3$ orbits of length $p^3(p+1)$ and two orbits of length $p^2(p+1)$;
(5) For $(G, \circ)$ of type 9, $x_1 \neq x_2$. When $x_1, x_2 \neq 0$ the stabiliser has order $(p-1)^2$, otherwise the stabiliser has order $p(p-1)^2$. Therefore, there are $\frac{q-3}{2}$ orbits of length $p^3(p+1)$ and one of length $p^2(p+1)$;

10.5. The case $|\ker(\gamma)| = p$ and $\gamma(A) \leq \mathrm{Inn}(G)$. Since here $|\gamma(G)| = pq$ and $\gamma(G)$ intersects $\iota(A)$ non-trivially, we have

$$\gamma(G) = \langle\, \iota(c), \iota(d)\beta \,\rangle$$

for some $c, d \in A$, with $A^\sigma = \langle\, \iota(c) \,\rangle$, and $\beta \in \mathrm{GL}(2,p)$, $\beta \neq 1$.

Let $b \in G$ (of order q) such that $\gamma(b) = \iota(d)\beta$. With respect to a suitable basis of A, the matrix associated to $\gamma(b)_{|A}$ is

$$T := \begin{bmatrix} \lambda^{x_1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix},$$

where x_1, x_2 are not both zero. Denote by $\{a_1, a_2\}$ such a basis, and keep in mind that when $[\gamma(b)_{|A}]$ is non-scalar there are $\frac{1}{2}p(p+1)$ choices for a pair $\{A_1, A_2\}$ of distinct one-dimensional subspaces of A.

Following Subsection 7.3, and recalling that for G of type 7 $k = 1$, here we find the following cases:

- Case A: $\ker(\sigma) = \langle a_1 \rangle$.
  (A1) $\nu = 0, \mu \neq 0, x_1 - x_2 = 1$;
  (A2) $\nu = 1, \mu \neq 0, x_1 = x_2$;
  (A3) $\nu = 1, \mu = 0$.
- Case B: $\ker(\sigma) = \langle a_1 a_2 \rangle$.
  (B2) $\mu + 1 - \nu = 0, x_1 = x_2$.

As explained in Subsection 7.3, the results in the cases (A1*), (A2*) and (A3*) can be obtained doubling the results we will obtain in the cases (A1), (A2) and (A3).

Notice that p divides both $|\ker(\gamma)|$ and $|\ker(\tilde{\gamma})|$ if and only if σ has both 0 and 1 as eigenvalues, that is, in all the cases above except (A1), where, since σ has only 0 as eigenvalue, $p \mid |\ker(\gamma)|$ but $p \nmid |\ker(\tilde{\gamma})|$.

10.5.1. Invariant Sylow *q*-subgroups. Following Subsubsection 7.3.1 and taking $k = 1$, we obtain the following.

Proposition 10.1. *The number of invariant Sylow q-subgroups is*

(A1) 1 *when* $x_1 \neq 0, 1$ *and* p *otherwise.*
(A2) 1 *when* $x_1 \neq -1$ *and* p *otherwise.*
(A3) 1 *when* $x_1 \neq 0$ *and* $x_2 \neq -1$, p^2 *when* $x_1 = 0$ *and* $x_2 = -1$, *and* p *otherwise.*
(B2) 1 *when* $x_1 \neq -1$ *and* p *otherwise.*

10.5.2. Computations. By Subsubsection 7.3.2 the action $Z_\circ$ of b on A in $(G, \circ)$ is given by

$$Z_\circ = (\sigma(1 - Z) + Z)T, \tag{10.1}$$

and we obtain the following.

Case A. Here $\ker(\sigma) = \langle a_1 \rangle$ and equality (7.10) yields

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} & 0 \\ \mu(1-\lambda)\lambda^{x_1} & \lambda^{x_2}(\nu(1-\lambda)+\lambda) \end{bmatrix}.$$

(A1)+(A1*) We have $p - 1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: if $x_1 = 0$ or $x_1 = -1$. If $x_1 = 0$ for each of the $(p-1)$ choices for σ we have p^2/p choices for B giving different GF's, so $p^2(p^2-1)$ groups. If $x_1 = -1$, then $x_1 = 1$ if and only if $q = 2$. If $q > 2$ there are $p^3(p^2-1)$ groups, otherwise if $q = 2$ there are $p^2(p^2-1)$ groups.

Type 7: does not arise.

Type 8: if $x_1 \neq 0, -1, (q-1)/2$, and these are always three independent conditions. We have $p^2(p^2-1)$ groups when $x_1 = 1$ and $p^3(p^2-1)(q-4)$ groups when $x_1 \neq 1$. They split in $p^2(p^2-1) + p^3(p^2-1)$ groups isomorphic to G_2, and $2p^3(p^2-1)$ groups isomorphic to G_s, for every $s \neq 2, s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. Since $(q-1)/2 = 1$ if and only if $q = 3$, we have $p^2(p^2-1)$ groups when $q = 3$ and $p^3(p^2-1)$ groups when $q > 3$.

(A2)+(A2*) We have $p-1$ choices for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: when $x_1 = -1$. Since $x_1 = -1$ there are $2p(p-1)$ groups.

Type 7: does not arise.

Type 8: if $x_1 \neq 0, -1, (q-1)/2$, and these are always three independent conditions. Since $x_1 \neq -1$, there are $2p^2(p-1)(q-3)$ groups. They split in $4p^2(p-1)$ groups isomorphic to G_s for every $s \in \mathcal{K}$.

Type 9: if $x_1 = (q-1)/2$. Since $x_1 \neq -1$, there are $2p^2(p-1)$ groups.

(A3)+(A3*) We have 1 choice for σ, and

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: if $1 + x_1 = x_2 = 0$. Since $x_1 \neq 0$ and $x_2 \neq -1$, there are $p^3(p+1)$ groups.

Type 6: if either $x_1 = -1$ and $x_2 \neq 0$ or $x_1 \neq -1$ and $x_2 = 0$. In the first case, there are $2p$ groups when $x_2 = -1$, otherwise, for $x_2 \neq -1$, there are $p^3(p+1)(q-2)$ groups. In the second case, since $x_2 = 0$, we have to take $x_1 \neq 0$ and there are $p^3(p+1)(q-2)$ groups.

Type 7: when $x_2 = 1 + x_1 \neq 0$. If $x_1 = 0$ and $x_2 = -1$, then $q = 2$ and there are $p(p+1)$ groups. In both the cases $x_1 \neq 0$, $x_2 = -1$, and $x_1 = 0$, $x_2 \neq -1$ there are $2p^2(p+1)$ groups. If $x_1 \neq 0$ and $x_2 \neq -1$ there are $p^3(p+1)(q-3)$ groups.

Type 8: when $x_1 \neq -1, x_2 - 1, -x_2 - 1, x_2 \neq 0$. The case $x_1 = 0$ and $x_2 = -1$ does not arise. If $x_1 \neq 0$ and $x_2 = -1$, the four conditions on x_1 are actually three conditions and there are $p^2(p+1)(q-3)$ groups. If $x_1 = 0$ and $x_2 \neq -1$ we get further $p^2(p+1)(q-3)$ groups.

Suppose now $x_1 \neq 0$, $x_2 \neq -1$. There are always four independent conditions on x_1 except when $x_2 = 1$; in the latter case the conditions become three. If $x_2 = 1$ then there is one invariant Sylow q-subgroup and β is scalar if and only if $x_1 = 1 = x_2$, so there are

$2p^2+p^3(p+1)(q-4)$ groups. If $x_2 \neq 1$, there is one invariant Sylow q-subgroup and β can always be scalar except when $x_2 = (q-1)/2$, thus there are $p^3(p+1)(q-4)$ groups when $x_2 = (q-1)/2$ and $2p^2(q-4) + p^3(p+1)(q-5)(q-4)$ when $x_2 \neq (q-1)/2$.
Summing up, we have just obtained $2p^2(p+1)(q-3) + 2p^2(q-3) + p^3(p+1)(q-4)(q-3)$ groups. They split in $4p^2(p+1) + 4p^2 + 2p^3(p+1)(q-4)$ groups isomorphic to G_s for every $s \in \mathcal{K}$.

Type 9: if $x_1 \neq -1$, $x_2 \neq 0$ and $1 + x_1 + x_2 = 0$. Here $x_1 = x_2$ if and only if $x_2 = (q-1)/2$. If $x_2 = -1$ then $x_1 = 0$ and there are $p(p+1)$ groups. If $x_2 = (q-1)/2 = x_1$ there are $2p^2$ groups. Otherwise $x_2 \neq 0, -1, (q-1)/2$ and there are $p^3(p+1)(q-3)$ groups.

Case B. Here $\ker(\sigma) = \langle a_1 a_2 \rangle$, $x_1 = x_2$.

(B2) Here $\mu + 1 = \nu$, and we have $p(p-1)$ choices for

$$\sigma = \begin{bmatrix} -\mu & -\mu - 1 \\ \mu & \mu + 1 \end{bmatrix}.$$

Equality (7.10) yields

$$Z_\circ = \begin{bmatrix} \lambda^{x_1+1} - \lambda^{x_1}\mu(1-\lambda) & -\lambda^{x_1}(\mu+1)(1-\lambda) \\ \lambda^{x_1}\mu(1-\lambda) & \lambda^{x_1+1} + \lambda^{x_1}(\mu+1)(1-\lambda) \end{bmatrix},$$

so that

$$Z_\circ \sim \begin{bmatrix} \lambda^{x_1+1} & 0 \\ 0 & \lambda^{x_1} \end{bmatrix}.$$

We obtain the following groups $(G, \circ)$.

Type 5: does not arise.

Type 6: if $x_1 = -1$, and there are $p^2(p-1)$ groups.

Type 7: does not arise.

Type 8: when $x_1 \neq -1, 0, (q-1)/2$. These are three independent conditions and there are $p^3(p-1)(q-3)$ groups.

Type 9: if $x_1 \neq -1, 0$, $x_1 = (q-1)/2$, and there are $p^3(p-1)$ groups, which split in $2p^3(p-1)$ groups isomorphic to G_s for every s.

As for the conjugacy classes, here an automorphism of G has the form $\varphi = \iota(x)\delta$, where $x \in A$ and $\delta \in \mathrm{GL}(2, p)$. Therefore, we can refer to Subsubsection 7.3.3 for the computation of the conjugacy classes.

Summing up all the results obtained for the kernel of size p, we have the following.

Recap 10.2. *For G of type 7 and γ a GF on G with kernel of size p and such that $\gamma(A) \leq \mathrm{Inn}(G)$, we list for each isomorphism class of groups ($(G, \circ)$), the number (n) and the lengths (l) of the conjugacy classes in* $\mathrm{Hol}(G)$.

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

$T(n)$ denotes the total number of conjugacy classes.

$(G,\circ)$	*Conditions*	n	l	$T(n)$
5		1	$p^3(p+1)$	1
6		1	$p^2(p^2-1)$	$2q-1$
		1	$p^3(p^2-1)$	
		1	$p^2(p+1)$	
		$2(q-2)$	$p^3(p+1)$	
7		2	$p^2(p+1)$	$q-1$
		$q-3$	$p^3(p+1)$	
8_{G_2}	$q>3$	1	$p^2(p^2-1)$	$2q$
		1	$p^3(p^2-1)$	
		4	$p^2(p+1)$	
		$2(q-3)$	$p^3(p+1)$	
8_{G_s}, $s\neq 2$	$q>3$	2	$p^3(p^2-1)$	$2q$
		4	$p^2(p+1)$	
		$2(q-3)$	$p^3(p+1)$	
9	$q>2$	1	$p(p+1)$	q
		$q-2$	$p^3(p+1)$	
	$q=3$	1	$p^2(p^2-1)$	
	$q>3$	1	$p^3(p^2-1)$	

In the row of 8_{G_s} we mean that for every $s\in\mathcal{K}$, $s\neq 2$ there are n classes of length l of regular subgroups isomorphic to G_s.

10.6. The case $|\ker(\gamma)| = p$ and $\gamma(A)\not\leq \mathrm{Inn}(G)$. Let $\ker(\gamma)=\langle a_1\rangle$. We claim that $\gamma(G)\cap\iota(A)=\{1\}$. Indeed, since $q\mid p-1$, $\gamma(G)$ (of order pq) has a unique subgroup of order p. Moreover A is the Sylow p-subgroup of both G and $(G,\circ)$, so that $\gamma(A)\leq\gamma(G)$ and the order of $\gamma(A)$ is a divisor of p^2. Therefore, the unique subgroup of order p of $\gamma(G)$ is necessarily $\gamma(A)$. Now, either $|\gamma(G)\cap\iota(A)|=1$, and we are done, or $|\gamma(G)\cap\iota(A)|=p$. In the latter case $\gamma(G)\cap\iota(A)=\gamma(A)$, namely $\gamma(A)\leq\mathrm{Inn}(G)$, contradiction.

Now, since $\gamma(G)$ intersects $\iota(A)$ trivially,

$$\gamma(G)=\langle\,\iota(c)\alpha,\beta\,\rangle,$$

where $c\in A$, $\alpha\in\mathrm{Aut}(G)$ has order p, and $\beta\in\mathrm{Aut}(G)$ has order q. So $\alpha_{|A}$ is an element of order p in $\mathrm{GL}(2,p)$, and $\beta_{|A}$ is an element of order q in $\mathrm{GL}(2,p)$. By Subsection 4.3.2 $\alpha_{|A}$ fixes $\langle a_1\rangle$. Moreover, by Subsection 4.3.4, $\langle a_1\rangle$ is an eigenspace for $\beta_{|A}$ too, and if $\langle a_2\rangle$ is another eigenspace for $\beta_{|A}$ for a suitable choice of a_2 we can write

$$\alpha_{|A}=\begin{bmatrix}1&0\\1&1\end{bmatrix},\quad \beta_{|A}=\begin{bmatrix}\lambda^{x_1}&0\\0&\lambda^{x_2}\end{bmatrix},$$

with respect to the basis a_1,a_2, where λ is an element of multiplicative order q in $\mathbb{Z}/p\mathbb{Z}$ and $0\leq x_1,x_2\leq q-1$ not both zero.

The subgroup $\langle\beta\rangle$ of $\mathrm{Aut}(G)$, of order q, acts on the set Ω of the Sylow q-subgroups of G, and since $|\Omega|=p^2$, this action has at least one fixed point. Suppose it is $\langle b\rangle$. We can suppose that $\langle b\rangle$ is fixed by α too, in fact otherwise it will be fixed by $\alpha':=\iota(x)\alpha$ for a suitable $x\in A$, and up to an appropriate adjustment of c, $\gamma(G)=\langle\,\iota(c)\alpha',\beta\,\rangle$.

Thus we assume in the following that $\alpha_{|A}$ and $\beta_{|A}$ are in the same copy of $\mathrm{GL}(2, p)$.

Note that the Sylow q-subgroups $\langle xb \rangle$ fixed by β are those for which x satisfies $x^{1-\beta} = 1$. Therefore, there is a unique fixed Sylow q-subgroup when $x_1, x_2 \neq 0$, and there are p when either $x_1 = 0$ or $x_2 = 0$.

10.6.1. The case $\gamma(G)$ abelian. Suppose first $\gamma(G)$ is abelian. Then $[\alpha, \beta] = 1$ modulo $\iota(A)$, so that $\beta_{|A}$ is a non-trivial scalar matrix, namely $x_1 = x_2 \neq 0$. We can assume that $\beta_{|A}$ is scalar multiplication by λ. Moreover by the discussion above $\langle b \rangle$ is the unique β-invariant Sylow q-subgroup, as $x_1, x_2 \neq 0$.

Since $\gamma(G)$ is abelian, β has to centralise $\iota(c)\alpha$, and since

$$b^{\beta\iota(c)\alpha} = c^{(-1+\lambda^{-1})\alpha} b, \qquad b^{\iota(c)\alpha\beta} = c^{(-1+\lambda^{-1})\alpha\beta} b,$$

and β does not have fixed points in A, we get $c = 1$.

Therefore,

$$\gamma(G) = \langle \alpha, \beta \rangle,$$

where both α and β fix $\langle b \rangle$ pointwise. In particular $\langle b \rangle$ is $\gamma(G)$-invariant.

We will have $\gamma(a_2) = \alpha^i$ for $i \neq 0$. Moreover, since b is fixed by $\gamma(G)$, $\gamma(b)^q = \gamma(b^q) = 1$, namely $\gamma(b) = \beta^j$. Now we show that necessarily $j = -1$.

In fact, in this case $(G, \circ)$ can be of type 5 or 6, as they are the only types which have an abelian quotient of order pq. We have

$$b^{\ominus 1} \circ a \circ b = b^{-1} a^{\gamma(b)} b = a^{\beta^j \iota(b)} = a^{\lambda^{j+1}}.$$

Denoting by $Z_\circ$ the action of $\iota(b)$ on A in $(G, \circ)$, and taking into account that

$$a_1^{\circ k} = a_1^k \text{ and } a_2^{\circ k} = a_1^{i(\frac{k(k-1)}{2})} a_2^k,$$

we have that $Z_\circ$ has to be scalar multpilication by 1. In this case $j = -1$, $(G, \circ)$ is of type 5 and $\gamma(b) = \beta^{-1} = \iota(b^{-1})$.

Now we show that such an assignment extends to a gamma function, namely if $a = a_1^s a_2^t$ the maps defined by

$$\gamma(ab^k) = \alpha^{it\lambda^k} \beta^{-k}$$

satisfy the GFE. Let $a' = a_1^x a_2^y$. Then

$$\gamma(ab^k)\gamma(a'b^m) = \alpha^{it\lambda^k + iy\lambda^m} \beta^{-(k+m)},$$

and

$$\begin{aligned} \gamma((ab^k)^{\gamma(a'b^m)} a'b^m) &= \gamma(a^{\alpha^{iy\lambda^m}\beta^{-m}} b^k a'b^m) \\ &= \gamma((a_2^t)^{\alpha^{iy\lambda^m}\beta^{-m}} a_2^{y\lambda^{-k}} b^{k+m}) \\ &= \gamma(a_2^{t\lambda^{-m} + y\lambda^{-k}} b^{k+m}) \\ &= \alpha^{i(t\lambda^{-m} + y\lambda^{-k})\lambda^{k+m}} \beta^{-(k+m)}. \end{aligned}$$

Since there are $p + 1$ choices for $\ker(\gamma)$, $p - 1$ choices for i, and p^2 choices for the Sylow q-subgroup fixed by β, we obtain $p^2(p^2 - 1)$ groups $(G, \circ)$.

As to the conjugacy classes, $B = \langle b \rangle$ is the unique $\gamma(B)$-invariant Sylow q-subgroup, so that by Lemma 2.11-(2) $\bar{B} = B^{\iota(a)}$ is the unique $\gamma^{\iota(a)}(\bar{B})$-invariant Sylow q-subgroup. Since $\iota(A)$ conjugates transitively the Sylow q-subgroups of G, all classes have order a multiple of p^2.

Suppose now $\delta \in \mathrm{GL}(2, p)$. Then $\gamma(a_1^{\delta^{-1}}) = 1$ if and only if $\delta_{12} = 0$. Moreover,

$$\gamma^{\delta}(a_2) = \delta^{-1}\gamma(a_2^{\delta_{22}^{-1}})\delta = \delta^{-1}\alpha^{i\delta_{22}^{-1}}\delta,$$

and it is equal to α^i when $\delta_{11} = \delta_{22}^2$. Since every δ stabilises β, as $[\delta, \beta] = 1$, we have that the stabiliser has order $p(p-1)$. Therefore, there is one class of length $p^2(p^2-1)$.

10.6.2. The case $\gamma(G)$ non-abelian. Suppose now $\gamma(G)$ is non-abelian. Here β normalises but does not centralise $\iota(c)\alpha$. Therefore,

$$(\iota(c)\alpha)^{\beta} = \iota(c^{\beta})\beta^{-1}\alpha\beta = \iota(c^{\beta})\alpha^{\lambda^{x_1-x_2}}$$

is an element of $\langle \iota(c)\alpha \rangle$, and $x_1 \neq x_2$. Since $(\iota(c)\alpha)^k = \iota(c^{1+\alpha^{-1}+\cdots+\alpha^{-(k-1)}})\alpha^k$, we have that $(\iota(c)\alpha)^{\beta} \in \langle \iota(c)\alpha \rangle$ if and only if

$$c^{\beta} = c^{1+\alpha^{-1}+\cdots+\alpha^{-(\lambda^{x_1-x_2}-1)}}.$$

Writing $c = a_1^u a_2^v$, the latter yields

$$\begin{cases} u(1-\lambda^{-x_2}) = \frac{1}{2}v\lambda^{-x_2}(1-\lambda^{x_1-x_2}) \\ v\lambda^{x_2} = v\lambda^{x_1-x_2} \end{cases} . \tag{10.2}$$

From the second equation we obtain either $v = 0$ or $x_1 \equiv 2x_2 \bmod q$.

First case. If $v = 0$ the first equation yields $u = 0$ or $x_2 = 0$.

If $x_2 = 0$ and $u \neq 0$, we have

$$\gamma(G) = \left\langle \iota(a_1^u)\alpha, \beta \right\rangle,$$

where $\alpha_{|\langle b \rangle}, \beta_{|\langle b \rangle} = 1$. In this case there are p Sylow q-subgroups fixed by β, namely $\left\langle a_2^t b \right\rangle$. Among these, those fixed by $\iota(a_1^u)\alpha$ too are

$$a_2^t b = (a_2^t b)^{\iota(a_1^u)\alpha} = (a_2^t)^{\alpha} a_1^{u(-1+\lambda^{-1})\alpha} b = (a_2^t)^{\alpha} a_1^{u(-1+\lambda^{-1})} b,$$

that is, those with t such that $(a_2^t)^{1-\alpha} = a_1^{u(-1+\lambda^{-1})}$, namely $t = u(1-\lambda^{-1})$.

Since $\langle a_2^{u(1-\lambda^{-1})} b\rangle$ is fixed by both $\iota(a_1)\alpha$ and β, we can suppose that $\gamma(G) = \langle \alpha, \beta \rangle$ with α and β fixing the same Sylow q-subgroup (which we still denote by $\langle b \rangle$).

By the discussion above, when $v = 0$ we can always suppose

$$\gamma(G) = \langle \alpha, \beta \rangle,$$

with α, β fixing $\langle b \rangle$. Moreover a Sylow q-subgroup $\langle yb \rangle$ is $\gamma(G)$-invariant if and only if $y^{\alpha} = y^{\beta} = y$. The latter has $y = 1$ as a unique solution except when $x_1 = 0$, which gives the p solutions $y \in \langle a_1 \rangle$.

We will have $\gamma(a_2) = \alpha^i$, $i \neq 0$, and replacing β with another element of order q in $\gamma(G)$ we can suppose $\gamma(b) = \beta$.

Now we show that such an assignment extends to a gamma function if and only if $x_1 = 2x_2 + 1$. If $a = a_1^s a_2^t$, consider the maps defined by

$$\gamma(ab^k) = \alpha^{it\lambda^{-kx_2}}\beta^k.$$

With computations similar to the previous case, for $a' = a_1^x a_2^y$ we find that

$$\gamma(ab^k)\gamma(a'b^m) = \alpha^{it\lambda^{-kx_2}+iy\lambda^{-mx_2-k(x_1-x_2)}}\beta^{k+m},$$

and

$$\gamma((ab^k)^{\gamma(a'b^m)}cb^m) = \alpha^{i(t\lambda^{-kx_2}+y\lambda^{-k-(k+m)x_2})}\beta^{k+m},$$

so that they are equal precisely when $x_1 = 2x_2 + 1$.

Since there are $p+1$ choices for the kernel $\langle a_1 \rangle$, p choices for $\langle a_2 \rangle$, $p-1$ for the image of a_2, $q-1$ for x_2 such that $\beta_{|A}$ is non-scalar, and p^2 for the Sylow q-subgroup $\langle b \rangle$ fixed by α and β, we have $p^3(p^2-1)(q-1)$ GF. They are all distinct if $\langle b \rangle$ is the unique $\gamma(G)$-invariant Sylow q-subgroup. Otherwise, when $x_2 = \frac{q-1}{2}$, the p choices for a $\gamma(G)$-invariant Sylow q-subgroups yield the same GF, so there are $p^2(p^2-1)$ distinct GF.

Note that we do not consider the choices for the images of b, as if $\gamma(b) = \alpha^i\beta^k$, then the choices for k correspond to the choices for x_2, and since $\alpha^i\beta^k$ will have eigenspaces $\langle a_1 \rangle, \langle a_3 \rangle$, for $a_3 \in A \setminus \langle a_1 \rangle$, the choices for i correspond to the choices for the second eigenspace, namely to the choice for $\langle a_2 \rangle$.

Since

$$b^{\ominus 1} \circ a \circ b = a^{\gamma(b)\iota(b)},$$

and $a_1^{\circ t} = a_1^t$, $a_2^{\circ t} = a_2^t$ modulo $\langle a_1 \rangle$ for all t, denoting by $Z_\circ$ the action of b on A in $(G, \circ)$, we have

$$Z_\circ \sim \begin{bmatrix} \lambda^{2x_2+2} & 0 \\ 0 & \lambda^{x_2+1} \end{bmatrix}.$$

If $q = 2$ the unique choice for x_1, x_2 compatible with the conditions β non-scalar and $x_1 = 2x_2 + 1$ is $x_1 = 1$ and $x_2 = 0$. Therefore $Z_\circ = \operatorname{diag}(1, \lambda)$, and $(G, \circ)$ is of type 6. Therefore:

Type 6: is for $q = 2$ and $x_2 = 0$. Since $x_1 \neq 0$ we obtain $p^3(p^2-1)$ groups.

Suppose now $q > 2$. In this case if $x_2 = -1$, then $x_1 = -1$ too so that β is scalar, against the assumption that $\gamma(G)$ is non-abelian. So suppose $x_2 \neq -1$, so that the groups $(G, \circ)$ can have only types 8 or 9.

Type 9: is when $\det(Z_\circ) = 1$, namely for $3(x_2 + 1) = 0$. If $q > 3$ then there are no groups of type 9. If $q = 3$ then for $x_2 = 0$ there is a unique invariant Sylow q-subgroup, and we have $p^3(p^2-1)$ groups. If $x_2 = 1$ there are p invariant Sylow q-subgroups, and there are $p^2(p^2-1)$ groups. Thus in total $p^2(p^2-1) + p^3(p^2-1)$ groups.

Type 8: is when $q > 3$ and $\det(Z_\circ) \neq 1$, namely $x_2 \neq -1$. For each choice of x_2 there are $p^3(p^2-1)$ groups, except when $x_2 = \frac{q-1}{2}$, in which there are $p^2(p^2-1)$ groups. Thus in total $p^2(p^2-1)+p^3(p^2-1)(q-2)$ groups.

Note that in this case $Z_\circ \sim \operatorname{diag}(\mu^2, \mu)$, where $\mu = \lambda^{x_2+1}$, therefore the groups $(G, \circ)$ are all isomorphic to G_2.

As to the conjugacy classes, when there is a unique Sylow q-subgroup B which is $\gamma(B)$-invariant, as in the previous case, all classes have order a multiple of p^2. Otherwise $x_1 = 0$ and there are p invariant Sylow q-subgroups. In this case $\iota(x)$ stabilises γ if and only if $\iota(x)$ commutes with both α and β, namely when $x \in \langle a_1 \rangle$.

Now suppose $\delta = (\delta_{ij}) \in \mathrm{GL}(2, p)$. As above, $\gamma^\delta(a_1) = 1$ and $\gamma^\delta(a_2) = \gamma(a_2)$ if and only if $\delta_{12} = 0$ and $\delta_{11} = \delta_{22}^2$. Moreover

$$\gamma^\delta(b) = \delta^{-1}\gamma(b)\delta = \delta^{-1}\beta\delta,$$

and β, which is non-scalar, is centralised by δ when δ is a diagonal matrix.

Therefore, the orbits have length $p^3(p^2-1)$ if $x_1 \neq 0$ and $p^2(p^2-1)$ if $x_1 = 0$.

Second case. Suppose now $v \neq 0$ and $x_1 \equiv 2x_2 \bmod q$. Then (10.2) yields $v = -2u$ and we have

$$\gamma(G) = \langle\, \iota(c)\alpha, \beta \,\rangle,$$

where α and β fix $\langle\, b \,\rangle$, and $c = a_1^u a_2^{-2u}$.

replacing a_1 with a suitable element in $\langle\, a_1 \,\rangle$ we can suppose $u = \frac{1}{2}$.

We will have $\gamma(a_2) = (\iota(c)\alpha)^i$ for some $i \neq 0$, and $\gamma(b) = (\iota(c)\alpha)^j\beta^k$, with $k \neq 0$, as it is an element of order q in $\gamma(G)$. replacing β with a suitable element in $\langle\, \beta \,\rangle$ we can suppose that $k = 1$.

Now we show that if the assignment above extends to a GF, then $i = 1$. In fact, denoting by M_i the matrix $1 + \alpha^{-1} + \dots + \alpha^{-(i-1)}$, we will have

$$\begin{aligned}
b^{\ominus 1} \circ a_2 \circ b &= (b^{\gamma(b)^{-1}\gamma(a_2)\gamma(b)})^{-1} a_2^{\gamma(b)} b \\
&= (b^{\iota(c^{M_i})\alpha^i\beta})^{-1} a_2^{\alpha^j\beta} b \\
&= b^{-1} c^{(1-\lambda^{-1})M_i\alpha^i\beta} a_2^{\alpha^j\beta} b \\
&= c^{(\lambda-1)M_i\alpha^i\beta} a_2^{\lambda\alpha^j\beta} \\
&= a_1^{(\frac{1}{2}(1-\lambda)i^2+j\lambda)\lambda^{2x_2}} a_2^{((1-\lambda)i+\lambda)\lambda^{x_2}}.
\end{aligned}$$

Applying γ to both the sides we obtain

$$\gamma(b)^{-1}\gamma(a_2)\gamma(b) = \gamma(a_2)^{((1-\lambda)i+\lambda)\lambda^{x_2}},$$

and comparing with

$$\gamma(b)^{-1}\gamma(a_2)\gamma(b) = \beta^{-1}\iota(c^{M_i})\alpha^i = \iota(c^{M_i\beta})\alpha^{i\lambda^{x_2}}$$

we obtain $(1-\lambda)i + \lambda = i$, so that $i = 1$.

Now we show that if the map γ extends to a GF then there always exists at least one Sylow q-subgroup B which is $\gamma(B)$-invariant.

Write $x = a_1^w a_2^z$ for an element in A. If γ extends to a GF, then

$$\gamma(xb) = \gamma(a_2^{z\lambda^{-x_2}})\gamma(b) = \iota(c^{M_K})\alpha^K\beta,$$

where $K := j + z\lambda^{x_2}$.

Since

$$(xb)^{\gamma(xb)} = (xb)^{\iota(c^{M_K})\alpha^K\beta} = (xc^{(-1+\lambda^{-1})M_K})^{\alpha^K\beta}b,$$

$(xb)^{\gamma(xb)}$ belongs to $\langle\, xb \,\rangle$ if and only if

$$x^{1-\alpha^K\beta} = c^{(-1+\lambda^{-1})M_K\alpha^K\beta}. \tag{10.3}$$

Writing x and c in the basis $\{a_1, a_2\}$ and looking at their second component in (10.3) we find

$$z(\lambda^{-1} - \lambda^{x_2}) = j\lambda^{x_2}(1 - \lambda^{-1}). \tag{10.4}$$

If $x_2 \neq -1$ then there is a unique solution for z and in this case the first component (10.3) yields

$$w(1 - \lambda^{2x_2}) = j^2\lambda^{2x_2}\frac{(1-\lambda^{-1})}{2(\lambda^{-1} - \lambda^{x_2})^2}(1 - \lambda^{2x_2}),$$

so that, since $q > 2$ (as $x_2 \neq 0, -1$), there is a unique invariant Sylow q-subgroup.

Suppose now $x_2 = -1$. By induction one can show that in this case if the map γ is a GF then

$$\gamma(b^m) = \gamma(a_2)^{j(m-(1+\lambda+\cdots+\lambda^{m-1}))}\gamma(b)^m.$$

Looking at the exponent of α in $\gamma(b^q) = 1$, we obtain that $jq = 0$, namely $j = 0$.

Therefore, (10.4) yields that there are p solutions for z. Moreover in this case

$$w(1 - \lambda^{-2}) = \frac{1}{2}z^2(1 + \lambda^{-1}),$$

so that there are p invariant Sylow q-subgroups when $q > 2$ and p^2 when $q = 2$.

Now, since the Sylow q-subgroup $\langle\, b \,\rangle$ is invariant, $\gamma(b) = \beta^k$.

With computations similar to the previous cases one can show that the assignment

$$\begin{cases} \gamma(a_2) = \iota(c)\alpha \\ \gamma(b) = \beta \end{cases}$$

extends to a GF, namely if $a = a_1^s a_2^t$ then the map defined as

$$\gamma(ab^k) = \iota(c^{M_{t\lambda^{-kx_2}}})\alpha^{t\lambda^{-kx_2}}\beta^k$$

satisfies the GFE.

Since there are $p+1$ choices for $\langle\, a_1 \,\rangle$, p choices for $\langle\, a_2 \,\rangle$, $p-1$ choices for a_2 in $\langle\, a_2 \,\rangle$, $q-1$ choices for x_2, and p^2 choices for the Sylow q-subgroup fixed by α and β, we have $p^3(p^2-1)(q-1)$ GF. They are all distinct if $\langle\, b \,\rangle$ is the unique invariant Sylow q-subgroup. Otherwise, when $x_2 = -1$, the p (respectively p^2 when $q = 2$) choices for an invariant Sylow q-subgroup yield the same GF, and so there are $p^2(p^2-1)$ (respectively $p(p^2-1)$) distinct GF.

We have

$$b^{\ominus 1} \circ a_1 \circ b = a_1^{\lambda^{2x_2+1}},$$
$$b^{\ominus 1} \circ a_2 \circ b = a_1^{\frac{1}{2}(1-\lambda)\lambda^{2x_2}} a_2^{\lambda^{x_2}},$$

and since $a_1^{\circ t} = a_1^t$ and $a_2^{\circ t} = a_2^t$ modulo $\langle a_1 \rangle$ for all t, denoting as usual by $Z_\circ$ the action of b on A in $(G, \circ)$, we have

$$Z_\circ \sim \begin{bmatrix} \lambda^{2x_2+1} & 0 \\ 0 & \lambda^{x_2} \end{bmatrix}.$$

Type 6: when $Z_\circ$ has an eigenvalue 1, namely for $x_2 = \frac{q-1}{2}$. In this case there are $p^3(p^2-1)$ groups.

Type 7: when $Z_\circ$ is scalar, namely for $x_2 = -1$. In this case there are $p^2(p^2-1)$ groups if $q > 2$ and $p(p^2-1)$ if $q = 2$. (Note that for $x_1 = -1$, $b^{\ominus 1} \circ a_2 \circ b = a_2^{\circ \lambda^{x_2}}$, so that $Z_\circ$ is actually a scalar matrix.)

Type 8: when $q > 3$ and $x_2 \neq -1, \frac{q-1}{2}, \frac{q-1}{3}$, so there are $p^3(p^2-1)(q-4)$ groups of type 8.

Here each group $(G, \circ)$ is isomorphic to $G_{2+x_2^{-1}}$ for a certain x_2. For each $s \neq 2$, $s \in \mathcal{K}$, there are $2p^3(p^2-1)$ groups isomorphic to G_s, namely those obtained for x_2 such that $2 + x_2^{-1} = s$ and $2 + x_2^{-1} = s^{-1}$, while there are $p^3(p^2-1)$ groups isomorphic to G_2, as they can be obtained just for x_2 such that $2 + x_2^{-1} = 2^{-1}$.

Type 9: when $x_2 \neq -1, \frac{q-1}{2}$ and $Z_\circ$ has determinant equal to 1, namely when $x_2 = \frac{q-1}{3}$ and $q > 3$. In this case we obtain $p^3(p^2-1)$ groups.

As to the conjugacy classes, with computations similar to the previous cases we obtain orbits of length $p^3(p^2-1)$ when $x_2 \neq -1$, otherwise $x_2 = -1$ and there is unique orbit of length $p^2(p^2-1)$ if $q > 2$, and $p(p^2-1)$ if $q = 2$.

10.7. The cases $|\ker(\gamma)| = q$ and $|\ker(\gamma)| = 1$ when $q = 2$. The case $q = 2$ was treated by Crespo in [11]. In the following, we show that we obtain the same results using the gamma functions.

As explained in Subsection 10.1, in the case $\gamma(A) \leq \operatorname{Inn}(G)$ and $q = 2$ we cannot assume that $p \mid |\ker(\gamma)|$. But, except for the case when both γ and $\tilde{\gamma}$ have kernel of size not divisible by p, we will use duality to swich to a more convenient kernel as we done for the type 9. In particular, here we have to consider also the kernels of size q and 1.

10.7.1. The case $|\ker(\gamma)| = q = 2$. With exactly the same argument used for the type 9 (see Subsection 8.3), we obtain p^2 groups of type 5, which form a unique orbit of length p^2.

10.7.2. The case $|\ker(\gamma)| = 1$. Also in this case we proceed as for the type 9 (see Subsection 8.7), namely we divide the GF's of this case according to the size of $\ker(\tilde{\gamma})$, and for those for which $|\ker(\tilde{\gamma})| \neq 1$ we use the previous computations applied to $\tilde{\gamma}$. The others, for which $|\ker(\tilde{\gamma})| = 1$, can be obtained using Proposition 2.8.

Arguing as in Subsection 8.7, we obtain the following.

- When $|\ker(\tilde{\gamma})| = p^2q$, $\gamma = \tilde{\tilde{\gamma}}$ corresponds to the left regular representation, and this gives one group of type 7.
- In the remaining cases for which $q \mid |\ker(\tilde{\gamma})|$, none of the corresponding γ's has trivial kernel.
- If $|\ker(\tilde{\gamma})| = p^2$, the p^2 GF's $\tilde{\gamma}$ corresponding to $(G, \circ)$ of type 5 are such that the corresponding γ have kernel of size q, and all the others $\tilde{\gamma}$ correspond to γ with kernel of size 1. Therefore, in the case of kernel of size p^2 we double all the GF's except those corresponding to $(G, \circ)$ of type 5.
- If $|\ker(\tilde{\gamma})| = p$, again, $\ker(\gamma)$ can have size 1 or q. By Subsection 10.5, the $\tilde{\gamma}$'s such that $p \mid |\ker(\tilde{\gamma})|$ and $p \nmid |\ker(\gamma)|$ are those of the cases A1 and A1*. Moreover, for every $\tilde{\gamma}$ belonging to these cases the corresponding γ are injective. Therefore, in the case of kernel of size p we double the GF's of cases A1 and A1*.
- When $|\ker(\tilde{\gamma})| = 1$, following the argument in Subsection 8.7, we obtain
$$2p(p-1)(p-2) \cdot \frac{p(p+1)}{2} = p^2(p^2-1)(p-2)$$
groups $(G, \circ)$ of type 6, which split in $p-2$ classes of length $p^2(p^2-1)$.

10.8. Results. Taking into account what we said in Subsection 10.1, we obtain the following.

Proposition 10.3. *Let G be a group of order p^2q, $p > 2$, of type 7. For each isomorphism class of groups (Γ), the number of regular subgroups in* $\mathrm{Hol}(G)$ *(RS), and the number (n) and the lengths (l) of the conjugacy classes in* $\mathrm{Hol}(G)$ *are listed in the following table.*

For groups of type 8 we denote by 8_{G_s}, where $s \in \mathcal{K}$, the isomorphism class of G_s.

$T(n)$ denotes the total number of conjugacy classes.

In the row of 8_{G_s} we mean that for every $s \in \mathcal{K}$, $s \neq 2$, there are
$$4p^2(p+1)(2p^2-5p+pq+2)$$
regular subgroups isomorphic to G_s.

Γ	*Conditions*	*RS*	n	l	$T(n)$
5		$p^3(3p+1)$	2	p^2	5
			1	$p^3(p+1)$	
			2	$p^2(p^2-1)$	
6	$q>2$	$4p^2(p+1)(p^2+pq-2p)$	4	$p^2(p+1)$	$4(q+1)$
			$4(q-2)$	$p^3(p+1)$	
			4	$p^2(p^2-1)$	
			4	$p^3(p^2-1)$	
	$q=2$	$p^3(p+1)(3p+1)$	4	$p^2(p+1)$	$10+p$
			$p+4$	$p^2(p^2-1)$	
			2	$p^3(p^2-1)$	
7	$q>2$	$2+p^2(2p^2+pq+2q-4)$	2	1	$3q-1$
			$2(q-2)$	p^2	
			2	$p^2(p+1)$	
			2	$p^2(p^2-1)$	
			$q-3$	$p^3(p+1)$	
	$q=2$	$2+p(p+1)(2p-1)$	2	1	5
			2	$p(p^2-1)$	
			1	$p(p+1)$	
8_{G_2}	$q>3$	$2p^2(p+1)(p^2q+pq-4p+2)$	$4(q-3)$	$p^3(p+1)$	$6q$
			8	$p^2(p+1)$	
			4	$p^2(p^2-1)$	
			$2q$	$p^3(p^2-1)$	
8_{G_s}, $s \neq 2$	$q>3$	$4p^2(p+1)(2p^2-5p+pq+2)$	$4(q-3)$	$p^3(p+1)$	$4(q+1)$
			8	$p^2(p+1)$	
			8	$p^3(p^2-1)$	
9	$q>3$	$4p^5+p^4(q-2)+p^3(2q-7)+3p^2+p$	$2q-5$	$p^3(p+1)$	$2(q+1)$
			2	$p^2(p+1)$	
			1	$p(p+1)$	
			4	$p^3(p^2-1)$	
	$q=3$	$p(p+1)(2p^3+3p^2-2p+1)$	1	$p^3(p+1)$	10
			2	$p^2(p+1)$	
			1	$p(p+1)$	
			2	$p^3(p^2-1)$	
			4	$p^2(p^2-1)$	

11. Type 10

In this case $q \mid p+1$, where $q>2$, and $G = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes_C \mathcal{C}_q$. The Sylow p-subgroup $A = \langle a_1, a_2 \rangle$ is characteristic and a generator b of a Sylow q-subgroup acts on A as a suitable power Z of a Singer cycle, namely $a^b = a^Z$ for $a \in A$. We know that Z has determinant 1 and two (conjugate) eigenvalues $\lambda, \lambda^p = \lambda^{-1} \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$.

The divisibility condition on p and q implies that $(G, \circ)$ can only be of type 5 or 10.

According to Subsections 4.1 and 4.4 of [8], we have

$$\mathrm{Aut}(G) = (\mathcal{C}_p \times \mathcal{C}_p) \rtimes (\mathcal{C}_{p^2-1} \rtimes \mathcal{C}_2),$$

where $\mathcal{C}_p \times \mathcal{C}_p = \iota(A)$, and for $\mu \in \mathcal{C}_{p^2-1}$ and $\psi \in \mathcal{C}_2$ we write

$$\mu : \begin{cases} a \mapsto a^M \\ b \mapsto b \end{cases}, \psi : \begin{cases} a \mapsto a^S \\ b \mapsto b^r \end{cases}, \tag{11.1}$$

where $M = uI + vZ \in \mathrm{GL}(2, p)$, for $u, v \in \mathbb{F}_p$ not both zero, and S, r are such that either $r = 1$ and $S = 1$, or $r = -1$ and

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The Sylow p-subgroup of $\mathrm{Aut}(G)$ has order p^2 and is characteristic and a Sylow q-subgroup is cyclic, so $\gamma(G)$ of order a divisor of $p^2 q$ is always contained in $\mathrm{Inn}(G)$.

Moreover

- since A is characteristic, it is also a Sylow p-subgroup of $(G, \circ)$, so $\gamma(A)$ is a subgroup of $\iota(A)$, the Sylow p-subgroup of $\mathrm{Aut}(G)$.
- $\gamma_{|A} : A \to \mathrm{Aut}(G)$ is a morphism, as for each $a \in A$ the automorphism $\iota(a)$ acts trivially on the abelian group A, and so

$$\gamma(a)\gamma(a') = \gamma(a^{\gamma(a')} a') = \gamma(aa').$$

  Therefore, $\gamma(a) = \iota(a^{-\sigma})$ for each $a \in A$, where $\sigma \in \mathrm{End}(A)$.

In the following assume $\gamma \neq 1$.

11.1. The case $|\ker(\gamma)| = p$. This case does not arise, in fact the group $(G, \circ)$ cannot have type 5, since $\mathrm{Inn}(G)$ does not contain an abelian subgroup of order pq. $(G, \circ)$ can neither be of type 10, since a group of type 10 has no normal subgroups of order p.

11.2. The case $|\ker(\gamma)| = p^2$. Here $\ker(\gamma) = A$ and $|\gamma(G)| = q$, so $\gamma(G) = \langle \iota(b) \rangle$ where b is a q-element of G. In this case $B = \langle b \rangle$ is the unique $\gamma(G)$-invariant Sylow q-subgroup, therefore by Proposition 2.6, each γ is the lifting of exactly one RGF defined on the unique $\gamma(G)$-invariant Sylow q-subgroup. So, for each choice of $B = \langle b \rangle$ (p^2 possibilities), we can define $\gamma(b) = \iota(b^{-s})$, with $1 \le s \le q - 1$ ($q - 1$ choices).

Since $[B, \gamma(B)] = 1$, by Lemma 2.5 the RGF's correspond to the morphisms.

For $s = 1$ we obtain a group of type 5 and for $s \neq 1$ ($q - 2$ choices) we obtain a group of type 10. In conclusion there are

(i) p^2 groups of type 5;
(ii) $p^2(q - 2)$ groups of type 10.

As to the conjugacy classes, here the kernel A is characteristic, so that every $\varphi \in \mathrm{Aut}(G)$ stabilises $\gamma_{|A}$.

All orbits here have length a multiple of p^2, as

$$\gamma^{\iota(x)}(b) = \iota(x^{-1})\gamma(b)\iota(x) = \iota(x^{-1+Z^s} b^{-s}) = \iota(x^{-1+Z^s})\gamma(b).$$

Now, let $\varphi = \mu\psi$, where μ and ψ are as in (11.1). Then

$$\gamma^{\varphi}(b) = \varphi^{-1}\iota(b^{-sr})\varphi = \psi^{-1}\iota(b^{-sr})\psi = \iota(b^{-s}) = \gamma(b),$$

so that the orbits have length exactly p^2.

11.3. The case $q \mid |\ker(\gamma)|$. In this case $(G, \circ)$ can only be of type 5, as a group of type 10 has no normal subgroups of order q or pq.

Let $B \leq \ker(\gamma)$. Since A is characteristic, by Proposition 2.6 each GF on G is the lifting of a RGF defined on A, and a RGF on A can be lifted to G if and only if B is invariant under $\{\gamma(a)\iota(a) \mid a \in A\}$.

Now, for each $a \in A$, $\gamma(a) = \iota(a^{-\sigma})$, where $\sigma \in \mathrm{End}(A)$, so that $\gamma(a)\iota(a) = \iota(a^{1-\sigma})$. Since every Sylow q-subgroup of G is self-normalising, necessarily $\sigma = 1$, so that for each $a \in A$

$$\gamma(a) = \iota(a^{-1}).$$

Since $[A, \gamma(A)] = 1$, by Lemma 2.5 the RGF's correspond to the morphisms. So, for each of the p^2 choices for B there is a unique RGF on A which lifts to G.

In conclusion we obtain p^2 groups of type 5. Note that for all the GF's of this case $|\ker(\gamma)| = q$, namely there are no GF's on G with $|\ker(\gamma)| = pq$.

As to the conjugacy classes, as in Subsection 8.3, since $\iota(A)$ conjugates transitively the p^2 Sylow q-subgroups of G, the p^2 GF's are conjugate.

11.4. The case $\ker(\gamma) = \{1\}$. As in Subsection 8.7, the GF's of this case can be divided into subclasses according to the size of $\ker(\tilde{\gamma})$.

In this case $\gamma(G) = \mathrm{Inn}(G) \cong (G, \circ)$, so that all the GF's correspond to groups of type 10.

Let $\gamma(a) = \iota(a^{-\sigma})$ for some $\sigma \in \mathrm{GL}(2, p)$.

Consider first the case $\sigma = 1$, namely $\gamma(a) = \iota(a^{-1})$. In this case $p^2 \mid |\ker(\tilde{\gamma})|$, since $\tilde{\gamma}(x) = \gamma(x^{-1})\iota(x^{-1})$ for all $x \in G$.

Consider the $\tilde{\gamma}$'s whose kernel has size exactly p^2. By the result of Subsection 11.2, the are p^2q such $\tilde{\gamma}$, and they split as follows.

- p^2 of these $\tilde{\gamma}$ correspond to $(G, \circ)$ of type 5; the corresponding γ have kernel of size q, so that they have already been considered in Subsection 11.3.
- $p^2(q-1)$ of these $\tilde{\gamma}$ correspond to $(G, \circ)$ of type 10; the corresponding γ have indeed kernels of size 1.

Therefore the γ with $|\ker(\gamma)| = 1$ and such that the corresponding $\tilde{\gamma}$ have $|\ker(\gamma)| \neq 1$ are $p^2(q-1)$ plus the right regular representation, and all of them correspond to groups of type 10.

Let now $\sigma \neq 1$. In this case 1 is not an eigenvalue of σ, in fact otherwise $p \mid |\ker(\tilde{\gamma})|$, but, as seen before, this implies $p^2 \mid |\ker(\tilde{\gamma})|$, and hence $\sigma = 1$. Therefore, here both σ and $(1 - \sigma^{-1})$ are invertible.

Let b be a q-element and let $\gamma(b) = \iota(a_0 b^{-s})$ for some $a_0 \in A$ and $s \not\equiv 0 \bmod q$. Then Subsection 4.2 yields (4.2), which in our notation here is

$$(\sigma^{-1} - 1)^{-1} Z^{-s} (\sigma^{-1} - 1) = Z^{1-s}. \tag{11.2}$$

Recall that Z has order q and has two conjugate eigenvalues λ and λ^{-1} in the extension $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. An easy computation shows that the corresponding eigenspaces are $\langle v_1 \rangle$ and $\langle v_2 \rangle$ with $v_1 = a_1 + \lambda a_2$, $v_2 = a_1 + \lambda^{-1} a_2$.

From (11.2) we get $\{\lambda^{-s}, \lambda^{s}\} = \{\lambda^{1-s}, \lambda^{-1+s}\}$, which is possible only for $\lambda^{-s} = \lambda^{-1+s}$: this gives the condition $2s \equiv 1 \bmod q$ and means that $\sigma^{-1} - 1$ exchanges the two eigenspaces of Z. Therefore, with respect to the basis $\{v_1, v_2\}$,

$$\sigma^{-1} - 1 = \begin{bmatrix} 0 & \nu^{p} \\ \nu & 0 \end{bmatrix},$$

where $\nu \in \mathbb{F}_{p^2}^*$. The condition that σ has not 0 and 1 as eigenvalues reads here as $\det(\sigma^{-1}-1) = \nu^{p+1} \neq 0$ and $\det(\sigma^{-1}) = 1-\nu^{p+1} \neq 0$, so we have $p^2-1-(p+1) = (p+1)(p-2)$ choice for $\sigma^{-1} - 1$ and hence for σ. Since $2s \equiv 1 \bmod q$, there are $(p+1)(p-2)$ choices for the couple (σ, s).

The next Proposition shows that all the GF's of this case can be constructed via gluing.

Proposition 11.1. *Let γ be a GF on a group G of type 10. If $|\ker(\gamma)| = 1$, then there is a unique Sylow q-subgroup B of G invariant under $\gamma(B)$.*

Proof. Let $B = \langle\, b \,\rangle$ be a Sylow q-subgroup of G. Then, a Sylow q-subgroup $\langle\, b^x \,\rangle$, where $x \in A$, is invariant when $(b^x)^{\gamma(b^x)} \in \langle\, b^x \,\rangle$, that is,

$$\gamma(b^x) \in \mathrm{Norm}_{\mathrm{Aut}(G)}(\iota(b^x)).$$

Since $\gamma(b^x)$ is a q-element, the latter means that

$$\gamma(b^x) \in \langle\, \iota(b^x) \,\rangle. \tag{11.3}$$

Since

$$\begin{aligned}
\gamma(b^x) &= \gamma(x^{-1+Z^{-1}}b) \\
&= \gamma(x^{(-1+Z^{-1})\iota(b^s)})\gamma(b) \\
&= \iota(x^{(1-Z^{-1})Z^s\sigma})\iota(a_0 b^{-s}) \\
&= \iota(b^{-s})\iota(x^{(1-Z^{-1})Z^s\sigma Z^{-s}} a_0^{Z^{-s}}),
\end{aligned}$$

(11.3) becomes

$$\iota(b^{-s})\iota(x^{(1-Z^{-1})Z^s\sigma Z^{-s}} a_0^{Z^{-s}}) = \iota(x^{-1+Z^{-1}}b)^{-s},$$

which is equivalent to

$$b^{-s}x^{(1-Z^{-1})Z^s\sigma Z^{-s}} a_0^{Z^{-s}} = b^{-s}x^{-Z^{-s}+1}.$$

Therefore, we are left with showing that the equation

$$x^{(1-Z^{-1})Z^s\sigma Z^{-s}} a_0^{Z^{-s}} = x^{-Z^{-s}+1} \tag{11.4}$$

has a unique solution x for each choice of $a_0 \in A$, namely that the matrix

$$D = (1 - Z^{-1})Z^s + (1 - Z^s)\sigma^{-1}$$

is invertible. One can easily compute D and $\det(D) = (1-\lambda^s)(1-\lambda^{-s})(1-\nu^{p+1})$, and since the latter is non-zero, D is invertible and (11.4) has a unique solution x. $\square$

Summarising, each γ admits a unique invariant Sylow q-subgroup, so that it is a gluing of a RGF $\gamma_B : B \to \mathrm{Aut}(G)$ and a RGF γ_A determined by σ, with the condition that equation (11.2) holds. Necessarily, $\gamma_B(b) = \iota(b^{-s})$ for some s, and by (11.2) we get $2s \equiv 1 \bmod q$. Therefore, for each B (p^2 choices), we have only one RGF on B and $(p+1)(p-2)$ choices for σ, so there are $p^2(p+1)(p-2)$ distinct GF, corresponding to groups of type 10.

As to the conjugacy classes, since each γ has a unique Sylow q-subgroup B which is $\gamma(B)$-invariant, by Lemma 2.11-(2), for $a \in A$, $\gamma^{\iota(a)}$ has $\bar{B} = B^{\iota(a)}$ as $\gamma(\bar{B})$-invariant Sylow p-subgroup. Now $\iota(A)$ conjugates transitively the Sylow p-subgroups of G, so that all classes have order a multiple of p^2.

Consider $\varphi = \mu\psi \in \mathrm{Aut}(G)$, where μ and ψ are as in (11.1). μ fixes b and centralises $\gamma(b)$, so that it stabilises $\gamma_{|B}$. Moreover ψ has order 2, and $b^\psi = b^r$, $\iota(b)^\psi = \iota(b^r)$, so that ψ stabilises $\gamma_{|B}$ as well.

As for $\gamma_{|A}$, we have $a^{\varphi^{-1}} = a^{\psi\mu^{-1}} = a^{SM^{-1}}$, so that

$$\gamma^\varphi(a) = \varphi^{-1}\gamma(a^{SM^{-1}})\varphi = \varphi^{-1}\iota(a^{-SM^{-1}\sigma})\varphi,$$

and it coincides with $\gamma(a) = \iota(a^{-\sigma})$ if and only if $a^{SM^{-1}\sigma MS} = a^\sigma$, namely if and only if $\sigma MS\sigma^{-1} = MS$. The latter can be written as

$$(\sigma^{-1} - 1)^{-1}MS(\sigma^{-1} - 1) = MS. \tag{11.5}$$

If $S = 1$, (11.5) yields $u + v\lambda^{-1} = u + v\lambda$, namely $v = 0$, so that there are $p - 1$ choices for μ. If $S \neq 1$, (11.5) yields $(u + v\lambda^{-1})\nu^{p-1} = u + v\lambda$, namely

$$\frac{u}{v} = \frac{\lambda - \lambda^{-1}\nu^{p-1}}{\nu^{p-1} - 1}.$$

Since it is fixed by the Frobenius endomorphism, it is actually in $\mathbb{F}_p$, and there are $p - 1$ choices for μ.

Therefore, the stabiliser has order $2(p-1)$, and there are $p-2$ orbits of length $p^2(p + 1)$.

We summarise, including the right and left regular representations.

Proposition 11.2. *Let G be a group of order p^2q, $p > 2$, of type 10. For each isomorphism class of groups (Γ), the number of regular subgroups in* Hol(G) *(RS), and the number (n) and the lengths (l) of the conjugacy classes in* Hol(G) *are listed in the following table.*

Γ	RS	n	l
5	$2p^2$	2	p^2
10	$2 + p^2(2(q-2) + (p+1)(p-2))$	2	1
		$2(q-2)$	p^2
		$p-2$	$p^2(p+1)$

12. Type 11

In this case $p \mid q - 1$ and $G = \mathcal{C}_p \times (\mathcal{C}_p \ltimes \mathcal{C}_q)$. Let $Z = \langle z \rangle$ be the center of G, and $B = \langle b \rangle$ the Sylow q-subgroup.

According to Subsection 4.6 of [8],

$$\mathrm{Aut}(G) = \mathrm{Hol}(\mathcal{C}_p) \times \mathrm{Hol}(\mathcal{C}_q),$$

so that a Sylow p-subgroup of $\mathrm{Aut}(G)$ is of the form $\mathcal{C}_p \times \mathcal{P}$, where $\mathcal{C}_p$ is generated by a central automorphism and $\mathcal{P}$ is a Sylow p-subgroup of $\mathrm{Hol}(\mathcal{C}_q)$. Therefore, a subgroup of order p^2 in $\mathrm{Aut}(G)$ is generated by an inner automorphism $\iota(a)$, for some non-central element a of order p, and the central automorphism

$$\psi : \begin{cases} z \mapsto z, \\ a \mapsto az, \\ b \mapsto b \end{cases} . \tag{12.1}$$

By Proposition 2.9 (see also [7, Corollary 2.25]) in counting the GF's we can suppose $B \le \ker(\gamma)$. Therefore, the image $\gamma(G)$ is contained in a subgroup of $\mathrm{Aut}(G)$ of order p^2, that is,

$$\gamma(G) \le \langle\, \iota(a), \psi \,\rangle,$$

for $a \in A \setminus Z$, and ψ as in (12.1).

In this case there exists at least one Sylow p-subgroup A of G which is $\gamma(G)$-invariant (see [7, Theorem 3.3]). More precisely $A = \langle\, a, z \,\rangle$ is $\gamma(G)$-invariant, and it is the unique $\gamma(G)$-invariant Sylow p-subgroup if $\gamma(G) \cap \mathrm{Inn}(G) \ne \{1\}$; otherwise $\gamma(G) \le \langle\, \psi \,\rangle$, and every Sylow p-subgroup is $\gamma(G)$-invariant.

We may thus apply Proposition 2.6, and look for the functions

$$\gamma' : A \to \mathrm{Aut}(G)$$

which satisfy the GFE (we will just write γ in the following). Since $(A, \circ)$ is abelian, we have

$$a^{\gamma(z)} z = a \circ z = z \circ a = z^{\gamma(a)} a = za,$$

so that $a^{\gamma(z)} = a$, namely

$$\gamma(z) = \iota(a)^s, \tag{12.2}$$

for some $0 \le s \le p-1$. We also have

$$\gamma(a) = \iota(a)^t \psi^u, \tag{12.3}$$

for some $0 \le t, u \le p-1$.

If both $s = 0$ and $u = t = 0$, then $\ker(\gamma) = G$ and we get the right regular representation.

Proposition 2.6 yields also that the RGF's on A with kernel of size 1, respectively p, correspond to the GF's on G with kernel of size q, respectively qp. In the following we suppose $\gamma(G) \ne \{1\}$.

12.1. The case $|\ker(\gamma)| = q$. Here $\gamma(G) = \langle\, \iota(a), \psi \,\rangle$ and $A = \langle\, a, z \,\rangle$ is the unique Sylow p-subgroup of G which is $\gamma(G)$-invariant. By the discussion above, we look for the RGF's γ on A extending the assignments (12.2), (12.3), and with trivial kernel, namely $s \ne 0$ and $u \ne 0$. By Lemma A.1 in the Appendix there is a unique RGF γ like that, and since there are q choices for A, we get $qp(p-1)^2$ maps.

As to the circle operation, for every $x \in A$, $x^{\ominus 1} \circ b \circ x = b^{\gamma(x)\iota(x)}$, so that

$$a^{\ominus 1} \circ b \circ a = b^{\iota(a^{t+1})\psi^u}, \ z^{\ominus 1} \circ b \circ z = b^{\iota(a^s z)}.$$

Since $b^{\circ k} = b^k$ and $s \neq 0$, all groups $(G, \circ)$ are of type 11.

As to the conjugacy classes, let $\varphi \in \mathrm{Aut}(G)$. Write $\varphi = \varphi_1 \varphi_2$, where $\varphi_1 \in \mathrm{Hol}(\mathcal{C}_p)$ and $\varphi_2 \in \mathrm{Hol}(\mathcal{C}_q)$, so that

$$\varphi_1 : \begin{cases} z \mapsto z^i \\ a \mapsto az^j \\ b \mapsto b \end{cases}, \varphi_2 : \begin{cases} z \mapsto z \\ a \mapsto b^m a \\ b \mapsto b^k \end{cases}. \tag{12.4}$$

Since the kernel B is characteristic, then $\gamma_{|B}$ is stabilised by every automorphism of G.

Moreover

$$\begin{aligned} \gamma^{\varphi}(a) &= \varphi^{-1}\gamma(az^{-ji^{-1}})\varphi \\ &= \varphi^{-1}\gamma(a)\gamma(z)^{-ji^{-1}}\varphi \\ &= \varphi^{-1}\iota(a^{t-sji^{-1}})\psi^u\varphi \\ &= (\iota(a^{t-sji^{-1}}))^{\varphi_2}(\psi^u)^{\varphi_1}, \end{aligned}$$

and

$$\begin{aligned} \gamma^{\varphi}(z) &= \varphi^{-1}\gamma(z^{i^{-1}})\varphi \\ &= \varphi^{-1}\iota(a^{si^{-1}})\varphi \\ &= (\iota(a^{si^{-1}}))^{\varphi_2}, \end{aligned}$$

so that φ stabilises γ if and only if $\varphi_1 = 1$ and $\varphi_2 \in \mathcal{C}_{q-1}$.

Therefore, the stabiliser has order $q-1$ and there are $p-1$ orbits of length $qp(p-1)$.

12.2. The case $|\mathbf{ker}(\gamma)| = pq$. Here $\gamma(G)$ is a subgroup of order p of $\langle \iota(a), \psi \rangle$. We look for the RGF's γ on A extending the assignments (12.2), (12.3), and with kernel of size p, namely $s = 0$ or $u = 0$.

Suppose first that $s = 0$, so that the kernel is ZB and $\gamma(a) = \iota(a)^t\psi^u$. By Lemma A.2 in the Appendix the RGF's on A with kernel of size p are precisely the morphisms.

(1) If $t = 0$, then $\gamma(a) = \psi^u$ and every Sylow p-subgroup is $\gamma(G)$- invariant. Therefore, here we obtain $p-1$ groups, and they are all of type 11 as B is $\gamma(B)$-invariant and

$$a^{\ominus 1} \circ b \circ a = b^{\iota(a)}.$$

(2) If $t \neq 0$, then $\gamma(a) = \iota(a)^t\psi^u$, and $A = \langle a, z \rangle$ is the unique $\gamma(G)$-invariant Sylow p-subgroup, so that we have q choices for A, $p-1$ for t and p for u, namely $qp(p-1)$ functions. Since

$$a^{\ominus 1} \circ b \circ a = b^{\iota(a)^{t+1}},$$

they correspond to qp groups of type 5 and $qp(p-2)$ groups of type 11.

As to the conjugacy classes, here the kernel ZB is charactertistic, so that $\gamma_{|ZB}$ is stabilised by every automorphism of G.

Now, since $a^{\varphi} \equiv a \bmod \ker(\gamma)$, we have

$$\gamma^{\varphi}(a) = \varphi^{-1}\gamma(a)\varphi = \varphi^{-1}\iota(a^t)\psi^u\varphi = (\iota(a^t))^{\varphi_2}(\psi^u)^{\varphi_1},$$

so that φ stabilises γ if and only if it centralises $\gamma(a)$.

If $t = 0$, the last condition is equivalent to say that $\varphi_1 \in \langle \psi \rangle$ and $\varphi_2 \in \operatorname{Hol}(\mathcal{C}_q)$, so that the $p-1$ groups of type 11 form one orbit of length $p-1$.

If $t \neq 0$, then φ centralises $\gamma(a)$ if and only if $\varphi_2 \in \mathcal{C}_{q-1}$, and either $u \neq 0$ and $\varphi_1 \in \langle \psi \rangle$, or $u = 0$ and $\varphi_1 \in \operatorname{Hol}(\mathcal{C}_p)$. In the first case the stabiliser has order $p(q-1)$, and there is one orbit of length $q(p-1)$ for the type 5, and $p-2$ orbits of length $q(p-1)$ for the type 11. In the second case the stabiliser has order $p(p-1)(q-1)$, and there is one orbit of length q for the type 5, and $p-2$ orbits of length q for the type 11.

Suppose now $u = 0$, so that $\gamma(a) = \iota(a^t)$ and $\gamma(z) = \iota(a^s)$. Here $\ker(\gamma) = \langle v \rangle$, where $v = z^e a^f$ is such that $tf + se = 0$. Up to changing the basis of A, we can appeal again to Lemma A.2, which yields that the RGF's here are exactly the morphisms. Again, $A = \langle a, z \rangle$ is the unique $\gamma(G)$-invariant Sylow p-subgroup, and we obtain $qp(p-1)$ functions. Since

$$z^{\ominus 1} \circ b \circ z = b^{\iota(a^s)}$$

they correspond to groups of type 11.

As to the conjugacy classes, since $B \leq \ker(\gamma)$ is characteristic, $\gamma_{|B}$ is stabilised by every automorphism φ. Moreover , let $\varphi = \varphi_1\varphi_2$ where φ_1, φ_2 are as in (12.4). We have

$$\begin{aligned}
\gamma^{\varphi}(a) &= \varphi^{-1}\gamma(az^{-ji^{-1}})\varphi = (\iota(a^{t-sji^{-1}}))^{\varphi_2},\\
\gamma^{\varphi}(z) &= \varphi^{-1}\gamma(z^{i^{-1}})\varphi = (\iota(a^{si^{-1}}))^{\varphi_2},
\end{aligned}$$

so that φ stabilises γ if and only if $\varphi_1 = \mathrm{id}$ and $\varphi_2 \in \mathcal{C}_{q-1}$, namely the stabiliser has order $q-1$, and there is one orbit of length $qp(p-1)$.

We summarise, including the right and left regular representations.

Proposition 12.1. *Let G be a group of order p^2q, $p > 2$, of type 11. For each isomorphism class of groups (Γ), the number of regular subgroups in* $\operatorname{Hol}(G)$ *(RS), and the number (n) and the lengths (l) of the conjugacy classes in* $\operatorname{Hol}(G)$ *are listed in the following table.*

Γ	RS	n	l
5	$2pq$	2	q
		2	$q(p-1)$
11	$2p(1+q(p^2-2))$	2	1
		2	$p-1$
		$2p$	$qp(p-1)$
		$2(p-2)$	$q(p-1)$
		$2(p-2)$	q

13. Conclusions

The proofs of Theorem 1.6 and Theorem 1.7 are obtained by piecing together the results of Propositions 5.2, 6.1, 8.3, 9.3, 10.3, 11.2, 12.1, and recalling that if the Sylow p-subgroups of the groups Γ and G are not isomorphic, then $e'(\Gamma, G) = e(\Gamma, G) = 0$ ([7, Corollary 3.4]).

To prove Theorem 1.5, we use Theorem 1.2 [4, Corollary p. 3220]. Therefore, for each pair of finite groups Γ, G with $|\Gamma| = |G|$, we have

$$e(\Gamma, G) = \frac{|\mathrm{Aut}(\Gamma)|}{|\mathrm{Aut}(G)|} e'(\Gamma, G).$$

The values of $e'(\Gamma, G)$ computed in Propositions 5.2, 6.1, 8.3, 9.3, 10.3, 11.2, 12.1 and the cardinalities of the automorphism groups given in Table 3 yield the values of $e(\Gamma, G)$.

Appendix A.

The following Lemma proves that the maps found in Subsections 5.2, 12.1 in the case $|\ker(\gamma)| = q$ are gamma functions.

Lemma A.1. *Let G be a group of type 5 or 11, B its Sylow q-subgroup, and $A = \langle a_1, a_2 \rangle$ a Sylow p-subgroup.*

Let $\gamma : A \to \mathrm{Aut}(G)$ a map such that

$$\begin{cases} \gamma(a_1) &= \eta_1 \\ \gamma(a_2) &= \eta_2, \end{cases} \tag{A.1}$$

where $\eta_{1|A} = 1$, $a_1^{\eta_2} = a_1$, $a_2^{\eta_2} = a_2 a_1^k$, $1 \leq k < p$.

Then

$$\gamma(a_1^n a_2^m) = \eta_1^{n-k(1+\cdots+(m-1))} \eta_2^m$$

is the unique RGF extending the assignment above.

Proof. By our assumptions A is clearly $\gamma(A)$-invariant. Moreover

$$\begin{aligned} \gamma((a_1^n a_2^m)^{\gamma(a_1^e a_2^f)} a_1^e a_2^f) &= \gamma((a_1^n a_2^m)^{\eta_1^{e-k(1+\cdots+(f-1))} \eta_2^f} a_1^e a_2^f) \\ &= \gamma(a_1^n (a_2^m)^{\eta_2^f} a_1^e a_2^f) \\ &= \gamma(a_1^{n+e+kfm} a_2^{m+f}) \\ &= \eta_1^{n+e+kfm-k(1+\cdots+(m+f-1))} \eta_2^{m+f}, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \gamma(a_1^n a_2^m) \gamma(a_1^e a_2^f) &= \eta_1^{n-k(1+\cdots+(m-1))} \eta_2^m \eta_1^{e-k(1+\cdots+(f-1))} \eta_2^f \\ &= \eta_1^{n-k(1+\cdots+(m-1))+e-k(1+\cdots+(f-1))} \eta_2^{m+f}. \end{aligned}$$

Therefore, γ satisfies the GFE if and only if

$$-k(\sum_{s=1}^{m+f-1} s) + fkm \equiv -k(\sum_{s=1}^{m-1} s + \sum_{s=1}^{f-1} s) \bmod p,$$

that is,

$$\sum_{s=m}^{m+f-1} s - fm \equiv \sum_{s=1}^{f-1} s \bmod p.$$

Since $m+(m+1)+\cdots+(m+f-1) = fm+(1+\cdots+f-1)$, the last condition holds true, and γ is a RGF on A.

Now let γ' be a RGF on A extending the assignment (A.1). Since $\eta_{1|A} = 1$, necessarily $a_1^{\circ n} = a_1^n$, so

$$\gamma'(a_1^n a_2^m) = \gamma'((a_2^m)^{\gamma'(a_1^n)^{-1}})\gamma'(a_1^n) = \gamma'(a_2^m)\gamma'(a_1)^n.$$

Moreover

$$\begin{aligned}\gamma'(a_2^m) &= \gamma'((a_2^{m-1})^{\gamma'(a_2)^{-1}})\gamma'(a_2)\\ &= \gamma'(a_1^{-k(m-1)}a_2^{m-1})\gamma'(a_2)\\ &= \gamma'(a_1)^{-k(m-1)}\gamma'(a_2^{m-1})\gamma'(a_2).\end{aligned}$$

By induction we obtain $\gamma'(a_2^m) = \gamma'(a_1)^{-k((m-1)+(m-2)+\cdots+1)}\gamma'(a_2)^m$, so that

$$\gamma'(a_1^n a_2^m) = \gamma'(a_1)^{n-k((m-1)+(m-2)+\cdots+1)}\gamma'(a_2)^m,$$

namely $\gamma' = \gamma$. □

The following Lemma proves that the maps γ in Subsections 5.2, 10.3 and 12.2, in the case $|\ker(\gamma)| = pq$, satisfy the assumptions of Lemma 2.5.

Lemma A.2. *Let G be a group of order p^2q, $A = \langle a_1, a_2 \rangle$ a Sylow p-subgroup of G, and $\gamma : A \to \mathrm{Aut}(G)$ a map such that*

$$\begin{cases}\gamma(a_1) = \varphi \text{ (possibly modulo } \iota(A))\\ \gamma(a_2) = 1\end{cases},$$

where $a_2^{\varphi} = a_2$, $a_1^{\varphi} = a_1 a_2^{k_\varphi}$ for a certain k_φ.

Then γ extends to a unique RGF on A if and only if γ is a morphism.

Proof. We show that

$$[A, \gamma(A)] = [A, \gamma(\langle a_1 \rangle)] \subseteq \langle a_2 \rangle,$$

and then, by Lemma 2.5, the RGF's on A with kernel $\langle a_2 \rangle$ correspond to the morphisms $A \to \mathrm{Aut}(G)$.

Note that if γ is a RGF or a morphism, then $\gamma(a_1^s) = \gamma(a_1)^s$, as $\ker(\gamma) = \langle a_2 \rangle$. Thus we have

$$(a_2^m a_1^t)^{-1}(a_2^m a_1^t)^{\gamma(a_1^s)} = (a_2^m a_1^t)^{-1+\varphi^s} = (a_1^t)^{-1+\varphi^s} = a_2^{tk_{\varphi^s}} \in \langle a_2 \rangle.$$

□

References

[1] ACRI, EMILIANO; BONATTO, MARCO. Skew braces of size p^2q I: Abelian type. *Algebra Colloq.* **29** (2022), no. 2, 297–320. MR4414157, Zbl 1495.16030, arXiv:2004.04291, doi: 10.1142/S1005386722000244. 95, 102

[2] ACRI, EMILIANO; BONATTO, MARCO. Skew braces of size p^2q II: Non-abelian type. *J. Algebra Appl.* **21** (2022), no. 3, Paper No. 2250062, 61 pp. MR4391827, Zbl 1495.16029, arXiv:2004.04232, doi: 10.1142/S0219498822500621. 95, 96, 102

[3] BACHILLER, DAVID. Counterexample to a conjecture about braces. *J. Algebra* **453** (2016), 160–176. MR3465351, Zbl 1338.16022, arXiv:1507.02137, doi: 10.1016/j.jalgebra.2016.01.011. 94

[4] BYOTT, NIGEL P. Uniqueness of Hopf Galois structure for separable field extensions. *Comm. Algebra* **24** (1996), no. 10, 3217–3228. MR1402555, Zbl 0878.12001, doi: 10.1080/00927879608825743. 94, 95, 183

[5] BYOTT, NIGEL P. Galois structure of ideals in wildly ramified abelian p-extensions of a p-adic field, and some applications. *J. Théor. Nombres Bordeaux.* **9** (1997), no. 1, 201–219. MR1469668, Zbl 0889.11040, doi: 10.5802/jtnb.196. 94

[6] CAMPEDEL, ELENA. Hopf–Galois structures and skew braces of order p^2q. Ph.D. thesis, *Università degli Studi di Milano-Bicocca*, 2022. https://boa.unimib.it/handle/10281/378739. 95

[7] CAMPEDEL, ELENA; CARANTI, ANDREA; DEL CORSO, ILARIA. Hopf–Galois structures on extensions of degree p^2q and skew braces of order p^2q: the cyclic Sylow p-subgroup case. *J. Algebra* **556** (2020), 1165–1210. MR4089566, Zbl 1465.12006, arXiv:1911.11567, doi: 10.1016/j.jalgebra.2020.04.009. 95, 96, 97, 98, 103, 104, 105, 107, 108, 109, 111, 180, 183

[8] CAMPEDEL, ELENA; CARANTI, ANDREA; DEL CORSO, ILARIA. The automorphism groups of groups of order p^2q. *Int. J. Group Theory* **10** (2021), no. 3, 149–157. MR4223627, Zbl 1500.20003, arXiv:1911.11567, doi: 10.22108/ijgt.2020.124183.1641. 97, 98, 109, 120, 135, 137, 148, 158, 175, 180

[9] CARTER, ROGER; FONG, PAUL. The Sylow 2-subgroups of the finite classical groups. *J. Algebra* **1** (1964), 139–151. MR0166271, Zbl 0123.02901, doi: 10.1016/0021-8693(64)90030-4. 113

[10] CHILDS, LINDSAY N. Taming wild extensions: Hopf algebras and local Galois module theory. Mathematical Surveys and Monographs, 80. *American Mathematical Society (AMS), Providence, RI*, 2000. viii+215 pp. ISBN: 0-8218-2131-8. MR1767499, Zbl 0944.11038, doi: 10.1090/surv/260. 94

[11] CRESPO, TERESA. Hopf Galois structures on field extensions of degree twice an odd prime square and their associated skew left braces. *J. Algebra* **565** (2021), 282–308. MR4150347, Zbl 1464.16025, arXiv:2003.01819, doi: 10.1016/j.jalgebra.2020.09.005. 102, 173

[12] GREITHER, CORNELIUS; PAREIGIS, BODO. Hopf Galois theory for separable field extensions. *J. Algebra* **106** (1987), no. 1, 239–258. MR878476, Zbl 0615.12026, doi: 10.1016/0021-8693(87)90029-9. 94

[13] GUARNIERI, LEANDRO; VENDRAMIN, LEANDRO. Skew braces and the Yang–Baxter equation. *Math. Comp.* **86** (2017), no. 307, 2519–2534. MR3647970, Zbl 1371.16037, arXiv:1511.03171, doi: 10.1090/mcom/3161. 95, 96

[14] HÖLDER, OTTO. Die Gruppen der Ordnungen p^3, pq^2, pqr, p^4. *Math. Ann.* **43** (1893), no. 2-3, 301–412. MR1510814, JFM 25.0201.02, doi: 10.1007/BF01443651. 97

[15] KOHL, TIMOTHY. Groups of order $4p$, twisted wreath products and Hopf–Galois theory. *J. Algebra* **314** (2007), 42–74. MR2331752, Zbl 1129.16031, doi: 10.1016/j.jalgebra.2007.04.001. 95

[16] Koch, Alan; Truman, Paul J. Opposite skew left braces and applications. *J. Algebra* **546** (2020), 218–235. MR4033084, Zbl 1435.16009, arXiv:1908.02682, doi: 10.1016/j.jalgebra.2019.10.033. 97

[17] Stefanello, Lorenzo; Trappeniers, Senne. On the connection between Hopf–Galois structures and skew braces. *Bull. Lond. Math. Soc.* **55** (2023), no. 4, 1726–1748. MR4623681, Zbl 7738097, arXiv:2206.07610, doi: 10.1112/blms.12815. 94

[18] Smoktunowicz, Agata; Vendramin, Leandro. On skew braces (with an appendix by N. Byott and L. Vendramin). *J. Comb. Algebra* **2** (2018), no. 1, 47–86. MR3763907, Zbl 1416.16037, arXiv:1705.06958, doi: 10.4171/JCA/2-1-3. 94, 95, 96

(E. Campedel) Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio, I-67100 L'Aquila, Italy
elena.campedel@univaq.it

(A. Caranti) Dipartimento di Matematica, Università degli Studi di Trento, via Sommarive 14, I-38123 Trento, Italy
andrea.caranti@unitn.it
https://caranti.maths.unitn.it/

(I. Del Corso) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo, 5, 56127 Pisa, Italy
ilaria.delcorso@unipi.it
http://people.dm.unipi.it/delcorso/

This paper is available via http://nyjm.albany.edu/j/2024/30-6.html.