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Strichartz estimates associated with the Grushin operator

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ABSTRACT. Let $G = -\Delta - |x|^2\partial_t^2$ denote the Grushin operator on $\mathbb{R}^{n+1}$. It is well known that the Grushin-Schrödinger equation is totally non-dispersive and hence the classical approach to obtain Strichartz estimates fails. In this paper, we prove a restriction theorem with respect to the scaled Hermite-Fourier transform on $\mathbb{R}^{n+2}$ for certain surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$ and as an application, we obtain anisotropic Strichartz estimates for the Grushin-Schrödinger equation and for the Grushin wave equation.

CONTENTS

1. Introduction

Consider the free Schrödinger equation on $\mathbb{R}^n$:

$$\begin{aligned} i\partial_s u(x,s) - \Delta u(x,s) = 0, \quad x \in \mathbb{R}^n,\ s \in \mathbb{R} \setminus \{0\}, \\ u(x,0) = f(x), \end{aligned} \tag{1.1}$$

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where Δ denotes the standard Laplacian on $\mathbb{R}^n$. It is well known that $e^{-is\Delta}f$ is the unique solution to the IVP (1.1) and can be written as

$$u(\cdot, s) = \frac{e^{i\frac{|\cdot|^2}{4s}}}{(4\pi i s)^{\frac{n}{2}}} * f(\cdot). \tag{1.2}$$

An application of Young's inequality in (1.2) gives the following dispersive estimate: For all $s \in \mathbb{R} \setminus \{0\}$,

$$\|u(\cdot, s)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{1}{(4\pi|s|)^{\frac{n}{2}}} \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.3}$$

Such an estimate is crucial in the study of semilinear and quasilinear equations which has wide applications in physical systems (see [5, 19] and the references therein). The dispersive estimate (1.3) yields the following remarkable estimate for the solution of (1.1) by Strichartz [29] (see also [24, 25]) in connection with Fourier restriction theory:

$$\|u\|_{L^q(\mathbb{R}, L^p(\mathbb{R}^n))} \leq C(p, q) \|f\|_{L^2(\mathbb{R}^n)}, \tag{1.4}$$

where (p, q) satisfies the scaling admissibility condition $\frac{2}{q} + \frac{n}{p} = \frac{n}{2}$ with $p, q \geq 2$ and $(n, q, p) \neq (2, 2, \infty)$. We refer to [9, 11, 13] for further study on Strichartz estimates and its connection with dispersive estimates.

In this work, we aim at investigating such phenomenon associated with the Grushin operator G on $\mathbb{R}^{n+1}$ defined by

$$G = -\Delta - |x|^2 \partial_t^2, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},$$

where $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

The studies of the Grushin operator date back to Baouendi and Grushin [8, 21, 20]. Since then, several authors studied the operator extensively in different contexts, involving classification of solutions to an elliptic equations, free boundary problems in partial differential equations, well-posedness problems in Sobolev spaces etc. [1, 14, 17, 23]. Even though numerous studies in the direction of PDEs associated with the Grushin operator are currently available, to the best of our knowledge, the study on Strichartz estimates for the Schrödinger and the wave equations associated with the Grushin operator has not been addressed in the literature so far.

Consider the following Grushin-Schrödinger equation:

$$\begin{aligned} i\partial_s u(x, t, s) - Gu(x,t,s) &= h(x, t, s), \quad s \in \mathbb{R},\ (x, t) \in \mathbb{R}^{n+1}, \\ u(x, t, 0) &= f(x, t). \end{aligned} \tag{1.5}$$

For f in $L^2(\mathbb{R}^{n+1})$, $u(x, t, s) = e^{-isG} f(x, t)$ is the unique global time solution to the above IVP (1.5) (with $h = 0$). Unlike the Euclidean case, the IVP (1.5) is totally non dispersive (see [18]) for $n = 1$. A similar conclusion is observed in the following proposition for $n \geq 1$.

Proposition 1.1. *There exists a function $f \in \mathcal{S}(\mathbb{R}^{n+1})$, the space of all Schwartz class functions on $\mathbb{R}^{n+1}$, such that the solution to the IVP (1.5) (for $h = 0$) with initial data f satisfies*

$$u(x,t,s) = f(x, t+sn), \quad \forall s \in \mathbb{R}, \quad \forall (x,t) \in \mathbb{R}^{n+1}. \tag{1.6}$$

Notice that $\|u(\cdot,s)\|_p = \|f\|_p$ for all $1 \leq p \leq \infty$, hence one cannot expect for a global dispersive estimate of the type (1.2). Due to loss of dispersion, the Euclidean strategy of finding Strichartz estimates fails, and the problem of obtaining Strichartz estimates is considerably more difficult. Similar situations have already been handled in the literature in different contexts. For instance, we refer to [10, 12, 22] for compact Riemannian manifolds, [4] for the Heisenberg group, [2] for the hyperbolic space and [6, 15] for the nilpotent Lie groups. In particular, Bahouri-Gérard-Xu [7] emphasized that the Schrödinger operator on the Heisenberg group $\mathbb{H}^d$ has no dispersion at all. Further, Bahouri-Barilari-Gallagher [4] derived anisotropic Strichartz estimates for the Schrödinger and the wave equations on the Heisenberg group involving the sublaplacian, only for the radial initial data, by adapting the Fourier transform restriction analysis initiated in [29] and [31].

Since the Grushin operator is closely linked to the sublaplacian on the Heisenberg group, we expect analogous results in the context of the Grushin operator. Following the strategy introduced in [4], we obtain anisotropic Strichartz estimates for the Grushin-Schrödinger equation (1.5) and the Grushin wave equation (1.14) for initial data that belongs to a more general class of functions.

For $1 \leq p, q, r \leq \infty$, consider the anisotropic Lebesgue spaces

$$L^r_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$$

with the mixed norm

$$\|f\|_{L^r_t L^q_s L^p_x} = \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^n} |f(x,t,s)|^p dx \right)^{\frac{q}{p}} ds \right)^{\frac{r}{q}} dt \right)^{\frac{1}{r}}.$$

Proposition 1.1 shows that the semigroup e^{-isG} is unbounded from $L^2(\mathbb{R}^{n+1})$ to $L^r_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ unless $r = \infty$. Therefore, we investigate the following question: can we obtain nontrivial time space estimates for the solution u of the IVP (1.5) such that $u \in L^\infty_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ for some non-trivial (p,q)?

We affirm this question by proving a restriction theorem for the scaled Hermite-Fourier transform (defined below) on specific surfaces in $\mathbb{N}^n_0 \times \mathbb{R}^* \times \mathbb{R}$ and adapting general methods to derive Strichartz estimates in [29].

The scaled Hermite-Fourier restriction theorem. For $f \in \mathcal{S}(\mathbb{R}^{n+2})$, the space of all Schwartz class functions on $\mathbb{R}^{n+2}$, let

$$f^{\lambda,\nu}(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,t,s)\, e^{i\lambda t} e^{i\nu s} dt ds, \ \lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}, \ \nu \in \mathbb{R}, \tag{1.7}$$

stand for the inverse Fourier transform of $f(x,t,s)$ in the (t,s) variable. We define the scaled Hermite-Fourier transform of f on $\mathbb{R}^{n+2}$ as

$$\hat{f}(\alpha,\lambda,\nu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t}e^{i\nu s} f(x,t,s)\Phi_\alpha^\lambda(x)dsdtdx = \langle f^{\lambda,\nu}, \Phi_\alpha^\lambda\rangle, \tag{1.8}$$

for any $(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$. Here Φ_α^λ is defined in Section 2 and $\langle\cdot,\cdot\rangle$ denotes the inner product on $\mathbb{R}^n$. Given a surface S in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$ endowed with an induced measure $d\sigma$, we define the restriction operator $\mathcal{R}_S : L^2(\mathbb{R}^{n+2}) \to L^2(S, d\sigma)$ as

$$\mathcal{R}_S f = \hat{f}|_S, \tag{1.9}$$

on the surface S and the operator dual to $\mathcal{R}_S$ (called the extension operator) as

$$\mathcal{E}_S(\Theta)(x,t,s) = \frac{1}{(2\pi)^2}\int_S e^{-i\nu s}e^{-i\lambda t}\Theta(\alpha,\lambda,\nu)\,\Phi_\alpha^\lambda(x)\; d\sigma, \tag{1.10}$$

$\Theta \in L^2(S,d\sigma)$. Consider the surface $S = \{(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} : \nu = (2|\alpha| + n)|\lambda|\}$ with a localized induced measure $d\sigma_{loc}$ (defined in Section 3) and let $S_{\sigma_{loc}}$ be the support of $d\sigma_{loc}$ in S. We obtain the following restriction theorem for scaled Hermite-Fourier transform for $S_{\sigma_{loc}}$.

Theorem 1.2 (Scaled Hermite-Fourier restriction theorem)**.** *Let $n \geq 1$.*

(1) *If $1 \leq q \leq p < 2$, then*

$$\|\mathcal{R}_{S_{\sigma_{loc}}} f\|_{L^2(S,d\sigma_{loc})} \leq C(p,q)\|f\|_{L_t^1 L_s^q L_x^p}, \tag{1.11}$$

for all functions $f \in \mathcal{S}(\mathbb{R}^{n+2})$.

(2) *For $n = 1$, the inequality (1.11) holds for all $f \in \mathcal{S}(\mathbb{R}^3)$, when $p = 2$ and $1 \leq q \leq 2$.*

(3) *For $n \geq 2$, the inequality (1.11) holds for all $f \in \mathcal{S}_{rad}(\mathbb{R}^{n+2})$, the space of all radial[2] Schwartz class functions on $\mathbb{R}^{n+2}$, when $p = 2$ and $1 \leq q \leq 2$.*

We refer to Liu and Song [26] for a similar restriction theorem associated with the Grushin operator (see Subsection 2.3).

Anisotropic Strichartz estimates. By duality, Theorem 1.2 can be reframed as follows: for any $2 < p' \leq q' \leq \infty$,

$$\|\mathcal{E}_{S_{\sigma_{loc}}}(\Theta)\|_{L_t^\infty L_s^{q'} L_x^{p'}} \leq C(p,q)\|\Theta\|_{L^2(S,d\sigma_{loc})} \tag{1.12}$$

holds for all $\Theta \in L^2(S, d\sigma_{loc})$.

Now, realizing the solution of (1.5) (with $h = 0$) as the extension operator $\mathcal{E}_{S_{\sigma_{loc}}}$ acting on a suitable function on S and using (1.12), we prove an anisotropic Strichartz estimate for the solution of the free Grushin-Schrödinger equation. More generally, we obtain the following result.

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[2]A function f on $\mathbb{R}^{n+2}$ (resp. $\mathbb{R}^{n+1}$) is said to be radial if $f(x,t,s) = f(|x|,t,s)$ (resp. $f(x,t) = f(|x|,t)$) for all $x \in \mathbb{R}^n$ and $t,s \in \mathbb{R}$.

Theorem 1.3. *Let $f \in L^2(\mathbb{R}^{n+1})$ and $h \in L^1_s(\mathbb{R}; L^2_{x,t}(\mathbb{R}^{n+1}))$. If (p,q) lies in the admissible set*

$$A = \left\{(p,q) : 2 < p \le q \le \infty \quad \text{and} \quad \frac{2}{q} + \frac{n}{p} = \frac{n+2}{2}\right\},$$

then the solution $u(x,t,s)$ of the IVP (1.5) is in $L^\infty_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ and satisfies the estimate:

$$\|u(x,t,s)\|_{L^\infty_t L^q_s L^p_x} \le C\left(\|f\|_{L^2(\mathbb{R}^{n+1})} + \|h\|_{L^1_s(\mathbb{R}; L^2_{x,t}(\mathbb{R}^{n+1}))}\right). \tag{1.13}$$

Moreover, at the endpoint $(p,q) = (2,2)$, the estimate (1.13) is valid for all functions f and h in one dimension, and for radial functions f and h when $n \ge 2$.

Remark 1.4. The Strichartz estimates (1.13) are not the usual ones in terms of order of Lebesgue norms. Note that the usual Strichartz estimate, i.e., the semigroup e^{-isG} is bounded from $L^2(\mathbb{R}^{n+1})$ to $L^q_s(\mathbb{R}; L^r_t(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ only when $(q,r,p) = (\infty, 2, 2)$, by Proposition 1.1.

Consider the following Grushin wave equation:

$$\begin{aligned} \partial_s^2 u(x,t,s) + Gu(x,t,s) &= h(x,t,s) \quad s \in \mathbb{R},\ (x,t) \in \mathbb{R}^{n+1}, \\ u(x,t,0) = f(x,t), &\quad \partial_s u(x,t,0) = g(x,t). \end{aligned} \tag{1.14}$$

The solution to the above IVP (1.14) (with $h = 0$) can be realized as the extension operator $\mathcal{E}_{S_w}$ acting on a suitable function on the surface S_w (defined in Remark 3.2). Using the scaled Hermite–Fourier restriction theorem for the surface S_w, we prove an anisotropic Strichartz estimate for the solution of the free Grushin wave equation. More generally, we obtain the following result.

Theorem 1.5. *Let*

$$f \in L^2(\mathbb{R}^{n+1}), \quad G^{-1/2}g \in L^2(\mathbb{R}^{n+1}), \quad G^{-1/2}h \in L^1_s(\mathbb{R}; L^2_{x,t}(\mathbb{R}^{n+1})).$$

If (p,q) lies in the admissible set

$$A_w = \left\{(p,q) : 2 < p \le q \le \infty \quad \text{and} \quad \frac{1}{q} + \frac{n}{p} = \frac{n+2}{2}\right\},$$

then the solution $u(x,t,s)$ of the IVP (1.14) is in $L^\infty_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ and satisfies the estimate:

$$\begin{aligned} &\|u(x,t,s)\|_{L^\infty_t L^q_s L^p_x} \\ &\quad \le C\left(\|f\|_{L^2(\mathbb{R}^{n+1})} + \|G^{-1/2}g\|_{L^2(\mathbb{R}^{n+1})} + \|G^{-1/2}h\|_{L^1_s(\mathbb{R}; L^2_{x,t}(\mathbb{R}^{n+1}))}\right). \end{aligned} \tag{1.15}$$

Remark 1.6. In [7], Bahouri-Gérard-Xu derived a (usual) Strichartz estimate for the wave equation associated with the sublaplacian on the Heisenberg group, we can expect an analogue result in the case of the Grushin operator. However, the above theorem may be viewed as an extension of Theorem 1.1 in [7] in the context of Grushin operator.

This paper is organized as follows: in Section 2, the spectral theory of the Grushin operator and some properties of scaled Hermite Fourier transform on $\mathbb{R}^{n+1}$ are discussed. In Section 3, the restriction theorem for the scaled Hermite Fourier transform is obtained. In Section 4, the anisotropic Strichartz estimates for the solutions to IVP (1.5) and IVP (1.14) (with $h = 0$) are derived. Finally, we prove Theorem 1.2 and Theorem 1.3 in Section 5.

2. Preliminaries

In this section, we discuss the spectral theory for the Grushin operator and the Fourier analysis tools associated with the Grushin operator.

2.1. The Grushin operator and its spectral theory: Let H_k denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \quad k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\},$$

and h_k denote the normalized Hermite functions on $\mathbb{R}$ defined by

$$h_k(x) = (2^k\sqrt{\pi}k!)^{-\frac{1}{2}} H_k(x)e^{-\frac{1}{2}x^2}, \quad k \in \mathbb{N}_0.$$

The higher dimensional Hermite functions denoted by Φ_α are then obtained by taking tensor product of one dimensional Hermite functions. Thus, for any multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we define $\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$. For $\lambda \in \mathbb{R}^*$, the scaled Hermite functions are defined by $\Phi_\alpha^\lambda(x) = |\lambda|^{\frac{n}{4}} \Phi_\alpha(\sqrt{|\lambda|}x)$, they are the eigenfunctions of the (scaled) Hermite operator $H(\lambda) = -\Delta + \lambda^2|x|^2$ with eigenvalues $(2|\alpha| + n)|\lambda|$, where $|\alpha| = \sum_{j=1}^n \alpha_j$, $\alpha \in \mathbb{N}_0^n$. For each $\lambda \in \mathbb{R}^*$, the family $\{\Phi_\alpha^\lambda : \alpha \in \mathbb{N}_0^n\}$ is then an orthonormal basis for $L^2(\mathbb{R}^n)$. For each $k \in \mathbb{N}$, let $P_k(\lambda)$ stand for the orthogonal projection of $L^2(\mathbb{R}^n)$ onto the eigenspace of $H(\lambda)$ spanned by $\{\Phi_\alpha^\lambda : |\alpha| = k\}$. More precisely, for $f \in L^2(\mathbb{R}^n)$,

$$P_k(\lambda)f = \sum_{|\alpha|=k} \langle f, \Phi_\alpha^\lambda\rangle \, \Phi_\alpha^\lambda \,, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denote the standard inner product in $L^2(\mathbb{R}^n)$. Then the spectral decomposition of $H(\lambda)$ is explicitly given as

$$H(\lambda)f = \sum_{k=0}^{\infty}(2k + n)|\lambda|P_k(\lambda)f. \tag{2.2}$$

Note that

$$P_k(\lambda)f(x) = P_k(1)(f \circ d_{|\lambda|^{-\frac{1}{2}}}) \circ d_{|\lambda|^{\frac{1}{2}}}(x), \tag{2.3}$$

where the dilations d_r on $\mathbb{R}^n$ is defined by $d_r(x) = rx$ for $r > 0$.

For a Schwartz function f on $\mathbb{R}^{n+1}$, let $f^{\lambda}(x) = \int_{\mathbb{R}} f(x,t)e^{i\lambda t}dt$ denotes the inverse Fourier transform of $f(x,t)$ in the t variable. Applying the operator G to the Fourier expansion $f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t} f^{\lambda}(x)d\lambda$, we see that

$$Gf(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t} H(\lambda) f^{\lambda}(x)d\lambda.$$

Using (2.2), the spectral decomposition of the Grushin operator is given by

$$Gf(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}\left(\sum_{k=0}^{\infty}(2k+n)|\lambda|P_k(\lambda)f^{\lambda}(x)\right)d\lambda. \tag{2.4}$$

This operator belongs to the wide class of subelliptic operators studied by Franchi et al. in [16]. Moreover, it is positive, self-adjoint, and hypoelliptic. The operator G possesses a natural family of anisotropic dilations, namely

$$\delta_r(x,t) = (rx, r^2 t) \quad \text{for } r > 0, \tag{2.5}$$

and this anisotropic dilation structure introduces homogeneous norm on $\mathbb{R}^{n+1}$ $\rho := \rho(x,t) = \left(\sum_{i=1}^{n}|x_i|^4 + t^2\right)^{\frac{1}{4}}$. With the norm ρ, we define the ball centered at $w_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$ and of radius $R \geq 0$ by $B(w_0, R) = \{(x,t) \in \mathbb{R}^{n+1} : \rho(x - x_0, t - t_0) < R\}$. We refer to [27] and the references therein for a detailed information about the Grushin operator.

2.2. The scaled Hermite-Fourier transform on $\mathbb{R}^{n+1}$: For a reasonable function f, the scaled Fourier-Hermite transform is defined by

$$\hat{f}(\alpha,\lambda) = \int_{\mathbb{R}^n}\int_{\mathbb{R}} e^{i\lambda t} f(x,t)\Phi_{\alpha}^{\lambda}(x)dtdx = \langle f^{\lambda}, \Phi_{\alpha}^{\lambda}\rangle, \quad (\alpha,\lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*. \tag{2.6}$$

If $f \in L^2(\mathbb{R}^{n+1})$, then $\hat{f} \in L^2(\mathbb{N}_0^n \times \mathbb{R}^*)$ and satisfies the Plancherel formula

$$\|f\|_{L^2(\mathbb{R}^{n+1})} = \frac{1}{2\pi}\|\hat{f}\|_{L^2(\mathbb{N}_0^n\times\mathbb{R}^*)}. \tag{2.7}$$

The inversion formula is given by

$$f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\lambda t}\sum_{\alpha\in\mathbb{N}_0^n}\hat{f}(\alpha,\lambda)\,\Phi_{\alpha}^{\lambda}(x)\;d\lambda. \tag{2.8}$$

If $f \in L^1(\mathbb{R}^{n+1})$, it can be seen that for $r > 0$,

$$\widehat{(f\circ\delta_r)}(\alpha,\lambda) = r^{-(\frac{n}{2}+2)}\hat{f}(\alpha, r^{-2}\lambda), \tag{2.9}$$

where δ_r is the anisotropic dilation on $\mathbb{R}^{n+1}$ in (2.5).
Replacing f by Gf in (2.8) and comparing with (2.4), we get

$$\widehat{(Gf)}(\alpha,\lambda) = (2|\alpha| + n)|\lambda|\,\hat{f}(\alpha,\lambda), \quad (\alpha,\lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*. \tag{2.10}$$

As in the Euclidean case, (2.10) allows us to solve (1.5) (with $h = 0$) explicitly. For $f \in L^2(\mathbb{R}^{n+1})$, taking the scaled Hermite-Fourier transform with respect to (x, t) variable in (1.5) with $h = 0$, we get

$$\begin{aligned} i\frac{d}{ds}\hat{u}(\alpha,\lambda,s) - (2|\alpha|+n)\,|\lambda|\,\hat{u}(\alpha,\lambda,s) &= 0, \\ \hat{u}(\alpha,\lambda,0) &= \hat{f}(\alpha,\lambda). \end{aligned} \tag{2.11}$$

Solving the ordinary differential equation (2.11), we get

$$\hat{u}(\alpha,\lambda,s) = e^{-is(2|\alpha|+n)|\lambda|}\hat{f}(\alpha,\lambda).$$

Now applying the inversion formula (2.8), the solution of the IVP (1.5) (with $h = 0$) can be written as

$$u(x,t,s) = e^{-isG}f(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}^*} e^{-i\lambda t}\sum_{\alpha\in\mathbb{N}^n} e^{-is(2|\alpha|+n)|\lambda|}\hat{f}(\alpha,\lambda)\Phi_\alpha^\lambda(x)d\lambda. \tag{2.12}$$

We proceed to prove Proposition 1.1.

***Proof of Proposition* 1.1 :** Fix a function $Q \in C_c^\infty((1,\infty))$ and consider

$$f(x,t) = \frac{1}{2\pi}\int_1^\infty e^{-i\lambda t}\Phi_0^\lambda(x)Q(\lambda)d\lambda. \tag{2.13}$$

Thus, $f \in \mathcal{S}(\mathbb{R}^{n+1})$ and comparing (2.13) with the inversion formula (2.8) we have

$$\hat{f}(\alpha,\lambda) = \begin{cases} 0, & \text{if } \alpha \neq 0, \lambda \in \mathbb{R}^* \\ Q(\lambda), & \text{if } \alpha = 0, \lambda \in \mathbb{R}^*. \end{cases}$$

By (2.8), the solution of the IVP (1.5) can be written as

$$u(x,t,s) = e^{-isG}f(x,t) = \frac{1}{2\pi}\int_1^\infty e^{-i\lambda(t+ns)}\Phi_0^\lambda(x)Q(\lambda)d\lambda = f(x,t+ns).$$

□

2.3. A restriction theorem for the scaled Hermite-Fourier transform on $\mathbb{R}^{n+1}$: For $\mu > 0$, consider the surface

$$\mathbb{S}^n(\mu) = \left\{(\alpha,\lambda) \in \mathbb{N}_0^n \times \mathbb{R}^* : (2|\alpha|+n)|\lambda| = \mu\right\},$$

with the measure $d\sigma_\mu$ on $\mathbb{S}^n(\mu)$ defined by

$$\int_{\mathbb{S}^n(\mu)}\Theta(\alpha,\lambda)\,d\sigma_\mu = \sum_{\alpha\in\mathbb{N}_0^n}\frac{1}{2|\alpha|+n}\left(\Theta(\alpha,\frac{\mu}{2|\alpha|+n}) + \Theta(\alpha,\frac{-\mu}{2|\alpha|+n})\right),$$

for suitable functions Θ on $\mathbb{S}^n(\mu)$. The surface $\mathbb{S}^n(\mu)$ can be viewed as an analogue of the sphere of radius μ in $\mathbb{N}_0^n \times \mathbb{R}^*$ with surface measure $d\sigma_\mu$, in the sense that for any $F \in L^1(\mathbb{N}_0^n \times \mathbb{R}^*)$, we have

$$\sum_{\alpha\in\mathbb{N}_0^n}\int_{\mathbb{R}^*}F(\alpha,\lambda)d\lambda = \int_0^\infty\left(\int_{\mathbb{S}^n(\mu)}F(\alpha,\lambda)\,d\sigma_\mu\right)d\mu.$$

In [26], Liu-Song derived a restriction theorem associated to Grushin operator on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, analogous to the seminal work of Müller [28]. Specifically, by setting $d_1 = n, d_2 = 1, q = p$ and $r = p'$, with $\frac{1}{p} + \frac{1}{p'} = 1$, Theorem 2 in [26] can be reframed as follows:

Theorem 2.1. [26] *If* $1 \le p < 2$*, then*

$$\|\hat{f}|_{\mathbb{S}^n(\mu)}\|_{L^2(\mathbb{S}^n(\mu),d\sigma_\mu)} \le C\mu^{n(\frac{1}{p}-\frac{1}{2})}\|f\|_{L^1_t L^p_x},$$

for all functions $f \in \mathcal{S}(\mathbb{R}^{n+1})$ *and* $\mu > 0$.

In order to obtain Strichartz estimates via the Fourier restriction method for evolution PDEs, one applies the result to specific surfaces in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, such as the paraboloid for the Schrödinger equation and the cone for the wave equation (see [29]).

When dealing with evolution equations associated to the Grushin operator G on $\mathbb{R}^{n+1}$, one is naturally lead to consider surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$. Consequently, restriction theorems in $\mathbb{N}_0^n \times \mathbb{R}^*$ alone are not sufficient. Thus, we adapt the scaled Hermite–Fourier transform on $\mathbb{R}^{n+2}$ (defined in (3.1)) and establish a restriction theorem (Theorem 1.2) for surfaces in $\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$.

3. Restriction theorem for the scaled Hermite–Fourier transform

For a Schwartz class function f on $\mathbb{R}^{n+2}$, the scaled Hermite–Fourier transform of f on $\mathbb{R}^{n+2}$ is defined by

$$\hat{f}(\alpha,\lambda,\nu) = \int_{\mathbb{R}^n}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t}e^{i\nu s} f(x,t,s)\Phi_\alpha^\lambda(x)\,ds\,dt\,dx = \langle f^{\lambda,\nu}, \Phi_\alpha^\lambda\rangle, \tag{3.1}$$

for any $(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R}$. If $f \in L^2(\mathbb{R}^{n+2})$ then $\hat{f} \in L^2(\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R})$ and satisfies the Plancherel formula

$$\|f\|_{L^2(\mathbb{R}^{n+2})} = \frac{1}{(2\pi)^2}\|\hat{f}\|_{L^2(\mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R})}. \tag{3.2}$$

The inversion formula is given by

$$f(x,t,s) = \frac{1}{(2\pi)^2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\nu s}e^{-i\lambda t}\sum_{\alpha\in\mathbb{N}_0^n}\hat{f}(\alpha,\lambda,\nu)\,\Phi_\alpha^\lambda(x)\ d\lambda d\nu. \tag{3.3}$$

3.1. A surface measure:

Let us consider the surface

$$S = \{(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} : \nu = (2|\alpha| + n)|\lambda|\}. \tag{3.4}$$

We endow S with the measure $d\sigma$ induced by the projection $\pi : \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} \to \mathbb{N}_0^n \times \mathbb{R}^*$ onto the first two factors, where $\mathbb{N}_0^n \times \mathbb{R}^*$ endowed with the measure

$d\mu \otimes d\lambda$, $d\mu$ and $d\lambda$ denote the counting measure on $\mathbb{N}_0^n$ and Lebesgue measure on $\mathbb{R}^*$ respectively. More explicitly, for any integrable function Θ on S, we have

$$\int_S \Theta \, d\sigma = \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} \Theta(\alpha, \lambda, (2|\alpha| + n)|\lambda|) \, d\lambda.$$

By construction, it is clear that if $\Theta = \hat{f} \circ \pi|_S$, where $\hat{f}$ is a function on $\mathbb{N}_0^n \times \mathbb{R}^*$, then for all $1 \leq p \leq \infty$

$$\|\Theta\|_{L^p(S,d\sigma)} = \|\hat{f}\|_{L^p(\mathbb{N}_0^n \times \mathbb{R}^*)}. \tag{3.5}$$

Our purpose here is to show that every (appropriate) function f (on $\mathbb{R}^{n+2}$) has a scaled Hermite-Fourier transform $\hat{f}$ that can be restricted to the surface S. In view of the Fourier restriction theorem due to Thomas [31], such restriction property is best dealt with compact subsets in the Euclidean space. Therefore, we consider the surface S endowed with the surface measure $d\sigma_{loc} = \psi(\nu) d\sigma$ defined by

$$\int_S \Theta \, d\sigma_{loc} = \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} \Theta(\alpha, \lambda, (2|\alpha| + n)|\lambda|) \, \psi((2|\alpha| + n)|\lambda|) \, d\lambda. \tag{3.6}$$

with ψ any smooth, even, compactly supported function in $\mathbb{R}$ with an L^∞ norm at most 1.

Let $S_{\sigma_{loc}}$ be the support of σ_{loc} in S, i.e., $S_{\sigma_{loc}} = \{(\alpha, \lambda, \nu) \in S : \psi(\nu) \neq 0\}$. The restriction operator, $\mathcal{R}_{S_{\sigma_{loc}}}$ and the extension operator, $\mathcal{E}_{S_{\sigma_{loc}}}$ with respect to the surface $(S, d\sigma_{loc})$ can be computed as $\mathcal{R}_{S_{\sigma_{loc}}} f = \hat{f}|_{S_{\sigma_{loc}}}$ and

$$(2\pi)^2 \mathcal{E}_{S_{\sigma_{loc}}}(\Theta)(x, t, s) = \tag{3.7}$$
$$\sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} e^{-i(2|\alpha|+n)|\lambda| s} e^{-i\lambda t} \Theta(\alpha, \lambda, (2|\alpha| + n)|\lambda|) \Phi_\alpha^\lambda(x) \psi((2|\alpha| + n)|\lambda|) d\lambda.$$

3.2. Proof of the scaled Hermite-Fourier restriction theorem: We prove each case in Theorem 1.2 separately. First, we prove the case $1 \leq q \leq p < 2$. Before proceeding to the proof, we need to observe the following:

Lemma 3.1. *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}^*$, then for all $1 \leq p \leq 2$,*

$$\|P_k(\lambda)\phi\|_{L^{p'}(\mathbb{R}^n)} \leq C |\lambda|^{\frac{n}{2}(1-\frac{2}{p'})} (2k + n)^{\frac{n-1}{2}(1-\frac{2}{p'})} \|\phi\|_{L^p(\mathbb{R}^n)}, \tag{3.8}$$

where p' is the conjugate exponent of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Since $\{P_k(\lambda)\}_{k \geq 0}$ are orthogonal projections on $L^2(\mathbb{R}^n)$,we have

$$\|P_k(\lambda)\phi\|_{L^2(\mathbb{R}^n)} \leq \|\phi\|_{L^2(\mathbb{R}^n)}. \tag{3.9}$$

Using the relation (2.3) and the $L^1 - L^\infty$ estimate in the proof of Proposition 4.4.2 in [30], we have

$$\|P_k(\lambda)\phi\|_{L^\infty(\mathbb{R}^n)} \leq |\lambda|^{\frac{n}{2}} (2k + n)^{\frac{n-1}{2}} \|\phi\|_{L^1(\mathbb{R}^n)}. \tag{3.10}$$

This estimate can also be found in the proof of Proposition 1 in [26]. Thus, the Lemma 3.1 follows by interpolating (3.9) and (3.10). □

***Proof of Theorem* 1.2 for the case** $1 \le q \le p < 2$**:** By duality argument, it is enough to show that the boundedness of the operator $\mathcal{E}_{S_{\sigma_{loc}}}$ from $L^2(S, d\sigma_{loc})$ to $L^\infty_t(\mathbb{R}; L^{q'}_s(\mathbb{R}; L^{p'}_x(\mathbb{R}^n)))$. Equivalently, we show that the operator $\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^*$ is bounded from $L^1_t(\mathbb{R}; L^q_s(\mathbb{R}; L^p_x(\mathbb{R}^n)))$ to $L^\infty_t(\mathbb{R}; L^{q'}_s(\mathbb{R}; L^{p'}_x(\mathbb{R}^n)))$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$.

Let $f \in \mathcal{S}(\mathbb{R}^{n+2})$. From (3.7) and (3.6), we have

$$\begin{aligned}
&(2\pi)^2 \mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f(x,t,s)\\
&= \sum_{\alpha \in \mathbb{N}_0^n} \int_{\mathbb{R}^*} e^{-i(2|\alpha|+n)|\lambda| s} e^{-i\lambda t} \hat{f}(\alpha, \lambda, (2|\alpha|+n)|\lambda|)\, \Phi^\lambda_\alpha(x) \psi((2|\alpha|+n)|\lambda|)\, d\lambda\\
&= \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{2|\alpha|+n} \int_{\mathbb{R}^*} e^{-i|\lambda| s} e^{-\frac{i\lambda t}{2|\alpha|+n}} \hat{f}(\alpha, \frac{\lambda}{2|\alpha|+n}, |\lambda|)\, \Phi^{\frac{\lambda}{2|\alpha|+n}}_\alpha(x) \psi(|\lambda|)\, d\lambda,
\end{aligned}$$

where the last term obtained by performing the change of variables $(2|\alpha| + n)\lambda \mapsto \lambda$ in each integral. Using (3.1), (2.1) and writing $a_k = \frac{1}{2k+n}$, we obtain

$$\begin{aligned}
&(2\pi)^2 \mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f(x,t,s)\\
&\quad = \sum_{k=0}^\infty \frac{1}{2k+n} \sum_{\pm} \int_0^\infty e^{-i\lambda s} e^{\mp i a_k \lambda t} P_k(a_k\lambda) f^{\pm a_k\lambda, \lambda}(x) \psi(\lambda)\, d\lambda\\
&\quad = C \sum_{k=0}^\infty \sum_{\pm} \frac{1}{2k+n} \mathcal{F}_{\lambda\to s}\left(e^{\mp i a_k \lambda t} P_k(a_k\lambda) f^{\pm a_k\lambda, \lambda}(x) \psi_+(\lambda)\right),
\end{aligned} \tag{3.11}$$

where $\psi_+(\lambda) = \psi(\lambda)\mathbf{1}_{\lambda > \mathbf{0}}$. For fixed $t \in \mathbb{R}$, Hausdorff–Young inequality on the right-hand side of (3.11) with respect to $s-$variable gives

$$\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f\|_{L^{q'}_s} \le C \sum_{k=0}^\infty \sum_{\pm} \frac{1}{2k+n} \|\psi_+(\lambda) e^{\mp i a_k \lambda t} P_k(a_k\lambda) f^{\pm a_k \lambda, \lambda}(x)\|_{L^q_\lambda}. \tag{3.12}$$

For any function g on $\mathbb{R}^{n+1}$ and for $q' \ge p' > 2$, applying Minkowski's inequality followed by Housdorff-Young inequality and again applying Minkowski's inequality, we get

$$\|\mathcal{F}_{\lambda\to s} g\|_{L^{q'}_s L^{p'}_x} \le \|\mathcal{F}_{\lambda\to s} g\|_{L^{p'}_x L^{q'}_s} \le C\|g\|_{L^{p'}_x L^q_\lambda} \le C\|g\|_{L^q_\lambda L^{p'}_x}. \tag{3.13}$$

In view of (3.13) and (3.12), we deduce that

$$\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^* f\|_{L^\infty_t L^{q'}_s L^{p'}_x} \le C \sum_{k=0}^\infty \sum_{\pm} \frac{1}{2k+n} \|\psi(\lambda) P_k(a_k\lambda) f^{\pm a_k\lambda, \lambda}(x)\|_{L^q_\lambda L^{p'}_x}.$$

But, by Lemma 3.1, we have

$$\|P_k(a_k\lambda)f^{\pm a_k\lambda,\lambda}\|_{L_x^{p'}} \leq C|a_k\lambda|^{\frac{n}{2}(1-\frac{2}{p'})}(2k+n)^{\frac{n-1}{2}(1-\frac{2}{p'})}\|\mathcal{F}_{s\to-\lambda}f(\cdot,\cdot,s)\|_{L_x^pL_t^1},$$

which implies that

$$\begin{aligned}
&\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^*f\|_{L_t^\infty L_s^{q'}L_x^{p'}} \\
&\leq C\sum_{k=0}^{\infty}\frac{1}{(2k+n)^{1+\frac{1}{2}(1-\frac{2}{p'})}}\left\|\|\mathcal{F}_{s\to-\lambda}f(\cdot,\cdot,s)\|_{L_x^pL_t^1}\psi(\lambda)\lambda^{\frac{n}{2}(1-\frac{2}{p'})}\right\|_{L_\lambda^q} \\
&\leq C\left\|\|\mathcal{F}_{s\to-\lambda}f(\cdot,\cdot,s)\|_{L_x^pL_t^1}\psi(\lambda)\lambda^{\frac{n}{2}(1-\frac{2}{p'})}\right\|_{L_\lambda^q} \\
&\leq C\|\mathcal{F}_{s\to-\lambda}f(\cdot,\cdot,s)\|_{L_\lambda^aL_x^pL_t^1}\|\psi(\lambda)\lambda^{\frac{n}{2}(1-\frac{2}{p'})}\|_{L_\lambda^b(\mathbb{R})}, \qquad (3.14)
\end{aligned}$$

where the last step is justified by an application of Hölder's inequality in (3.14) with $a \geq 2$, $\frac{1}{a}+\frac{1}{a'}=1$ and $\frac{1}{a}+\frac{1}{b}=\frac{1}{q}$. Then, taking $a=q'$ and applying Minkowski's inequality followed by Hausdorff-Young inequality in $\lambda-$ variable, we get

$$\|\mathcal{E}_{S_{\sigma_{loc}}}(\mathcal{E}_{S_{\sigma_{loc}}})^*f\|_{L_t^\infty L_s^{q'}L_x^{p'}} \leq C\|\psi(\lambda)\lambda^{\frac{n}{2}(1-\frac{2}{p'})}\|_{L_\lambda^b(\mathbb{R})}\|f\|_{L_x^pL_t^1L_s^q}. \qquad (3.15)$$

Thus, (1.11) follows from (3.15) by Minkowski's integral inequality for all $1 \leq q \leq p < 2$. □

***Proof of Theorem* 1.2 for the case $n=1, p=2, 1\leq q\leq 2$:** Note that for $n=1$,

$$\|\mathcal{R}_{S_{\sigma_{loc}}}f\|^2_{L^2(S,d\sigma_{loc})} = \frac{1}{(2\pi)^2}\sum_{\pm}\sum_{k=0}^{\infty}\int_0^\infty \frac{1}{2k+1}\left\|P_k(\pm a_k\lambda)f^{\pm a_k\lambda,\lambda}\right\|^2_{L^2(\mathbb{R})}\psi(\lambda)d\lambda. \qquad (3.16)$$

Consider the Hilbert space $L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$, with respect to the inner product $\langle\tilde{\alpha},\tilde{\beta}\rangle' = \sum_{k=0}^{\infty}\int_{\mathbb{R}^+}\langle\tilde{\alpha}(k,\lambda),\tilde{\beta}(k,\lambda)\rangle\psi(\lambda)d\lambda$, for all $\tilde{\alpha}$, $\tilde{\beta} \in L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$, where $\mathbb{R}_+$ denote the set of all positive reals. In view of (3.16) it is enough to prove that the operator T defined on $\mathcal{S}(\mathbb{R}^3)$ by

$$Tf = \frac{1}{(2k+1)^{\frac{1}{2}}}P_k(a_k\lambda)f^{a_k\lambda,\lambda},$$

is bounded from $L_t^1(\mathbb{R};L_s^q(\mathbb{R};L_x^2(\mathbb{R}^n)))$ into $L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$ or equivalently that its adjoint T^* is bounded from $L^2(\mathbb{N}_0\times\mathbb{R}^+;L^2(\mathbb{R}))$ into $L_t^\infty(\mathbb{R};L_s^{q'}(\mathbb{R};L_x^{p'}(\mathbb{R}^n)))$ to obtain (1.11).

For $\tilde{\alpha} \in L^2(\mathbb{N}_0 \times \mathbb{R}^+; L^2(\mathbb{R}))$, the operator T^* can be computed to be

$$T^*(\tilde{\alpha})(x,t,s) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^+} \frac{1}{(2k+1)^{\frac{1}{2}}} e^{-ia_k\lambda t} e^{-i|\lambda|s} P_k(a_k\lambda)(\tilde{\alpha}(k,\lambda))(x)\psi(\lambda)d\lambda.$$

Using Minkowski's inequality together with the Hausdorff-Young inequality (see (3.13)), for any fixed $t \in \mathbb{R}$, we have

$$\|T^*(\tilde{\alpha})(\cdot,t,\cdot)\|_{L_s^{q'}L_x^2} \leq C\|g\|_{L_\lambda^q L_x^2},$$

where

$$g(x,\lambda) = \psi(\lambda)\sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\frac{1}{2}}} P_k(a_k\lambda)(\tilde{\alpha}(k,\lambda))(x).$$

Now

$$\|g(\cdot,\lambda)\|_{L^2(\mathbb{R})}^2 = \psi(\lambda)^2 \sum_{k,l\geq 0} \frac{1}{(2k+1)^{\frac{1}{2}}(2l+1)^{\frac{1}{2}}} \langle P_k(a_k\lambda)\tilde{\alpha}(k,\lambda), P_l(a_k\lambda)\tilde{\alpha}(l,\lambda)\rangle$$

$$\leq C\psi(\lambda)^2 \sum_{k\leq l} \frac{\|\tilde{\alpha}(k,\lambda)\|_{L^2(\mathbb{R})}\|\tilde{\alpha}(l,\lambda)\|_{L^2(\mathbb{R})}}{(2k+1)^{\frac{3}{4}}(2l+1)^{\frac{3}{4}}} \int_{\mathbb{R}} \left|h_k\left(\frac{x}{\sqrt{2k+1}}\right)\right|\left|h_l\left(\frac{x}{\sqrt{2l+1}}\right)\right| dx, \tag{3.17}$$

where the last line obtained by Cauchy–Schwarz inequality and a change of variable $x \mapsto \lambda x$. Using Proposition 6.2 (see appendix), (3.17) turns out to be

$$\|g(\cdot,\lambda)\|_{L^2(\mathbb{R})}^2 \leq C\psi(\lambda)^2 \sum_l \|\tilde{\alpha}(l,\lambda)\|_{L^2(\mathbb{R})} \left(\frac{1}{l}\sum_{k=0}^{l} \|\tilde{\alpha}(k,\lambda)\|_{L^2(\mathbb{R})}\right).$$

By Hardy's inequality (see [3]), we get

$$\|g(\cdot,\lambda)\|_{L^2(\mathbb{R})} \leq C\psi(\lambda)\left(\sum_{k=0}^{\infty} \|\tilde{\alpha}(k,\lambda)\|_{L^2(\mathbb{R})}^2\right)^{\frac{1}{2}}.$$

Further, applying Hölder's inequality, we have

$$\|g\|_{L_\lambda^q L_x^2} \leq C\|\psi(\lambda)\|_{L_\lambda^{\frac{2q}{2-q}}(\mathbb{R}^+)} \|\tilde{\alpha}\|_{L^2(\mathbb{N}_0\times\mathbb{R}^+)}.$$

□

Proposition 6.2 plays a decisive role in the proof presented above. However, we could not find such estimate for the higher dimensional Hermite functions ($n \geq 2$). Nonetheless, we prove the restriction inequality (1.11) for $n \geq 2$ and $p = 2$ for the radial functions. Recall that a function f on $\mathbb{R}^{n+2}$ is said to be radial if $f(x,t,s) = f(|x|,t,s)$ for all $x \in \mathbb{R}^n$ and $t, s \in \mathbb{R}$. If f is radial on $\mathbb{R}^{n+2}$, then $f^{\lambda,\nu}$ is radial on $\mathbb{R}^n$ for any $\lambda \in \mathbb{R}^*$ and $\nu \in \mathbb{R}$. Thus, by Corollary 3.4.1 in [30] and the relation (2.3), for all $k \in \mathbb{N}_0$, we get

$$P_{2k+1}(\lambda)(f^{\lambda,\nu}) = 0 \quad \text{and} \quad P_{2k}(\lambda)(f^{\lambda,\nu})(x) = R_{2k}(f^{\lambda,\nu})L_k^{\frac{n}{2}-1}(|\lambda||x|^2)e^{-\frac{|\lambda|}{2}|x|^2},$$

where

$$R_{2k}(f^{\lambda,\nu}) = \frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})}|\lambda|^{\frac{n}{2}} \int_{\mathbb{R}^n} f^{\lambda,\nu}(x) L_k^{\frac{n}{2}-1}(|\lambda||x|^2) e^{-\frac{|\lambda|}{2}|x|^2} dx$$

and L_k^δ denote the Laguerre polynomials of type $\delta(> -1)$ defined by $L_k^\delta(r) = \frac{1}{k!} e^r r^{-\delta} \frac{d^k}{dx^k}(e^{-r} r^{k+\delta})$ for $r > 0$.

***Proof of Theorem* 1.2 for the case $n \geq 2, p = 2, 1 \leq q \leq 2$:** Let $f \in \mathcal{S}_{rad}(R^{n+2})$. To prove (1.11) for $n \geq 2$ and $p = 2$ (proceeding as in (3.16) for $n = 1$ case), it suffices to show

$$\sum_{k=0}^{\infty} \int_0^\infty \left(\left| R(k, \frac{\lambda}{4k+n}, \lambda) \right|^2 + \left| R(k, \frac{-\lambda}{4k+n}, \lambda) \right|^2 \right) \lambda^{\frac{n}{2}} \phi(\lambda) d\lambda \leq C \|f\|^2_{L^1_t L^q_s L^p_x}, \tag{3.18}$$

where

$$R(k, \lambda, \nu) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})(4k+n)^{\frac{n}{2}+1}} \right)^{\frac{1}{2}} \int_{\mathbb{R}^n} f^{\lambda,\nu}(x) L_k^{\frac{n}{2}-1}(|\lambda||x|^2) e^{-\frac{|\lambda|}{2}|x|^2} dx. \tag{3.19}$$

Consider the operator $T : \mathcal{S}_{rad}(R^{n+2}) \to L^2(\mathbb{N}_0 \times \mathbb{R}^+)$ defined by

$$(Tf)(k, \lambda) = R(k, a_{2k}\lambda, \lambda),$$

where f is related to R through (3.19) and the space $L^2(\mathbb{N}_0 \times \mathbb{R}^+)$ endowed with the measure $\ell^2(\mathbb{N}_0) \otimes L^2(\mathbb{R}^+, \lambda^{\frac{n}{2}} \phi(\lambda) d\lambda)$. To prove (3.18), it is enough to show the adjoint T^* is bounded from $L^2(\mathbb{N}_0 \times \mathbb{R}^+)$ into $L^\infty_t(\mathbb{R}; L^{q'}_s(\mathbb{R}; L^{p'}_x(\mathbb{R}^n)))$. For $\alpha \in L^2(\mathbb{N}_0 \times \mathbb{R}^+)$, the operator T^* is given by

$$T^*(\alpha)(x, t, s) = \sum_{k=0}^{\infty} \int_{\mathbb{R}^+} \alpha(k, \lambda) e^{-i a_k \lambda t} e^{-i|\lambda| s} \mathcal{L}_k(a_{2k}\lambda)(x) \lambda^{\frac{n}{2}} \psi(\lambda) d\lambda,$$

with

$$\mathcal{L}_k(\lambda)(x) = \left(\frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})(4k+n)^{\frac{n}{2}+1}} \right)^{\frac{1}{2}} L_k^{\frac{n}{2}-1}(|\lambda||x|^2) e^{-\frac{|\lambda|}{2}|x|^2}.$$

Again using Minkowski's inequality together with the Hausdorff-Young inequality (see (3.13)), for any fixed $t \in \mathbb{R}$, we have

$$\|T^*(\alpha)(\cdot, t, \cdot)\|_{L^{q'}_s L^2_x} \leq C \|g\|_{L^q_\lambda L^2_x},$$

where $g(x,\lambda) = \lambda^{\frac{n}{2}}\psi(\lambda)\sum_{k=0}^{\infty}\alpha(k,\lambda)\,\mathcal{L}_k\,(a_{2k}\lambda)\,(x)$. By an obvious change of variable, we get

$$\|g(\cdot,\lambda)\|^2_{L^2(\mathbb{R})} \le \lambda^n\psi(\lambda)^2\sum_{k,l\ge 0}|\alpha(k,\lambda)||\alpha(l,\lambda)|\int_{\mathbb{R}^n}|\mathcal{L}_k(a_k)(x)|\,|\mathcal{L}_l(a_l)(x)|\,dx. \tag{3.20}$$

Now, by Lemma 4.2 in [28], there exists $C > 0$ such that for all $k, l \in \mathbb{N}_0$,

$$\int_{\mathbb{R}^n}\left|\mathcal{L}_k\left(\frac{1}{4k+n}\right)(x)\right|\left|\mathcal{L}_l\left(\frac{1}{4l+n}\right)(x)\right|dx \le \frac{C}{\max(k,l)}. \tag{3.21}$$

Note that the above result is stated in Lemma 4.2 of [28] for even n, but a same idea works for odd n as well. Once we have (3.21), applying Hardy's inequality in (3.17) and after using Hölders inequality (arguing as in the proof of $n = 1$ case), we obtain (3.18). □

Remark 3.2. We consider the surfaces

$$S_{\pm} = \{(\alpha,\lambda,\nu) \in \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} \,:\, \nu^2 = (2|\alpha| + n)|\lambda|, \pm\nu > 0\}, \tag{3.22}$$

to obtain Strichartz estimate for the wave equation (1.14). The induced measure $d\sigma_{\pm}$ by the projection $\pi \,:\, \mathbb{N}_0^n \times \mathbb{R}^* \times \mathbb{R} \to \mathbb{N}_0^n \times \mathbb{R}^*$ onto the first two factors, for the surfaces $S_{\pm}$ are given by

$$\int_{S_{\pm}}\Theta\,d\sigma_{\pm} = \sum_{\alpha\in\mathbb{N}_0^n}\int_{\mathbb{R}^*}\Theta(\alpha,\lambda,\pm\sqrt{(2|\alpha|+n)|\lambda|}\,)\,d\lambda,$$

for any integrable function Θ on $S_{\pm}$.

Arguing as in the proof of Theorem (1.2), the restriction inequality (1.11) can be archived for the surface $S_w = S_+ \cup S_-$ endowed with the corresponding localized measure.

4. Anisotropic Strichartz estimates for the homogenous case

We consider the following class of functions : A function $f \in \mathcal{S}(\mathbb{R}^{n+1})$ is said to be frequency localized in a ball $\mathcal{B}_R$, center at 0 of radius R if there exists a smooth, even function ψ supported in $(-1,1)$ and equal to 1 near 0 such that

$$f = \psi(\,-R^{-2}\,G\,)\,g, \tag{4.1}$$

for some $g \in \mathcal{S}(\mathbb{R}^{n+1})$, which equivalent to saying that for all $(\alpha,\lambda) \in \mathbb{N}_0^n \times \mathbb{R}^*$,

$$\hat{f}(\alpha,\lambda) = \psi(R^{-2}(|\alpha| + n)|\lambda|)\hat{g}(\alpha,\lambda). \tag{4.2}$$

Note that (4.1) is defined using functional calculus for G. By construction it is clear that any function $f \in \mathcal{S}(\mathbb{R}^{n+1})$ can be approximated by frequency localized functions in L^2 sense. Now we are in position to prove Theorem 1.3 for $h = 0$.

***Proof of Theorem* 1.3 for $h = 0$:** First, suppose $f \in \mathcal{S}(\mathbb{R}^{n+1})$ is frequency localized in the unit ball $\mathcal{B}_1$, i.e., there exists a smooth, even function ψ supported in $(-1,1)$ such that $\hat{f}(\alpha,\lambda)=\psi((|\alpha|+n)|\lambda|)\hat{g}(\alpha,\lambda)$ for some $g\in\mathcal{S}(\mathbb{R}^{n+1})$. Let $\Theta=\hat{g}\circ\pi|_S$ and the localized measure on S be $d\sigma_{loc}=\psi d\sigma$ defined in (3.6). In view of (2.12) and (3.7) we can write

$$e^{-isG}f(x,t)=\mathcal{E}_{S_{\sigma_{loc}}}(\Theta)(x,t,s).$$

By the restriction inequality (1.12), we have for $2<p\le q\le\infty$

$$\|e^{-isG}f\|_{L_t^\infty L_s^q L_x^p}\le C\|\Theta\|_{L^2(S,d\sigma_{loc})}=C\|\hat{f}\circ\pi|_S\|_{L^2(S,d\sigma)}=C\|f\|_{L^2(\mathbb{R}^{n+1})},\quad(4.3)$$

where the last equality is obtained by (3.5) and the Plancherel formula (3.2).

Next, assume that f is frequency localized in the ball $\mathcal{B}_R$. By (2.9) one can check that the function $f_R:=f\circ\delta_{R^{-1}}$ is frequency localized in $\mathcal{B}_1$ and hence applying (4.3) we get

$$\|e^{-isG}f_R(x,t)\|_{L_t^\infty L_s^q L_x^p}\le C\|f_R\|_{L^2(\mathbb{R}^{n+1})}=CR^{\frac{n}{2}+1}\|f\|_{L^2(\mathbb{R}^{n+1})}.\quad(4.4)$$

Again using (4.3), we have $e^{-isG}f_R(x,t)=e^{-iR^{-2}sG}f(R^{-1}x,R^{-2}t)$, thus from (4.4) we obtain

$$\|e^{-isG}f\|_{L_t^\infty L_s^q L_x^p}=R^{-\frac{2}{q}-\frac{n}{p}}\|e^{-iR^{-2}sG}f(R^{-1}x,R^{-2}t)\|_{L_t^\infty L_s^q L_x^p}\le CR^{\frac{n+2}{2}-\frac{2}{q}-\frac{n}{p}}\|f\|_{L^2(\mathbb{R}^{n+1})}.\quad(4.5)$$

So, if f is frequency localized in the ball $\mathcal{B}_R$, then

$$\|e^{-isG}f\|_{L_t^\infty L_s^q L_x^p}\le C\|f\|_{L^2(\mathbb{R}^{n+1})},$$

provided $\frac{2}{q}+\frac{n}{p}=\frac{n+2}{2}$ and hence the estimate (1.13) (with $h=0$) follows by density of frequency localized functions in $L^2(\mathbb{R}^{n+1})$.

Using Theorem 1.2 for $(p,q)=(2,2)$ and following the preceding argument, we can derive Theorem 1.3 (with $h=0$) at the point $(2,2)$. □

***Proof of Theorem* 1.5 for $h=0$:** Let $f,\ g\in\mathcal{S}(\mathbb{R}^{n+1})$ with $G^{-1/2}g\in L^2(\mathbb{R}^{n+1})$. Using (2.10) and the inversion formula (2.8), the solution of (1.14) (with $h=0$) is given by

$$u(x,t,s)=\sum_{\pm}\frac{1}{2\pi}\int_{\mathbb{R}^*}e^{-i\lambda t}\sum_{\alpha\in\mathbb{N}^n}e^{\mp is\sqrt{(2|\alpha|+n)|\lambda|}}\widehat{\varphi_\pm}(\alpha,\lambda)\Phi_\alpha^\lambda(x)d\lambda,\quad(4.6)$$

where $\widehat{\varphi_\pm}=\frac{1}{2}\left(\hat{f}\mp i\widehat{G^{-1/2}g}\right)$.

Let the surface $S_w=S_+\cup S_-$ endowed with the measure $d\sigma_\pm$, where $S_\pm, d\sigma_\pm$ are defined in Remark 3.2 and $\Theta=\widehat{\varphi_\pm}\circ\pi|_{S_\pm}$ on each sheet. With this, (4.6) can be written as $u(x,t,s)=\mathcal{E}_{S_w}(\Theta)(x,t,s)$. Assume that $\varphi\pm$ are frequency localized in $\mathcal{B}_1$. Proceeding as in the proof of Theorem 1.3 for the surface $(S_w,d\sigma_\pm)$ and using (3.5), we obtain

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p}\le C\|\Theta\|_{L^2(S,d\sigma_\pm)}=\|\widehat{\varphi_\pm}\|_{L^2(\mathbb{N}_0^n\times\mathbb{R}^*)}=\|\varphi_\pm\|_{L^2(\mathbb{R}^{n+1})},\quad(4.7)$$

for $2<p\le q\le\infty$.

If $\varphi_\pm$ are frequency localized in $\mathcal{B}_R$, then the functions $\varphi_{\pm,R} = \varphi_\pm \circ \delta_{R^{-1}}$ are frequency localized in $\mathcal{B}_1$ and give rise to the solution

$$u_R(x,t,s) = u(R^{-1}x, R^{-2}t, R^{-1}s).$$

Thus, using (4.7) we obtain

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \le CR^{\frac{n+2}{2}-\frac{1}{q}-\frac{n}{p}} \|\varphi_\pm\|_{L^2(\mathbb{R}^{n+1})}.$$

By Plancherel formula, we have

$$\|\varphi_\pm\|^2_{L^2(\mathbb{R}^{n+1})} = \|\varphi_+\|^2_{L^2(\mathbb{R}^{n+1})} + \|\varphi_-\|^2_{L^2(\mathbb{R}^{n+1})} = \|f\|^2_{L^2(\mathbb{R}^{n+1})} + \|G^{-1/2}g\|^2_{L^2(\mathbb{R}^{n+1})}.$$

Hence, we conclude that if f, g are frequency localized in $\mathcal{B}_R$, then

$$\|u(x,t,s)\|_{L_t^\infty L_s^q L_x^p} \le C\left(\|f\|_{L^2(\mathbb{R}^{n+1})} + \|G^{-1/2}g\|_{L^2(\mathbb{R}^{n+1})}\right)$$

provided $\frac{1}{q} + \frac{n}{p} = \frac{n+2}{2}$. Thus, Theorem 1.5 for $h = 0$ follows by density argument. □

5. The inhomogeneous case

The solution of inhomogeneous Grushin–Schrödinger equation (1.5) is given by Duhamel's formula:

$$u(x,t,s) = e^{-isG}f(x,t) - i\int_0^s e^{-i(s-s')G}g(x,t,s')ds'. \tag{5.1}$$

***Proof of Theorem* 1.3:** Let $v(x,t,s) = i\int_0^s e^{-i(s-s')G}g(x,t,s')ds'$. Clearly, we have

$$\|v(\cdot,\cdot,\cdot)\|_{L_t^\infty L_s^q L_x^p} \le \int_{\mathbb{R}} \|e^{-i(\cdot)G}e^{is'G}g(\cdot,\cdot,s')\|_{L_t^\infty L_s^q L_x^p} ds'. \tag{5.2}$$

First, assume that, for all s', $g(\cdot,\cdot,s')$ is frequency localized in unit ball $\mathcal{B}_1$ in $\mathbb{R}^{n+1}$. For each s', using (4.3) and the unitarity of $e^{is'G}$, (5.2) yields

$$\|v\|_{L_t^\infty L_s^q L_x^p} \le C\int_{\mathbb{R}} \|e^{is'G}g(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds' = C\int_{\mathbb{R}} \|g(\cdot,\cdot,s')\|_{L^2(\mathbb{R}^{n+1})}ds'. \tag{5.3}$$

Now assume, for all s, $g(\cdot,\cdot,s)$ is frequency localized in $\mathcal{B}_R$. Letting

$$g_R = R^{-2}g(\cdot,\cdot,R^{-2}s)\circ\delta_{R^{-1}} \quad \text{and} \quad v_R(x,s,t) = i\int_0^s e^{-i(s-s')G}g_R(x,t,s')ds',$$

we find that $g_R(\cdot,\cdot,s)$ is frequency localized in ball $\mathcal{B}_1$ for all s and $v_R(x,t,s) = v(R^{-1}x, R^{-2}t, R^{-2}s)$. Applying (5.3) to g_R and using

$$\|v_R\|_{L_t^\infty L_s^q L_x^p} = R^{\frac{2}{q}+\frac{n}{p}}\|v\|_{L_t^\infty L_s^q L_x^p}$$

with

$$\|g_R\|_{L^1(\mathbb{R};L^2(\mathbb{R}^{n+1}))} = R^{\frac{n}{2}+1}\|g\|_{L^1(\mathbb{R};L^2(\mathbb{R}^{n+1}))},$$

we obtain

$$\|v\|_{L_t^\infty L_s^q L_x^p} \leq CR^{\frac{n+2}{2}-\frac{2}{q}-\frac{n}{p}} \|g\|_{L_s^1(\mathbb{R};L_{x,t}^2(\mathbb{R}^{n+1}))}. \tag{5.4}$$

Taking $\frac{2}{q} + \frac{n}{p} = \frac{n+2}{2}$ and using density of frequency localized functions in $L_s^1(\mathbb{R}; L_{x,t}^2(\mathbb{R}^{n+1}))$, (5.4) turns out to be

$$\|v\|_{L_t^\infty L_s^q L_x^p} \leq C\|g\|_{L_s^1(\mathbb{R};L_{x,t}^2(\mathbb{R}^{n+1}))}, \tag{5.5}$$

and holds for all $g \in L^1(\mathbb{R}; L^2(\mathbb{R}^{n+1}))$. Combining the estimate for the first term in (5.1) from Theorem 1.3 together with (5.5), we get (1.13).

We can derive Theorem 1.3 at the point $(2, 2)$ using Theorem 1.2 for $(p, q) = (2, 2)$ and arguing as before. □

For the inhomogeneous Grushin wave equation (1.14), one can apply Duhamel's principle and follow similar arguments as those used for the inhomogeneous Grushin-Schrödinger equation (1.5) discussed above to establish Theorem 1.5. The details of the proof are left to the reader.

6. Appendix

Let us recall a simplified pointwise estimate for the Hermite functions $\{h_k\}_{k\in\mathbb{N}_0}$ (see [17], Corollary 2.8). For $k \in \mathbb{N}_0$, we denote $\lambda_k = \sqrt{2k+1}$.

Lemma 6.1 (Rough pointwise estimates for Hermite functions)**.** *There exits $C > 0$ such that for any $k \in \mathbb{N}_0$ and $x \in \mathbb{R}$,*

$$|h_k(x)| \leq C \begin{cases} \lambda_k^{-\frac{1}{2}} & \text{if} \quad |x| \leq \frac{\lambda_k}{2} \\ \left(\lambda_k^{\frac{2}{3}} + |x^2 - \lambda_k^2|\right)^{-\frac{1}{4}} & \text{if} \quad \frac{\lambda_k}{2} \leq |x| \leq 2\lambda_k \\ e^{-\frac{x^2}{8}} & \text{if} \quad |x| \geq 2\lambda_k. \end{cases}$$

Using the previous lemma, we derive the following proposition, which plays a crucial role in proving the endpoint case for $n = 1$ in Theorem 1.2.

Proposition 6.2. *There exists $C > 0$ such that for any $k, l \in \mathbb{N}$,*

$$\frac{1}{(2k+1)^{\frac{3}{4}}(2l+1)^{\frac{3}{4}}} \int_{\mathbb{R}} \left| h_k\left(\frac{x}{\sqrt{2k+1}}\right)\right| \left| h_l\left(\frac{x}{\sqrt{2l+1}}\right)\right| dx \leq \frac{C}{\max\{k,l\}}. \tag{6.1}$$

Proof. Let $k \leq l$. We split the region of the integration in (6.1) into three parts and estimate each part separately.

(1) In the region $\{x \in \mathbb{R} : |x| \leq 2\lambda_k^2\}$, applying Hölder inequality and using the estimate $\left\|h_k\left(\lambda_k^{-1}\cdot\right) h_l\left(\lambda_l^{-1}\cdot\right)\right\|_{L^2(\mathbb{R})} \leq \frac{\lambda_k^{\frac{1}{2}}}{(2l+1)^{\frac{1}{4}}}$, in Corollary 5.2 of [17], we

obtain

$$\int_{|x|\le 2\lambda_k^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le 2\lambda_k \left\|h_k\left(\lambda_k^{-1}\cdot\right) h_l\left(\lambda_l^{-1}\cdot\right)\right\|_{L^2(\mathbb{R})} \le \frac{(2k+1)^{\frac{3}{4}}}{(2l+1)^{\frac{1}{4}}}.$$

(2) In the region $\{x \in \mathbb{R} : 2\lambda_k^2 \le |x| \le 2\lambda_l^2\}$, we use the pointwise estimates in Lemma 6.1.

Case I: Assume $\frac{1}{2}\lambda_l^2 \le 2\lambda_k^2$. Then

$$\begin{aligned}
\int_{2\lambda_k^2\le|x|\le 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx &\le C\int_{2\lambda_k^2}^{2\lambda_l^2} e^{-\frac{x^2}{8\lambda_k^2}} \frac{1}{\left(\lambda_l^{\frac{2}{3}} + |\lambda_l^{-2}x^2 - \lambda_l^2|\right)^{\frac{1}{4}}} dx \\
&= C\,\lambda_l^{\frac{3}{2}} \int_{2\frac{\lambda_k^2}{\lambda_l^2}}^{2} e^{-\frac{\lambda_l^4}{8\lambda_k^2}x^2} \frac{1}{\left(\lambda_l^{-\frac{4}{3}} + |x^2-1|\right)^{\frac{1}{4}}} dx \\
&\le C\,\lambda_l^{\frac{3}{2}} e^{-\frac{\lambda_l^4}{32\lambda_k^2}} \int_{\frac{1}{2}}^{2} \frac{1}{|x^2-1|^{\frac{1}{4}}} dx \\
&\le \frac{C}{(2l+1)^{\frac{1}{4}}},
\end{aligned}$$

where the second equality is obtained by changing the variable $x \mapsto \lambda_l^2 x$ and the last inequality follows from the fact that $e^{-\frac{\lambda_l^4}{32\lambda_k^2}} \le \frac{32\lambda_k^2}{\lambda_l^4}$.

Case II: Assume $2\lambda_k^2 \le \frac{1}{2}\lambda_l^2$. Then

$$\begin{aligned}
\int_{2\lambda_k^2\le|x|\le\frac{1}{2}\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx &\le C\,\lambda_l^{-\frac{1}{2}} \int_{2\lambda_k^2}^{\frac{1}{2}\lambda_l^2} e^{-\frac{x^2}{8\lambda_k^2}} dx \\
&\le C\,\lambda_l^{-\frac{1}{2}} \int_{2\lambda_k^2}^{\frac{1}{2}\lambda_l^2} \frac{8\lambda_k^2}{x^2} dx \\
&\le \frac{C}{(2l+1)^{\frac{1}{4}}},
\end{aligned}$$

and arguing as in the Case I, we obtain $\int_{\frac{1}{2}\lambda_l^2\le|x|\le 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le \frac{C}{(2l+1)^{\frac{1}{4}}}$. Thus,

$$\int_{2\lambda_k^2\le|x|\le 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx \le \frac{C}{(2l+1)^{\frac{1}{4}}}.$$

(3) In the region $\{x \in \mathbb{R} : |x| \geq 2\lambda_l^2\}$, again we use the Lemma 6.1. We obtain

$$\int_{|x|\geq 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx \leq C\int_{2\lambda_l^2}^{\infty} e^{-\left(\frac{\lambda_k^2+\lambda_l^2}{8\lambda_k^2\lambda_l^2}\right)x^2} dx.$$

Then writing $A = \frac{\lambda_k^2+\lambda_l^2}{8\lambda_k^2\lambda_l^2}$ and $X = 2\lambda_l^2$, we have

$$\int_X^{\infty} e^{-Ax^2} dx \leq \frac{1}{2AX}\int_X^{\infty} 2Axe^{-Ax^2} dx = \frac{1}{2AX} e^{-AX^2} \leq \frac{1}{2AX}\left(\frac{1}{\sqrt{AX}}\right).$$

Thus,

$$\int_{|x|\geq 2\lambda_l^2} \left|h_k\left(\lambda_k^{-1}x\right)\right|\left|h_l\left(\lambda_l^{-1}x\right)\right| dx \leq C\left(\frac{\lambda_k^2}{\lambda_k^2+\lambda_l^2}\right)^{\frac{3}{2}} \frac{1}{\lambda_l} \leq \frac{C}{(2l+1)^{\frac{1}{2}}}.$$

After combining the estimates obtained in each case, we get (6.1). □

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