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New branching formulae for classical groups and relations among them

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Abstract. We find the branching laws for the classical pairs $\mathrm{GL}(m, \mathbb{C}) \subset \mathrm{GL}(n, \mathbb{C})$, $\mathrm{Sp}(2m, \mathbb{C}) \subset \mathrm{Sp}(2n, \mathbb{C})$, $\mathrm{SO}(q, \mathbb{C}) \subset \mathrm{SO}(p, \mathbb{C})$ for all $m \leq n$, and all $q \leq p$, generalizing the well-known results of classical branching laws which exist for $m = n - 1$, and $q = p - 1$. Our approach provides a common proof applicable to all these groups. We also compare the branching multiplicities among these pairs.

Contents

1. Introduction

Branching rules are descriptions of how irreducible representations of a group G decompose under restriction to a subgroup H. We are interested in the cases when G and H are classical groups over complex numbers. We take G to be general linear groups $\mathrm{GL}(n) = \mathrm{GL}(n, \mathbb{C})$, symplectic groups $\mathrm{Sp}(2n) = \mathrm{Sp}(2n, \mathbb{C})$, or orthogonal groups $\mathrm{SO}(p) = \mathrm{SO}(p, \mathbb{C})$. We consider the following pair of groups $H \subset G$ in this paper,

$$\mathrm{GL}(m) \subset \mathrm{GL}(n), \quad \mathrm{Sp}(2m) \subset \mathrm{Sp}(2n), \qquad \mathrm{SO}(q) \subset \mathrm{SO}(p).$$

We provide formulae (see Theorem 2.3 - 2.5) expressing branching multiplicities as determinants of certain combinatorial matrices. They are usually not multiplicity-free. We use the Weyl Character formula to prove all the formulae

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for branching multiplicities as mentioned here. The notation used in this paper as well as the proofs follow [GW09, Chapter 8].

Our proof of the branching laws depends on a different version of the Weyl dimension formula which is our Theorem 4.2, and is a determinantal formula, to which our branching laws reduce to when branching from any group G to the trivial subgroup.

The results for the pair of classical groups ($H \subset G$) listed above for the special case $m = n - 1$ and $q = p - 1$ have been known for a long time. In his book [Wey50, V.18], H. Weyl provided the classical branching description for the pair $\mathrm{GL}(n-1) \subset \mathrm{GL}(n)$. Concerning the pair $\mathrm{Sp}(2n-2) \subset \mathrm{Sp}(2n)$, Zelobenko [Ž73] and Hegerfeldt [Heg67] have established conditions under which the multiplicity is non-zero. The multiplicity formula in this case is due to Whippman [Whi65] (for $n = 2, 3$) and Miller [Mil66] (for the general case of n). For the pair $\mathrm{SO}(p-1) \subset \mathrm{SO}(p)$, the branching rules are in the book [Mur63, IX.9] by F.D. Murnaghan.

After finishing the paper, when we sent it to Prof. Okada, he informed us that Theorems 2.3-2.5 can be derived from an unpublished preprint [Oka89] of his from 1989. He proved his result using Lindström–Gessel–Viennot lemma which counts the number of tuples of non-intersecting lattice paths. He also mentioned the paper [OS19] for similar results.

2. The main theorems

In this section, we provide the statements of our main theorems. The theorems involve certain binomial coefficients, which we will define before going into the details.

Definition 2.1. *Let $k \geq 0$ be a integer and $x \in \mathbb{R}$. We define binomial coefficients $\binom{x}{k}$ as follows:*

$$\binom{x}{k} = \begin{cases} \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}, & \text{if } k \geq 1 \\ 1, & \text{if } k = 0. \end{cases}$$

Definition 2.2. *Let $k \geq 0$ be a integer and $n \in \mathbb{Z}$. We define $\left\{ n \atop k \right\}$ as follows:*

$$\left\{ n \atop k \right\} = \begin{cases} \binom{n}{k}, & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

The dominant weights of the groups $\mathrm{GL}(n)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{SO}(2n)$ are parameterized by sequences of integers $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ satisfying the following conditions.

$$\begin{array}{ll} \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, & \text{for } \mathrm{GL}(n), \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0, & \text{for } \mathrm{Sp}(2n), \mathrm{SO}(2n+1), \\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq |\lambda_n|, & \text{for } \mathrm{SO}(2n). \end{array}$$

For our branching problem, it is sufficient to consider $\lambda_n \geq 0$ for all groups which we tacitly assume everywhere. All the groups $\mathrm{GL}(n)$, $\mathrm{Sp}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{SO}(2n)$ have rank n.

Let $H \subset G$ be one of the pairs as in the Introduction. Let

$$\underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0) \quad \text{and} \quad \underline{\mu} = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m \geq 0)$$

be two sequences of integers. In case $(H \subset G) = (\mathrm{SO}(q) \subset \mathrm{SO}(p))$, then $n = \lfloor \frac{p}{2} \rfloor$ and $m = \lfloor \frac{q}{2} \rfloor$. Let $\Pi_{\underline{\lambda}}$ and $\Psi_{\underline{\mu}}$ denote the irreducible highest weight representations of G and H with the highest weights $\underline{\lambda}$ and $\underline{\mu}$, respectively. Consider the following restriction,

$$\Pi_{\underline{\lambda}}|_H \quad = \sum_{\underline{\mu}} m(\underline{\lambda}, \underline{\mu}) \Psi_{\underline{\mu}}. \tag{1}$$

Set $\lambda_{n+i} = 0, \mu_{m+i} = 0$ for $i \geq 1$. Let

$$u_{ij} = \lambda_i - \mu_j + j - i \quad \text{for } \ 1 \leq i, j \leq n.$$

Theorem 2.3. *Let* $\mathrm{GL}(m) \subset \mathrm{GL}(n)$*, with* $0 \leq m \leq n-1$*. Then,*

(i) The multiplicity $m(\underline{\lambda}, \underline{\mu})$ *is nonzero if and only if*

$$\lambda_i \geq \mu_i \geq \lambda_{i+n-m} \quad \text{for} \quad 1 \leq i \leq m.$$

(ii) The multiplicity

$$m(\underline{\lambda}, \underline{\mu}) = \det\left[M_{ij}\right],$$

where

$$M_{ij} = \begin{cases} \left\{ {u_{ij} + n - m - 1 \atop n - m - 1} \right\}, & \text{if} \qquad 1 \leq j \leq m, \\ \dbinom{u_{ij} + n - j}{n - j}, & \text{if} \quad m+1 \leq j \leq n. \end{cases}$$

Theorem 2.4. *Let* $\mathrm{Sp}(2m) \subset \mathrm{Sp}(2n)$*, with* $0 \leq m \leq n-1$*. Then,*

(i) The multiplicity $m(\underline{\lambda}, \underline{\mu})$ *is nonzero if and only if*

$$\lambda_i \geq \mu_i \geq \lambda_{i+2n-2m} \quad \text{for} \quad 1 \leq i \leq m.$$

(ii) The multiplicity

$$m(\underline{\lambda}, \underline{\mu}) = \det\left[M_{ij}\right],$$

where

$$M_{ij} = \begin{cases} \left\{ {u_{ij} + 2n - 2m - 1 \atop 2n - 2m - 1} \right\}, & \text{if} \qquad 1 \leq j \leq m, \\ \dbinom{u_{ij} + 2n - 2j + 1}{2n - 2j + 1}, & \text{if} \quad m+1 \leq j \leq n. \end{cases}$$

Theorem 2.5. *Let* $\mathrm{SO}(q) \subset \mathrm{SO}(p)$ *with* $0 \leq q \leq p-1$. *Let* $n = \lfloor \frac{p}{2} \rfloor$ *and* $m = \lfloor \frac{q}{2} \rfloor$. *We define* l *as follows:*

$$l = \begin{cases} n-m, & \text{if} \quad p = 2n+1, \\ n-m-1, & \text{if} \quad p = 2n. \end{cases}$$

Then,

(i) *The multiplicity* $m(\underline{\lambda}, \underline{\mu})$ *is nonzero if and only if*

$$\lambda_i \geq \mu_i \geq \lambda_{i+p-q} \quad \text{for} \quad 1 \leq i \leq m,$$

(ii) *The multiplicity*

$$m(\underline{\lambda}, \underline{\mu}) = 2^l \det\left[M_{ij}\right],$$

where

$$M_{ij} = \begin{cases} \left\{ {u_{ij} + p - q - 1 \atop p - q - 1} \right\}, & \text{if} \qquad 1 \leq j \leq m, \\ \binom{u_{ij} + p - 2j - \frac{1}{2}}{p - 2j}, & \text{if} \quad m+1 \leq j \leq n. \end{cases}$$

Remark 2.6. *Putting* $m = 0$ *in Theorems* 2.3 - 2.5 *gives the corresponding Weyl dimension formula (Theorem* 4.2*) for general linear, symplectic and orthogonal groups, respectively. (In Theorem* 2.5*, for odd orthogonal groups, we assume that both* p *and* q *are odd and for even orthogonal groups, we assume that both* p *and* q *are even.) We provide details specifically for the case of general linear groups; similar procedures can be followed for other groups. Putting* $m = 0$ *in Theorem* 2.3*, we have for any* $1 \leq j \leq n$*,*

$$M_{ij} = \binom{u_{ij} + n - j}{n - j} = \binom{\lambda_i - \mu_j + j - i + n - j}{n - j} = \binom{\lambda_i + n - i}{n - j}.$$

Hence in this $m = 0$ *case,* $m(\underline{\lambda}, \underline{\mu}) = \det\left[M_{ij}\right] = \det\left[\binom{\lambda_i + n - i}{n - j}\right]$ *which matches with the dimension formula given in Theorem* 4.2 *for general linear groups.*

In fact, we first prove these dimension formulae (different looking than the usual Weyl dimension formula) which goes into the proof of Theorems 2.3 - 2.5*.*

Remark 2.7. *Put* $m = n-1$ *in Theorem* 2.3*. The inequality* $\lambda_i \geq \mu_i \geq \lambda_{i+1}$ *gives rise to* $\lambda_1 \geq \mu_1 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ *and the formula becomes*

$$M_{ij} = \begin{cases} \left\{ {u_{ij} \atop 0} \right\}, & \text{if} \quad 1 \leq j \leq n-1, \\ \binom{u_{ij}}{0}, & \text{if} \quad j = n. \end{cases}$$

Since $u_{ii} = \lambda_i - \mu_i \geq 0$*, we get* $M_{ii} = 1$ *for* $1 \leq i \leq n$*. Since* $u_{ij} = (\lambda_i - \mu_j) + (j - i) < 0$ *for* $1 \leq j < i \leq n$*, we get* $M_{ij} = 0$ *for* $1 \leq j < i \leq n$*. Therefore, the corresponding matrix becomes an upper triangular matrix where all the diagonal entries are equal to 1. Hence, the multiplicity is 1 when we have the interlacing* *condition. So, we obtain the branching rule for* GL$(n-1) \subset$ GL(n). *A similar argument can be given for other pairs.*

3. Preliminaries

Let $H \subset G$ be classical groups with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. Let T_G and T_H be maximal algebraic tori in G and H respectively with $T_H \subset T_G$. Let $\mathfrak{t}_{\mathfrak{g}}$ and $\mathfrak{t}_{\mathfrak{h}}$ be the corresponding Lie algebras. Let $\Phi_{\mathfrak{g}}$ and $\Phi_{\mathfrak{h}}$ be the roots of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Let $\Phi_{\mathfrak{g}}^+$ and $\Phi_{\mathfrak{h}}^+$ be a system of positive roots for $\mathfrak{g}$ and $\mathfrak{h}$ respectively such that the restriction of a positive root of $\mathfrak{g}$ to $\mathfrak{t}_{\mathfrak{h}}$ is either zero or positive. After fixing the simple roots [GW09, Subsection 2.4.3] we can describe associated positive roots $\Phi_{\mathfrak{g}}^+$ as follows:

$$\Phi_{\mathfrak{g}}^+ = \begin{cases} \{\varepsilon_i - \varepsilon_j \,:\, 1 \le i < j \le n\}, & \text{for GL}(n),\\ \{\varepsilon_i \pm \varepsilon_j \,:\, 1 \le i < j \le n\}, & \text{for SO}(2n),\\ \{\varepsilon_i \pm \varepsilon_j \,:\, 1 \le i < j \le n\} \cup \{2\varepsilon_i \,:\, 1 \le i \le n\}, & \text{for Sp}(2n),\\ \{\varepsilon_i \pm \varepsilon_j \,:\, 1 \le i < j \le n\} \cup \{\varepsilon_i \,:\, 1 \le i \le n\}, & \text{for SO}(2n+1). \end{cases} \tag{2}$$

We define $\rho_{\mathfrak{g}}, R_{\mathfrak{g}}$ for G as follows:

$$\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha \in \Phi_{\mathfrak{g}}^+} \alpha, \qquad R_{\mathfrak{g}} = \prod_{\alpha \in \Phi_{\mathfrak{g}}^+} (1 - \mathrm{e}^{-\alpha}). \tag{3}$$

We denote $\rho_{\mathfrak{g}} = \sum_{i=1}^n \rho_i \varepsilon_i$. The description of ρ_i for each group G is as follows:

$$\rho_i = \begin{cases} n-i, & \text{for} \quad \text{GL}(n), \text{SO}(2n),\\ n-i+1, & \text{for} \quad \text{Sp}(2n),\\ n-i+\frac{1}{2}, & \text{for} \quad \text{SO}(2n+1). \end{cases} \tag{4}$$

This choice of ρ for GL(n) has the advantage of being a positive dominant weight for GL(n). The Weyl Character formula is also valid for GL(n) with this ρ-shift taken as $\rho = \sum_{i=1}^n (n-i)\varepsilon_i$, see [GW09, Corollary 7.1.2]. Thus, we can use $\rho = \sum_{i=1}^n (n-i)\varepsilon_i$ instead of the half-sum of the positive roots (which is not an integral weight of GL(n)).

Definition 3.1. *(Weyl Denominator)*

$$\Delta_{\mathfrak{g}} = \mathrm{e}^{\rho_{\mathfrak{g}}} \prod_{\alpha \in \Phi_{\mathfrak{g}}^+} (1 - \mathrm{e}^{-\alpha}) = \mathrm{e}^{\rho_{\mathfrak{g}}} \cdot R_{\mathfrak{g}}. \tag{5}$$

We denote the Weyl group of G as $W_{\mathfrak{g}}$. The description of $W_{\mathfrak{g}}$ is as follows:

$$W_{\mathfrak{g}} = \begin{cases} \mathfrak{S}_n, & \text{for} \quad \text{GL}(n),\\ (\mathbb{Z}/2\mathbb{Z})^n \rtimes \mathfrak{S}_n, & \text{for} \quad \text{Sp}(2n), \text{SO}(2n+1),\\ (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes \mathfrak{S}_n, & \text{for} \quad \text{SO}(2n), \end{cases}$$

where $(\mathbb{Z}/2\mathbb{Z})^n = \langle \sigma_1, \dots, \sigma_n \rangle$ and σ_i acts as $\sigma_i \varepsilon_j = (-1)^{\delta_{ij}} \varepsilon_j$.

Theorem 3.2. *(Weyl Character Formula). Let $\underline{\lambda}$ be a dominant integral weight of $\mathfrak{t}_{\mathfrak{g}}$ and $\Pi_{\underline{\lambda}}$ the corresponding finite-dimensional irreducible G-module. Then*

$$\Delta_{\mathfrak{g}} \cdot \mathrm{ch}\left(\Pi_{\underline{\lambda}}\right) = \sum_{s\in W_{\mathfrak{g}}} \mathrm{sgn}(s) \mathrm{e}^{s\cdot(\underline{\lambda}+\rho_{\mathfrak{g}})}.$$

Now, we state the Weyl dimension formula for all groups. For proofs and further details, see [GW09, Subsection 7.1.2] and [FH91, Chapter 24]. $\underline{\lambda}$ corresponds to $\sum_i \lambda_i \varepsilon_i$.

Theorem 3.3. *(Weyl Dimension Formula) For a sequence of integers $\underline{\lambda} = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0)$ and an irreducible representation $\Pi_{\underline{\lambda}}$ with the highest weight $\underline{\lambda}$, the dimension of $\Pi_{\underline{\lambda}}$ is given by the following formula.*

$$\prod_{1\leq i<j\leq n} \frac{(\lambda_i + j - \lambda_j - i)}{j-i}, \qquad \textit{for } \mathrm{GL}(n),$$

$$\prod_{1\leq i<j\leq n} \frac{(\lambda_i + n - i)^2 - (\lambda_j + n - j)^2}{(n-i)^2 - (n-j)^2}, \qquad \textit{for } \mathrm{SO}(2n),$$

$$\prod_{1\leq i<j\leq n} \frac{(\lambda_i + n + 1 - i)^2 - (\lambda_j + n + 1 - j)^2}{(n+1-i)^2 - (n+1-j)^2} \prod_{1\leq i\leq n} \frac{\lambda_i + n + 1 - i}{n+1-i}, \quad \textit{for } \mathrm{Sp}(2n),$$

$$\prod_{1\leq i<j\leq n} \frac{(\lambda_i + n + \frac{1}{2} - i)^2 - (\lambda_j + n + \frac{1}{2} - j)^2}{(n+\frac{1}{2}-i)^2 - (n+\frac{1}{2}-j)^2} \prod_{1\leq i\leq n} \frac{\lambda_i + n + \frac{1}{2} - i}{n+\frac{1}{2}-i}, \quad \textit{for } \mathrm{SO}(2n+1).$$

Proposition 3.4. *For $\rho_{\mathfrak{g}} = \sum_{i=1}^n \rho_i \varepsilon_i$, half the sum of positive roots, we have:*

$$\prod_{1\leq i<j\leq n} (\rho_i - \rho_j) = \prod_{j=1}^{n-1} (n-j)! \qquad \textit{when } G = \mathrm{GL}(n),$$

$$\prod_{1\leq i<j\leq n} (\rho_i^2 - \rho_j^2) = \frac{1}{2^{n-1}} \prod_{j=1}^{n-1} (2n-2j)! \qquad \textit{when } G = \mathrm{SO}(2n),$$

$$\prod_{1\leq i<j\leq n} (\rho_i^2 - \rho_j^2) \prod_{1\leq i\leq n} \rho_i = \frac{1}{2^n} \prod_{j=1}^{n} (2n-2j+1)! \quad \textit{when } G = \mathrm{SO}(2n+1),$$

$$\prod_{1\leq i<j\leq n} (\rho_i^2 - \rho_j^2) \prod_{1\leq i\leq n} \rho_i = \prod_{j=1}^{n} (2n-2j+1)! \qquad \textit{when } G = \mathrm{Sp}(2n).$$

We omit the straightforward proof.

Definition 3.5. *Recall that the determinant for an $n \times n$ matrix $A = (a_{ij})$ is:*

$$\det(A) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{j=1}^{n} a_{\sigma(j),j}.$$

Proposition 3.6.

$$\det\left[x_i^{n-j}\right] = \prod_{1\le i<j\le n} (x_i - x_j),$$

$$\det\left[x_i^{2n-2j}\right] = \prod_{1\le i<j\le n} (x_i^2 - x_j^2),$$

$$\det\left[x_i^{2n-2j+1}\right] = \prod_{1\le i<j\le n} (x_i^2 - x_j^2) \prod_{1\le i\le n} x_i.$$

Proof. The first identity arises from the determinant of the Vandermonde matrix. The second matrix corresponds to a Vandermonde-type determinant where the values of x_i are substituted with x_i^2. A factor x_i can be pulled out of the i-th column in the third case. □

4. Weyl dimension formula as determinant

The following lemma establishes a relationship between determinants of matrices with entries containing factorial terms and matrices with entries as binomial coefficients. This lemma plays a crucial role in formulating and proving the main theorems presented in this paper.

Lemma 4.1. *We have the following three equalities of determinants of $n\times n$ matrices:*

$$\det\left[\frac{x_i^{n-j}}{(n-j)!}\right] = \det\left[\binom{x_i}{n-j}\right],$$

$$\det\left[\frac{x_i^{2n-2j+1}}{(2n-2j+1)!}\right] = \det\left[\binom{x_i+n-j}{2n-2j+1}\right],$$

$$\det\left[\frac{x_i^{2n-2j}}{(2n-2j)!}\right] = \det\left[\binom{x_i+n-j-\frac{1}{2}}{2n-2j}\right].$$

Proof. The first equality is established by applying column operations to the matrix defined by the (i,j)-th entry equal to $\frac{x_i^{n-j}}{(n-j)!}$. For $1\le j\le n-1$, one can write

$$\binom{x}{n-j} = \frac{x^{n-j}}{(n-j)!} + \sum_{j+1\le k\le n-1} a_{j,k}\frac{x^{n-k}}{(n-k)!},$$

where $a_{j,k}$ is a constant. We perform a column operation on the j-th column, where $1\le j\le n-1$, given by

$$C_j' = C_j + \sum_{j+1\le k\le n-1} a_{j,k} C_k,$$

where C_j and C_j' denote the old and new j-th column, respectively. Following this operation, the (i,j)-th entry transforms into $\binom{x_i}{n-j}$. This establishes the first equality in the lemma.

For the proof of the other two equalities, we need the following two identities, respectively, where $1 \le j \le n$:

$$\binom{x+n-j}{2n-2j+1} = \frac{x^{2n-2j+1}}{(2n-2j+1)!} + \sum_{j+1\le k\le n} b_{j,k}\frac{x^{2n-2k+1}}{(2n-2k+1)!},$$

$$\binom{x+n-j-\frac{1}{2}}{2n-2j} = \frac{x^{2n-2j}}{(2n-2j)!} + \sum_{j+1\le k\le n} c_{j,k}\frac{x^{2n-2k}}{(2n-2k)!}.$$

Here, again, $b_{j,k}$ and $c_{j,k}$ are constants. Note that $\binom{x+n-j}{2n-2j+1}$ is an odd function of x and $\binom{x+n-j-\frac{1}{2}}{2n-2j}$ is an even function of x. □

The above lemma is necessary for deriving the following theorem, which expresses the dimension of an irreducible representation with highest weight $\underline{\lambda} = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$ as a determinant of a certain combinatorial $n \times n$ matrix.

Theorem 4.2. *(New formulation of Weyl Dimension)*

$$\dim(\Pi_{\underline{\lambda}}) = d \cdot \det\left[h_{ij}\right]$$

$$h_{ij} = \begin{cases} \binom{\lambda_i+n-i}{n-j}, & \textit{for } \mathrm{GL}(n), \\ \binom{\lambda_i-i+2n-j+1}{2n-2j+1}, & \textit{for } \mathrm{Sp}(2n), \\ \binom{\lambda_i-i+2n-j+\frac{1}{2}}{2n-2j+1}, & \textit{for } \mathrm{SO}(2n+1), \\ \binom{\lambda_i-i+2n-j-\frac{1}{2}}{2n-2j}, & \textit{for } \mathrm{SO}(2n). \end{cases} \qquad d = \begin{cases} 1 & \textit{for } \mathrm{GL}(n) \\ 1 & \textit{for } \mathrm{Sp}(2n) \\ 2^n & \textit{for } \mathrm{SO}(2n+1) \\ 2^{n-1} & \textit{for } \mathrm{SO}(2n). \end{cases}$$

Proof. We only prove this for the general linear group; the proofs for other groups follow similarly. Using Proposition 3.6 and Proposition 3.4, Theorem 3.3 gives the following expression for the Weyl dimension formula for $\mathrm{GL}(n)$

$$\prod_{1\le i<j\le n} \frac{(\lambda_i+j-\lambda_j-i)}{j-i} = \frac{\prod_{1\le i<j\le n}\left[(\lambda_i+\rho_i)-(\lambda_j+\rho_j)\right]}{\prod_{1\le i<j\le n}(\rho_i-\rho_j)} = \frac{\det\left[(\lambda_i+\rho_i)^{n-j}\right]}{\prod_{j=1}^{n-1}(n-j)!}.$$

Now we insert $(n-j)!$ inside the determinant in j-th column and then apply Lemma 4.1.

$$\frac{\det\left[(\lambda_i+\rho_i)^{n-j}\right]}{\prod_{j=1}^{n-1}(n-j)!} = \det\left[\frac{(\lambda_i+\rho_i)^{n-j}}{(n-j)!}\right] = \det\left[\binom{\lambda_i+n-i}{n-j}\right].$$

This completes the proof of the Theorem for the general linear groups. Similar arguments can be given for other classical groups. □

5. Partition function

Before going into the proof of Theorem 2.3-2.5, we derive the necessary partition function for each pair in this section. Note that we follow the same notation and definitions as in [GW09, Subsection 8.2.1]. We use the following notations as defined earlier:

$$\begin{aligned} \mathrm{GL}_m^n &:= (\,\mathrm{GL}(m) \subset \mathrm{GL}(n)\,), \quad 0 \le m \le n-1; \\ \mathrm{Sp}_{2m}^{2n} &:= (\,\mathrm{Sp}(2m) \subset \mathrm{Sp}(2n)\,), \quad 0 \le m \le n-1; \\ \mathrm{SO}_q^p &:= (\,\mathrm{SO}(q) \subset \mathrm{SO}(p)\,), \quad 0 \le q \le p-1. \end{aligned}$$

Here $p = 2n$ or $2n+1$ and $q = 2m$ or $2m+1$. Let $\overline{\alpha}$ be the element of $\mathfrak{t}_{\mathfrak{h}}^*$ obtained by restricting α from the Lie algebra $\mathfrak{t}_{\mathfrak{g}}$ to the Lie algebra $\mathfrak{t}_{\mathfrak{h}}$. Define $\overline{\Phi_{\mathfrak{g}}^+}$ as $\{\bar{\alpha} : \alpha \in \Phi_{\mathfrak{g}}^+\} \setminus \{0\}$. Since $\overline{\varepsilon_{m+i}} = 0$, for $i \ge 1$ we have $\overline{\Phi_{\mathfrak{g}}^+}$ as follows:

$$\overline{\Phi_{\mathfrak{g}}^+} = \begin{cases} \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le m\} \cup \{\ \varepsilon_i \ : 1 \le i \le m\}, & \text{for } \mathrm{GL}_m^n, \\ \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le m\} \cup \{\varepsilon_i, 2\varepsilon_i : 1 \le i \le m\}, & \text{for } \mathrm{Sp}_{2m}^{2n}, \\ \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le m\} \cup \{\ \varepsilon_i \ : 1 \le i \le m\}, & \text{for } \mathrm{SO}_{2m+1}^{2n+1}, \mathrm{SO}_{2m}^{2n+1}, \\ \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le m\} \cup \{\ \varepsilon_i \ : 1 \le i \le m\}, & \text{for } \mathrm{SO}_{2m+1}^{2n}, \mathrm{SO}_{2m}^{2n}. \end{cases}$$

Note that $\Phi_{\mathfrak{h}}^+$ is a subset of $\overline{\Phi_{\mathfrak{g}}^+}$. For each $\beta \in \overline{\Phi_{\mathfrak{g}}^+}$, let S_β represent $\{\alpha \in \Phi_{\mathfrak{g}}^+ : \bar{\alpha} = \beta\}$. Now, consider the following definitions:

$$\Sigma_0 = \left\{\beta : \beta \in \Phi_{\mathfrak{h}}^+ \text{ and } \left|S_\beta\right| > 1\right\}, \quad \Sigma_1 = \overline{\Phi_{\mathfrak{g}}^+} \setminus \Phi_{\mathfrak{h}}^+.$$

Consider the set $\Sigma = \Sigma_0 \cup \Sigma_1$. For each $\beta \in \Sigma$, we define the multiplicity m_β as follows:

$$m_\beta = \begin{cases} \left|S_\beta\right|, & \text{if } \beta \in \Sigma_1, \\ \left|S_\beta\right| - 1, & \text{if } \beta \in \Sigma_0. \end{cases}$$

Define the partition function $\wp_\Sigma$ on $\mathfrak{t}_{\mathfrak{h}}^*$ by the formal identity (which follows from the geometric series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$)

$$\frac{1}{R_\Sigma} := \prod_{\beta \in \Sigma} \left(1 - \mathrm{e}^{-\beta}\right)^{-m_\beta} = \sum_{\xi} \wp_\Sigma(\xi)\mathrm{e}^{-\xi}. \tag{6}$$

$\wp_\Sigma(\xi)$ is the number of ways of writing $\xi = \sum_{\beta\in\Sigma} c_\beta \beta \quad (c_\beta \in \mathbb{N})$ where each β that occurs is counted with multiplicity m_β.

We describe the set Σ and the multiplicity m_β of an element β in Σ for all pairs $H \subset G$.

Proposition 5.1. *Consider all branching pairs $H \subset G$. Then*

$$\Sigma = \{\varepsilon_i \,:\, 1 \le i \le m\},\ (r :=)m_{\varepsilon_i} = \begin{cases} n-m, & \textit{for } \mathrm{GL}_m^n, \\ 2n-2m, & \textit{for } \mathrm{Sp}_{2m}^{2n}, \\ 2n+1-q, & \textit{for } \mathrm{SO}_q^{2n+1}, q = 2m \textit{ or } 2m+1, \\ 2n-q, & \textit{for } \mathrm{SO}_q^{2n}, \quad q = 2m \textit{ or } 2m+1. \end{cases}$$

Observe that m_{ε_i} remains constant for $1 \le i \le m$ for each group; therefore, we denote this constant by r. We do not provide a detailed proof for each pair, as it becomes evident through subsequent calculations.

Lemma 5.2. *Consider the same Σ and r as in the preceding proposition. Let $\underline{\xi} \in \mathfrak{t}_{\mathfrak{h}}^*$ be defined as $\underline{\xi} = \sum_{j=1}^m \xi_j \varepsilon_j$. Then,*

$$\wp_\Sigma\left(\underline{\xi}\right) = \prod_{j=1}^m \left\{ \begin{matrix} \xi_j + r - 1 \\ r-1 \end{matrix} \right\}.$$

Proof. Using the definition of partition function from equation (6) to the previously mentioned Σ and r in the preceding proposition, our partition function becomes:

$$\prod_{j=1}^m (1 - \mathrm{e}^{-\varepsilon_j})^{-r} = \sum_{\underline{\xi}\in\mathfrak{t}_{\mathfrak{h}}^*} \wp_\Sigma(\underline{\xi}) \mathrm{e}^{-\underline{\xi}}.$$

So, $\wp_\Sigma(\underline{\xi})$ represents the coefficients of $\mathrm{e}^{-\underline{\xi}} = \prod_{j=1}^m (\mathrm{e}^{-\varepsilon_j})^{\xi_j}$ in $\prod_{j=1}^m (1-\mathrm{e}^{-\varepsilon_j})^{-r}$. Since the ε_j's are linearly independent for $1 \le j \le m$, it suffices to determine the coefficients of $(\mathrm{e}^{-\varepsilon_j})^{\xi_j}$ in the expression of $(1-\mathrm{e}^{-\varepsilon_j})^{-r}$ and then take the product over j. Specifically, we need to find the coefficients of z^{ξ_j} in the expression of $(1-z)^{-r}$. Consider the following power series identity,

$$\frac{1}{(1-z)^r} = \sum_{l\ge 0} \binom{l+r-1}{r-1} z^l.$$

The coefficients mentioned above are some binomial coefficient. One can prove this identity by induction. Assuming it is true for $r-1$, differentiate both sides of the expression for value $r-1$ to obtain expressions for value r. Hence, if $\xi_j \ge 0$, the coefficients of z^{ξ_j} are $\binom{\xi_j+r-1}{r-1}$; otherwise, the coefficient are zero.

$$\wp_\Sigma\left(\underline{\xi}\right) = \begin{cases} \prod_{j=1}^m \binom{\xi_j+r-1}{r-1}, & \text{if } \ \xi_j \ge 0 \ \text{ for } \ 1 \le j \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Now using definition 2.2, $\wp_\Sigma\left(\underline{\xi}\right)$ becomes:

$$\wp_\Sigma\left(\underline{\xi}\right) = \prod_{j=1}^{m} \left\{ \begin{matrix} \xi_j + r - 1 \\ r - 1 \end{matrix} \right\}.$$

This completes the proof of this Lemma. □

6. Proof of Theorems 2.3 - 2.5

In this section, we prove the multiplicity formulae given in Theorem 2.3-2.5. Our proof strategies follow the approach used in the restriction from $\mathrm{Sp}(2n)$ to $\mathrm{Sp}(2n-2)$ as in [GW09, Subsection 8.3.4].

Recall the notation $R_{\mathfrak{h}}$ from Section 3. The main idea of the proof is to use the Weyl character formula (Theorem 3.2) for both H and G to get two expressions for $R_{\mathfrak{h}} \cdot \Pi_{\underline{\lambda}}|_H$. By equating the coefficient of $\mathrm{e}^{\underline{\mu}}$ in these two expression, we find the multiplicity $m(\underline{\lambda}, \underline{\mu})$. We divide the proof into several parts.

First Part We find the coefficient of $\mathrm{e}^{\underline{\mu}}$ in $R_{\mathfrak{h}} \cdot \Pi_{\underline{\lambda}}|_H$ using the Weyl character formula for H in this part. Using the Weyl character formula for H to $\Psi_{\underline{\mu}}$, equation (1) becomes:

$$R_{\mathfrak{h}} \cdot \Pi_{\underline{\lambda}}|_H = \sum_{\underline{\mu}} \sum_{w \in W_{\mathfrak{h}}} m(\underline{\lambda}, \underline{\mu})\, \mathrm{sgn}(w) \mathrm{e}^{w\cdot\left(\underline{\mu}+\rho_{\mathfrak{h}}\right)-\rho_{\mathfrak{h}}}. \tag{7}$$

Consider two dominant weights $\underline{\mu}$ and $\underline{\mu}'$ that appear in the sum. Since each weight is conjugate under the Weyl group $W_{\mathfrak{h}}$ to exactly one dominant weight, the equation

$$w \cdot \left(\underline{\mu} + \rho_{\mathfrak{h}}\right) - \rho_{\mathfrak{h}} = \underline{\mu}'$$

implies $\underline{\mu}' = \underline{\mu}$. Additionally, as $\underline{\mu}+\rho_{\mathfrak{h}}$ is regular, the equation $w\cdot\left(\underline{\mu} + \rho_{\mathfrak{h}}\right)-\rho_{\mathfrak{h}} = \underline{\mu}$ implies $w = 1$. Consequently, the coefficient of $\mathrm{e}^{\underline{\mu}}$ in equation (7) is $m(\underline{\lambda}, \underline{\mu})$.

Second Part We find an expression of the Weyl denominator at the end of this part, which is needed in the next part. Consider the set $\Phi_{\mathfrak{g}}^+$ (see equation (2)) of positive roots of G. Recall the notation $\overline{\alpha}$. Define P and Q to be subsets of $\Phi_{\mathfrak{g}}^+$ defined as follows:

$$P = \{\alpha \in \Phi_{\mathfrak{g}}^+ \,:\, \overline{\alpha} = 0\}, \qquad Q = \{\alpha \in \Phi_{\mathfrak{g}}^+ \,:\, \overline{\alpha} \neq 0\}.$$

So $\Phi_{\mathfrak{g}}^+$ is the disjoint union of P and Q. We define ρ_S, R_S for a set $S \subset \Phi_{\mathfrak{g}}^+$ as follows analogously in equation (3).

$$\rho_S = \frac{1}{2}\sum_{\alpha\in S} \alpha, \qquad R_S = \prod_{\alpha \in S} \left(1 - \mathrm{e}^{-\alpha}\right).$$

Hence,

$$R_{\mathfrak{g}} = R_P \cdot R_Q, \quad \text{and} \quad \mathrm{e}^{\rho_{\mathfrak{g}}} = \mathrm{e}^{\rho_P} \cdot \mathrm{e}^{\rho_Q}. \tag{8}$$

Here is the exact description of P for each pairs.

$$P = \begin{cases} \{\varepsilon_i - \varepsilon_j \,:\, m+1 \le i < j \le n\}, & \text{for } \mathrm{GL}_m^n, \\ \{\varepsilon_i \pm \varepsilon_j \,:\, m+1 \le i < j \le n\}, & \text{for } \mathrm{SO}_{2m+1}^{2n}, \mathrm{SO}_{2m}^{2n}, \\ \{\varepsilon_i \pm \varepsilon_j \,:\, m+1 \le i < j \le n\} \cup \{2\varepsilon_i \,:\, m+1 \le i \le n\}, & \text{for } \mathrm{Sp}_{2m}^{2n}, \\ \{\varepsilon_i \pm \varepsilon_j \,:\, m+1 \le i < j \le n\} \cup \{\varepsilon_i \,:\, m+1 \le i \le n\}, & \text{for } \mathrm{SO}_{2m+1}^{2n+1}, \mathrm{SO}_{2m}^{2n+1}. \end{cases}$$

Observe that the set P can be considered as the set of positive roots of K, where K is as follows:

$$K \cong \begin{cases} \mathrm{GL}(n-m), & \text{for} \quad \mathrm{GL}_m^n, \\ \mathrm{Sp}(2n-2m), & \text{for} \quad \mathrm{Sp}_{2m}^{2n}, \\ \mathrm{SO}(2n-2m+1), & \text{for} \quad \mathrm{SO}_{2m+1}^{2n+1}, \mathrm{SO}_{2m}^{2n+1}, \\ \mathrm{SO}(2n-2m), & \text{for} \quad \mathrm{SO}_{2m+1}^{2n}, \mathrm{SO}_{2m}^{2n}. \end{cases}$$

Note that we can write $R_P = R_{\mathfrak{k}}$ and $\mathrm{e}^{\rho_P} = \mathrm{e}^{\rho_{\mathfrak{k}}}$ as $P = \Phi_{\mathfrak{k}}^+$. Therefore, by equation (8), $R_{\mathfrak{g}} = R_{\mathfrak{k}} \cdot R_Q$ and $\mathrm{e}^{\rho_{\mathfrak{g}}} = \mathrm{e}^{\rho_{\mathfrak{k}}} \cdot \mathrm{e}^{\rho_Q}$. Hence, using equation (5), the denominator $\Delta_{\mathfrak{g}}$ becomes

$$\Delta_{\mathfrak{g}} = \mathrm{e}^{\rho_Q} \cdot R_Q \cdot \mathrm{e}^{\rho_{\mathfrak{k}}} \cdot R_{\mathfrak{k}}. \tag{9}$$

Third Part In this part, we use the Weyl character formula for G and break the Weyl numerator into the cosets over $W_{\mathfrak{k}} \setminus W_{\mathfrak{g}}$. Further, we use the Weyl dimension formula (Theorem 4.2) and get an expression of the multiplicity of $m(\underline{\lambda}, \underline{\mu})$ as an alternating sum over $W_{\mathfrak{k}} \setminus W_{\mathfrak{g}}$ after finding the coefficient of $\mathrm{e}^{\underline{\mu}}$ in a certain equation.

Using the Weyl character formula for G to $\Pi_{\underline{\lambda}}$ and equation (9) we obtain the following:

$$\Pi_{\underline{\lambda}} = \frac{1}{\mathrm{e}^{\rho_Q} \cdot R_Q \cdot \mathrm{e}^{\rho_{\mathfrak{k}}} \cdot R_{\mathfrak{k}}} \sum_{s \in W_{\mathfrak{g}}} \mathrm{sgn}(s) \mathrm{e}^{s \cdot (\underline{\lambda} + \rho_{\mathfrak{g}})}. \tag{10}$$

We break down this Weyl numerator into the cosets over $W_{\mathfrak{k}} \setminus W_{\mathfrak{g}}$ as follows:

$$\sum_{s \in W_{\mathfrak{g}}} \mathrm{sgn}(s) \mathrm{e}^{s \cdot (\underline{\lambda} + \rho_{\mathfrak{g}})} = \sum_{s \in W_{\mathfrak{k}} \setminus W_{\mathfrak{g}}} \mathrm{sgn}(s) \left\{ \sum_{w \in W_{\mathfrak{k}}} \mathrm{sgn}(w) \mathrm{e}^{(ws) \cdot (\underline{\lambda} + \rho_{\mathfrak{g}})} \right\}. \tag{11}$$

We are not considering sgn as a function on coset space $W_{\mathfrak{k}} \setminus W_{\mathfrak{g}}$ but as a function on a representative of each coset, see [GW09, Subsection 8.3.4]. Let $\gamma = s \cdot (\underline{\lambda} + \rho_{\mathfrak{g}})$ for $s \in W_{\mathfrak{g}}$. We denote $\gamma = \sum_{i=1}^n \gamma_i \varepsilon_i$. Note that $s \cdot \varepsilon_i = \varepsilon_{s^{-1}(i)}$. So,

$$\gamma_i = \lambda_{s(i)} + \rho_{s(i)}. \tag{12}$$

As $\overline{\varepsilon_{m+i}} = 0$ for $i \ge 1$, we can express $\overline{\gamma}$ as $\overline{\gamma} = \sum_{i=1}^m \gamma_i \varepsilon_i$. Let $\widetilde{\gamma} = \sum_{i=m+1}^n \gamma_i \varepsilon_i$. This allows us to express $\gamma = \overline{\gamma} + \widetilde{\gamma}$. Further, $w(\gamma) = \overline{\gamma} + w(\widetilde{\gamma})$ for $w \in W_{\mathfrak{k}}$ as

$w(i) = i$, for $1 \le i \le m$. Thus,

$$\sum_{w\in W_{\mathfrak{k}}} \operatorname{sgn}(w) e^{(ws)\cdot(\underline{\lambda}+\rho_{\mathfrak{g}})} = e^{\overline{\gamma}} \left\{ \sum_{w\in W_{\mathfrak{k}}} \operatorname{sgn}(w) e^{w\cdot\widetilde{\gamma}} \right\}. \tag{13}$$

Equation (10) becomes the following after combining equations (13), (11)

$$\Pi_{\underline{\lambda}} = \frac{1}{e^{\rho_Q}\cdot R_Q} \sum_{s\in W_{\mathfrak{k}}\backslash W_{\mathfrak{g}}} \operatorname{sgn}(s) e^{\overline{\gamma}} \left\{ \frac{\sum_{w\in W_{\mathfrak{k}}} \operatorname{sgn}(w) e^{w\cdot\widetilde{\gamma}}}{e^{\rho_{\mathfrak{k}}}\cdot R_{\mathfrak{k}}} \right\}. \tag{14}$$

Note that $\widetilde{\gamma} = \sum_{i=m+1}^{n} \gamma_i \varepsilon_i$ may not be a strictly dominant integral weight, but it is a Weyl conjugate of a strictly dominant integral weight. The expression within the parentheses in (14) corresponds to the Weyl character formula for K. Now when we restrict $\Pi_{\underline{\lambda}}$ from $\mathfrak{t}_{\mathfrak{g}}$ to $\mathfrak{t}_{\mathfrak{h}}$, the expression inside the parentheses provides dimension, say D. Observe that $\overline{Q} = \overline{\Phi_{\mathfrak{g}}^{+}}$, implies $e^{\overline{\rho_Q}} = e^{\overline{\rho_{\mathfrak{g}}}}$. Further $\overline{\Phi_{\mathfrak{g}}^{+}} = \Phi_{\mathfrak{h}}^{+} \cup \Sigma$ implies $\overline{R_Q} = R_{\mathfrak{h}} \cdot R_{\Sigma}$. Hence, taking the restriction of the equation (14) we get,

$$\Pi_{\underline{\lambda}}|_H = \frac{1}{e^{\overline{\rho_{\mathfrak{g}}}}\cdot R_{\mathfrak{h}}\cdot R_{\Sigma}} \sum_{s\in W_{\mathfrak{k}}\backslash W_{\mathfrak{g}}} \operatorname{sgn}(s) e^{\overline{\gamma}} D. \tag{15}$$

We have the following description of D using the Weyl dimension formula (Theorem 4.2) for the group K,

$$D = d\cdot \det\left[g_{ij}\right], \tag{16}$$

for $m+1 \le i, j \le n$, where

$$g_{ij} = \begin{cases} \binom{\gamma_i}{n-j}, & \text{for} \quad \mathrm{GL}(n-m), \\ \binom{\gamma_i+n-j}{2n-2j+1}, & \text{for} \quad \mathrm{Sp}(2n-2m), \\ \binom{\gamma_i+n-j}{2n-2j+1}, & \text{for} \quad \mathrm{SO}(2n-2m+1), \\ \binom{\gamma_i+n-j-\frac{1}{2}}{2n-2j}, & \text{for} \quad \mathrm{SO}(2n-2m), \end{cases} \tag{17}$$

$$d = \begin{cases} 1 & \text{for } \mathrm{GL}(n-m) \\ 1 & \text{for } \mathrm{Sp}(2n-2m) \\ 2^{n-m} & \text{for } \mathrm{SO}(2n-2m+1) \\ 2^{n-m-1} & \text{for } \mathrm{SO}(2n-2m). \end{cases}$$

Using equation (6), the equation (15) can be written as:

$$R_{\mathfrak{h}} \cdot \Pi_{\underline{\lambda}}|_H = \sum_{s\in W_{\mathfrak{k}}\backslash W_{\mathfrak{g}}} \sum_{\xi\in\mathfrak{t}_{\mathfrak{h}}^*} \mathrm{sgn}(s) D \ \wp_\Sigma(\xi)\ \mathrm{e}^{\overline{\gamma}-\xi-\overline{\rho_{\mathfrak{g}}}}, \tag{18}$$

where $\gamma = s\cdot(\lambda+\rho_{\mathfrak{g}})$. Equate the coefficient of $\mathrm{e}^{\underline{\mu}}$ in (18) with that in (7) to obtain:

$$m(\underline{\lambda},\underline{\mu}) = \sum_{s\in W_{\mathfrak{k}}\backslash W_{\mathfrak{g}}} \mathrm{sgn}(s) D\ \wp_\Sigma(\overline{\gamma}-\underline{\mu}-\overline{\rho_{\mathfrak{g}}}). \tag{19}$$

Fourth Part In this part, we write the summation in equation (19) over $\mathfrak{S}_{n-m}\backslash \mathfrak{S}_n$. Then we use the Weyl dimension formula for K (Theorem 4.2) and partition value (Lemma 5.2) to get the final formula.

We can identify $W_{\mathfrak{k}}\setminus W_{\mathfrak{g}}$ as follows:

$$\begin{aligned} &\mathfrak{S}_{n-m}\setminus\mathfrak{S}_n, && \text{for } \mathrm{GL}_m^n,\\ &((\mathbb{Z}/2\mathbb{Z})^{n-m}\rtimes\mathfrak{S}_{n-m})\setminus((\mathbb{Z}/2\mathbb{Z})^{n}\rtimes\mathfrak{S}_{n}), && \text{for } \mathrm{Sp}_{2m}^{2n}, \mathrm{SO}_{2m+1}^{2n+1}, \mathrm{SO}_{2m}^{2n+1},\\ &\left((\mathbb{Z}/2\mathbb{Z})^{n-m-1}\rtimes\mathfrak{S}_{n-m}\right)\setminus\left((\mathbb{Z}/2\mathbb{Z})^{n-1}\rtimes\mathfrak{S}_{n}\right), && \text{for } \mathrm{SO}_{2m+1}^{2n}, \mathrm{SO}_{2m}^{2n}.\end{aligned}$$

A set of representatives of $W_{\mathfrak{k}}\backslash W_{\mathfrak{g}}$ can be taken to be a pair of representatives of $\mathfrak{S}_{n-m}\backslash\mathfrak{S}_n$ and $(\mathbb{Z}/2\mathbb{Z})^{n-m}\backslash(\mathbb{Z}/2\mathbb{Z})^n$. A set of representatives of $(\mathbb{Z}/2\mathbb{Z})^{n-m}\backslash(\mathbb{Z}/2\mathbb{Z})^n$ is given by $(\mathbb{Z}/2\mathbb{Z})^m$ when writing $\sigma\in(\mathbb{Z}/2\mathbb{Z})^m$. We already have the sum in (19) over $\mathfrak{S}_{n-m}\setminus\mathfrak{S}_n$ for GL_m^n. We need to do for other pairs. Given $s\in W_{\mathfrak{k}}\setminus W_{\mathfrak{g}}$, we can write $s=\sigma\cdot\bar{v}$, where $\sigma\in(\mathbb{Z}/2\mathbb{Z})^m$ and $\bar{v}\in\mathfrak{S}_{n-m}\setminus\mathfrak{S}_n$. Choose any representative $v\in\mathfrak{S}_n$ for $\bar{v}$. Thus,

$$\overline{v\cdot(\lambda+\rho_{\mathfrak{g}})} = \sum_{j=1}^m(\lambda_{v(j)}+\rho_{v(j)})\varepsilon_j.$$

Note that each coordinate of $\overline{v\cdot(\lambda+\rho_{\mathfrak{g}})}$ is positive. Further, if $\overline{\rho_{\mathfrak{g}}}=(\rho_1,\rho_2,\dots,\rho_m)$, each coordinate of $\overline{\rho_{\mathfrak{g}}}$ is also positive. As every coordinate of $\underline{\mu}$ is non-negative, if $\sigma\in(\mathbb{Z}/2\mathbb{Z})^m$ and $\sigma\neq 1$, it implies that at least one component of $\overline{(\sigma v)\cdot(\lambda+\rho_{\mathfrak{g}})}-\mu-\overline{\rho_{\mathfrak{g}}}$ is negative. Hence by Lemma 5.2,

$$\wp_\Sigma\left(\overline{(\sigma v)\cdot(\lambda+\rho_{\mathfrak{g}})}-\mu-\overline{\rho_{\mathfrak{g}}}\right)=0 \quad \text{for } \sigma\neq 1, \bar{v}\in(\mathfrak{S}_{n-m}\setminus\mathfrak{S}_n).$$

Thus, we may take $\sigma=1$, and we can express the summation over $\mathfrak{S}_{n-m}\setminus\mathfrak{S}_n$. Therefore, equation (19) becomes:

$$\begin{aligned} m(\underline{\lambda},\underline{\mu}) &= \sum_{s\in\mathfrak{S}_{n-m}\backslash\mathfrak{S}_n}\mathrm{sgn}(s)D\wp_\Sigma(\overline{\gamma}-\underline{\mu}-\overline{\rho_{\mathfrak{g}}})\\ &= \sum_{s\in\mathfrak{S}_{n-m}\backslash\mathfrak{S}_n}\mathrm{sgn}(s)\left(d\sum_{\sigma\in\mathfrak{S}_{n-m}}\mathrm{sgn}(\sigma)\prod_{j=m+1}^n g_{\sigma(j),j}\right)\wp_\Sigma(\overline{\gamma}-\underline{\mu}-\overline{\rho_{\mathfrak{g}}}).\end{aligned}$$

The last equality is due to equation (16). Using the value of g_{ij} (in equation (17)) and h_{ij} (in Theorem 4.2) and by equation (12), we have $g_{\sigma(j),j} = h_{s\sigma(j),j}$. Hence,

$$\begin{aligned} m(\underline{\lambda}, \underline{\mu}) &= d \sum_{s \in \mathfrak{S}_{n-m} \backslash \mathfrak{S}_n} \operatorname{sgn}(s) \sum_{\sigma \in \mathfrak{S}_{n-m}} \operatorname{sgn}(\sigma) \prod_{j=m+1}^{n} h_{s\sigma(j),j} \ \wp_\Sigma(\overline{\gamma} - \underline{\mu} - \overline{\rho_{\mathfrak{g}}}), \\ &= d \sum_{s \in \mathfrak{S}_n} \operatorname{sgn}(s) \prod_{j=m+1}^{n} h_{s(j),j} \ \wp_\Sigma(\overline{\gamma} - \underline{\mu} - \overline{\rho_{\mathfrak{g}}}), \\ &= d \sum_{s \in \mathfrak{S}_n} \operatorname{sgn}(s) \prod_{j=m+1}^{n} h_{s(j),j} \left(\prod_{j=1}^{m} f_{s(j),j} \right). \end{aligned}$$

We have used Lemma 5.2 in the last equality, where

$$f_{s(j),j} = \left\{ \begin{matrix} u_{s(j),j} + r - 1 \\ r - 1 \end{matrix} \right\},$$

as $\overline{\gamma} - \underline{\mu} - \overline{\rho_{\mathfrak{g}}} = \sum_{j=1}^{m} \left(\lambda_{s(j)} - \mu_j + j - s(j) \right) \varepsilon_j = \sum_{j=1}^{m} u_{s(j),j} \varepsilon_j$. Note that both h_{ij} and f_{ij} reduce to M_{ij} defined in Section 2 in the respective cases. Therefore, we get the desired matrix as in the statement of the Theorems 2.3-2.5. Hence, this gives us the corresponding multiplicity $m(\underline{\lambda}, \underline{\mu})$ formulae of Theorems 2.3-2.5.

Fifth Part Now we find the corresponding interlacing condition for all pair and we prove this by induction. We only give details for the general linear groups; for other pairs follow similarly. We prove the multiplicity $m(\underline{\lambda}, \underline{\mu})$ is nonzero if and only if

$$\lambda_i \geq \mu_i \geq \lambda_{i+n-m} \quad \text{for} \quad 1 \leq i \leq m. \tag{20}$$

By assumption it is true for $\mathrm{GL}(m+1) \subset \mathrm{GL}(n)$ and $\mathrm{GL}(m) \subset \mathrm{GL}(m+1)$ and together these two provide the condition (20) for $\mathrm{GL}(m) \subset \mathrm{GL}(n)$.

This completes the proof of Theorems 2.3-2.5.

7. Comparison of multiplicities

We follow the same notation as in Section 5. Define $\ell(\underline{\lambda})$, the length of $\underline{\lambda}$, to be the largest integer s such that $\lambda_s \neq 0$. In this section, we consider the branching for the following pairs (For consistency of the four pairs, note that $2m - n = n - 2(n-m)$):

$$\begin{aligned} \mathrm{GL}_{2m-n}^{n} &= (\ \mathrm{GL}(2m-n) \subset \mathrm{GL}(n)\ ), & \mathrm{SO}_{2m+1}^{2n+1} &= (\ \mathrm{SO}(2m+1) \subset \mathrm{SO}(2n+1)\ ), \\ \mathrm{Sp}_{2m}^{2n} &= (\ \mathrm{Sp}(2m) \subset \mathrm{Sp}(2n)\ ), & \mathrm{SO}_{2m}^{2n} &= (\ \mathrm{SO}(2m) \subset \mathrm{SO}(2n)\ ). \end{aligned}$$

The corollary derived from Theorems 2.3 - 2.5 is as follows.

Corollary 7.1. *(1) Let $\frac{n}{2} \leq m < n$. For fixed pair $(\underline{\lambda}, \underline{\mu})$ with $\ell(\underline{\mu}) \leq 2m - n$, the branching multiplicity $m(\underline{\lambda}, \underline{\mu})$ is independent of the pairs considered.*

(2) *Let* $\frac{n}{2} \leq m < n$. *For fixed pair* $(\underline{\lambda}, \underline{\mu})$ *with* $\ell(\underline{\lambda}) \leq 2m - n$, *the branching multiplicity* $m(\underline{\lambda}, \underline{\mu})$ *is independent of the pairs considered.*

(3) *Let* $0 \leq m < n$. *For fixed pair* $(\underline{\lambda}, \underline{\mu})$ *with* $\ell(\underline{\lambda}) \leq m$, *the branching multiplicity* $m(\underline{\lambda}, \underline{\mu})$ *is independent of the pairs* Sp^{2n}_{2m}, $\mathrm{SO}^{2n+1}_{2m+1}$, SO^{2n}_{2m}.

Proof. We will prove part (3) first, as it helps prove part (2). Since part (1) can be proved similarly, we will omit its proof.

Assume $0 \leq m < n$. By Theorem 2.4-2.5, $m(\underline{\lambda}, \underline{\mu})$ is some scalar times the determinant of a $n \times n$ matrix, say M. Under the condition $\ell(\underline{\lambda}) \leq m$, M becomes a block matrix

$$M = \left(\begin{array}{c|c} A_{m\times m} & * \\ \hline O_{n-m\times m} & B_{(n-m)\times(n-m)} \end{array}\right).$$

It turns out that $A_{m\times m}$ is the same for all the three pairs Sp^{2n}_{2m}, $\mathrm{SO}^{2n+1}_{2m+1}$, SO^{2n}_{2m}. $B_{(n-m)\times(n-m)}$ is an upper triangular matrix with diagonal entries 1 in the case Sp^{2n}_{2m}, hence $m(\underline{\lambda}, \underline{\mu}) = \det A$ for Sp^{2n}_{2m}.

For the case $\mathrm{SO}^{2n+1}_{2m+1}$, $\det B = \frac{1}{2^{n-m}}$ by using Theorem 4.2 (for $\mathrm{SO}(2n-2m+1)$). Hence, $m(\underline{\lambda}, \underline{\mu}) = 2^{n-m} \cdot \det A \cdot \det B = 2^{n-m} \cdot \det A \cdot \frac{1}{2^{n-m}} = \det A$. Similarly for the case SO^{2n}_{2m}, $m(\underline{\lambda}, \underline{\mu}) = 2^{n-m-1} \cdot \det A \cdot \frac{1}{2^{n-m-1}} = \det A$. So $m(\underline{\lambda}, \underline{\mu}) = \det A$ for all three pairs Sp^{2n}_{2m}, $\mathrm{SO}^{2n+1}_{2m+1}$, SO^{2n}_{2m} under the condition $\ell(\underline{\lambda}) \leq m$. This proves part (3) of the corollary.

Now we prove part (2). Assume $\frac{n}{2} \leq m < n$ and $\ell(\underline{\lambda}) \leq 2m - n$. Since $2m - n < m$ as we assume $\frac{n}{2} \leq m < n$, we can use part (3) and hence $m(\underline{\lambda}, \underline{\mu})$ is independent of the pairs Sp^{2n}_{2m}, $\mathrm{SO}^{2n+1}_{2m+1}$, SO^{2n}_{2m}. Hence, $m(\underline{\lambda}, \underline{\mu})$ will be independent of all four pairs if $m(\underline{\lambda}, \underline{\mu})$ is independent of the pairs Sp^{2n}_{2m} and GL^{n}_{2m-n}. For GL^{n}_{2m-n}, the multiplicity matrix M will be a block diagonal matrix with $\det M = \det A_{(2m-n)\times(2m-n)} \cdot B_{(2n-2m)\times(2n-2m)} = \det A_{(2m-n)\times(2m-n)}$ as $B_{(2n-2m)\times(2n-2m)}$ is an upper triangular matrix with 1 on the diagonal. Now $\det A_{(2m-n)\times(2m-n)}$ for GL^{n}_{2m-n} is the same as $\det A_{m\times m}$ for Sp^{2n}_{2m}. This holds because the determinant of $m\times m$ matrix is same as the principal $(2m-n)\times(2m-n)$ minor for symplectic groups case. This proves part (2). □

(H,G) \ $\underline{\mu}$	(0,0)	(1,0)	(1,1)	(2,0)	(2,1)	(3,0)	(3,1)
Sp^8_4	45	40	10	16	4	4	1
SO^9_5	45	40	10	16	4	4	1
SO^8_4	45	40	10	16	4	4	1

TABLE 1. Calculation of $m(\underline{\lambda}, \underline{\mu})$ for $\underline{\lambda} = (3, 1, 0, 0)$.

Part (1) of the Corollary 7.1 determines multiplicities that are always independent across all pairs for a given value of $\underline{\lambda}$. In part (2), the corollary gives a criteria for $\underline{\lambda}$ that leads to the independence of all multiplicities among pairs. One can find similar results in [Mil66, Theorem 2], proving branching multiplicity is independent of the pairs GL_{n-1}^{n+1} and Sp_{2n-2}^{2n}. As a corollary, Miller proves that multiplicities are independent of the pairs GL_{n-2}^{n} and Sp_{2n-2}^{2n}, when $\ell(\underline{\mu}) \leq n-2$. Table 1 verifies part (3) of Corollary 7.1.

We can describe the multiplicities as a product formula, when $m = n-1$ for the mentioned pairs at the start of this section.

Corollary 7.2. *The multiplicity $m(\underline{\lambda}, \underline{\mu})$ is nonzero if and only if*

$$\lambda_j \geq \mu_j \geq \lambda_{j+2} \quad \text{for } j = 1, \dots, n-1$$

When these inequalities are satisfied, let

$$x_1 \geq y_1 \geq x_2 \geq y_2 \geq \cdots \geq x_{n-1} \geq y_{n-1} \geq x_n$$

be the non-increasing rearrangement of $\{\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_{n-1}\}$. *Then,*

$$m(\underline{\lambda}, \underline{\mu}) = \begin{cases} \left[\prod_{j=1}^{n-1}(x_j - y_j + 1)\right], & \text{for } \mathrm{GL}_{n-2}^{n}, \\ \left[\prod_{j=1}^{n-1}(x_j - y_j + 1)\right](x_n + 1), & \text{for } \mathrm{Sp}_{2n-2}^{2n}, \\ \left[\prod_{j=1}^{n-1}(x_j - y_j + 1)\right](2x_n + 1), & \text{for } \mathrm{SO}_{2n-1}^{2n+1}, \\ \left[\prod_{j=1}^{n-1}(x_j - y_j + 1)\right], & \text{for } \mathrm{SO}_{2n-2}^{2n}. \end{cases}$$

In Corollary 7.2, $m(\underline{\lambda}, \underline{\mu})$ is known in this form for the pair Sp_{2n-2}^{2n} in [GW09, Theorem 8.1.5]. Such a corollary for all pairs considered above seems new. It is a direct consequence of Theorem 2.3 (in case of GL_{n-2}^{n}) and Theorem 2.5 (in case of $\mathrm{SO}_{2n-1}^{2n+1}$ and of SO_{2n-2}^{2n}).

Proof. $c(\underline{\lambda}, \underline{\mu}) := \left[\prod_{j=1}^{n-1}(x_j - y_j + 1)\right](x_n + 1)$ for Sp_{2n-2}^{2n} is known. We use it to prove other equality. If we compare the multiplicity matrices of SO_{2n-2}^{2n} and Sp_{2n-2}^{2n}, we have all columns are identical except the last column. Let us denote the multiplicity by $d(\underline{\lambda}, \underline{\mu})$ for SO_{2n-2}^{2n}. In SO_{2n-2}^{2n}, last column of multiplicity matrix has all entry 1, hence we have

$$c(\underline{\lambda}, \underline{\mu}) = \begin{cases} (\mu_{n-1} + 1)d(\underline{\lambda}, \underline{\mu}) & \text{if } 0 \leq \mu_{n-1} \leq \lambda_n, \\ (\lambda_n + 1)d(\underline{\lambda}, \underline{\mu}) & \text{if } \lambda_n < \mu_{n-1} \leq \lambda_{n-1}. \end{cases}$$

When $0 \leq \mu_{n-1} \leq \lambda_n$, then $x_n = \mu_{n-1}$ and when $\lambda_n < \mu_{n-1} \leq \lambda_{n-1}$, then $x_n = \lambda_{n-1}$. Hence, $c(\underline{\lambda}, \underline{\mu}) = (x_n + 1)d(\underline{\lambda}, \underline{\mu})$ gives the result for the pair SO_{2n-2}^{2n}.

Let us denote the multiplicity by $b(\underline{\lambda},\underline{\mu})$ for $\mathrm{SO}^{2n+1}_{2n-1}$. In a similar way, we have

$$\frac{1}{2}b(\underline{\lambda},\underline{\mu}) = \begin{cases} (\mu_{n-1}+\frac{1}{2})d(\underline{\lambda},\underline{\mu}) & \text{if } 0 \le \mu_{n-1} \le \lambda_n, \\ (\lambda_n+\frac{1}{2})d(\underline{\lambda},\underline{\mu}) & \text{if } \lambda_n < \mu_{n-1} \le \lambda_{n-1}. \end{cases}$$

Hence, $b(\underline{\lambda},\underline{\mu}) = 2(x_n+\frac{1}{2})d(\underline{\lambda},\underline{\mu}) = (2x_n+1)d(\underline{\lambda},\underline{\mu})$ gives the result for $\mathrm{SO}^{2n+1}_{2n-1}$.

To prove for GL^n_{n-2}, we compare the multiplicity matrices of GL^n_{n-2} with SO^{2n}_{2n}. We showed all columns are the same for both matrices. The first $(n-2)$ columns and the last column are identical. The $(n-1)$-th column is also the same as $\left\{{\lambda_i+n-i \atop 1}\right\} = \binom{\lambda_i+n-i}{1}$, since $\lambda_i+n-i \ge 0$. Hence, multiplicity for GL^n_{n-2} is same with multiplicity for SO^{2n}_{2n}. This completes the proof of Corollary 7.2. □

Remark 7.3. *The multiplicity for the restriction problem from* Sp(2*n*) *to* Sp(2*n* − 2) *need not be 1, and is a bit complicated . The content of the above corollary in this case is that if* $\underline{\mu} = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} \ge 0)$*, and* $\underline{\lambda} = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0)$ *are highest weight for* Sp(2*n* − 2) *and* Sp(2*n*)*, then if* $\mu_{n-1} = 0$*, forcing* $x_n = 0$ *in the notation of Corollary* 7.2 *and therefore*

$$m_{\mathrm{Sp}(2n-2)}(\underline{\lambda},\underline{\mu}) = m_{\mathrm{GL}(n-2)}(\underline{\lambda},\underline{\mu}),$$

where now $\underline{\lambda}$ *is considered as a highest weight of* GL(*n*) *and* $\underline{\mu}$ *as a highest weight of* GL(*n* − 2) *as* $\mu_{n-1} = 0$*. More generally, we have the following equalities:*

$$m_{\mathrm{SO}(2n-2)}(\underline{\lambda},\underline{\mu}) = m_{\mathrm{Sp}(2n-2)}(\underline{\lambda},\underline{\mu}) = m_{\mathrm{SO}(2n-1)}(\underline{\lambda},\underline{\mu}) = m_{\mathrm{GL}(n-2)}(\underline{\lambda},\underline{\mu}),$$

where $\underline{\mu} = (\mu_1 \ge \mu_2 \ge \cdots \ge \mu_{n-1} \ge 0)$ *with* $\mu_{n-1} = 0$*.*

In Corollary 7.2, observe that x_n can be either λ_n or μ_{n-1} depending on the specific inequalities. For the pair GL^n_{n-2}, note that x_n must be zero since $\mu_{n-1} = 0$. Further, note that the multiplicity formula $m(\underline{\lambda},\underline{\mu})$ for the pair SO^{2n}_{2n-2} does not depend on x_n.

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