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A complete solution to the Cauchy dual subnormality problem for torally expansive toral 3-isometric weighted 2-shifts

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Abstract. In this paper, we present a complete solution to the Cauchy dual subnormality problem (for short CDSP) for torally expansive toral 3-isometric weighted 2-shifts. This solution is obtained by solving a couple of Hausdorff moment problems arising from 2-variable polynomials of lower bi-degree.

Contents

1. Introduction

The Cauchy dual subnormality problem (for short CDSP) in n-variables asks whether the Cauchy dual of an m-isometric n-tuple is jointly subnormal. This problem for 2-isometries has received significant attention, with extensive studies revealing intriguing links to moment theory and complex analysis (see [5, 10, 15]; for solutions to CDSP for various classes of m-isometries, see [3, Proposition 1.7], [5, Proposition 1.3],[10, Theorem 3.4], [15, Theorem 2.1], [18, Theorem 4.6] etc; for Brownian-type operators, see [16, Theorem 1.2]). In this paper, we present a complete solution to the Cauchy dual subnormality problem for torally expansive toral 3-isometric weighted 2-shifts. A special case of this (the case of separate 2-isometries) has been obtained in [6, Theorem 4.9]. Moreover, we present several families of Hausdorff moment net arising from the reciprocal of polynomials of bi-degree $(2, 1)$ and $(2, 2)$. Before we state the main result of this paper, let us fix some notations and recall the relevant notions.

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Let n be a positive integer and X be a set. The notation X^n represents the Cartesian product of X with itself, taken n times. Denote by $\mathbb{Z}_+$ and $\mathbb{R}_+$, the set of nonnegative integers and nonnegative real numbers, respectively. Let $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$. Set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $(\beta)_\alpha = \prod_{j=1}^n (\beta_j)_{\alpha_j}$, where $(\beta_j)_0 = 1$, $(\beta_j)_1 = \beta_j$ and

$$(\beta_j)_{\alpha_j} = \beta_j(\beta_j - 1)\cdots(\beta_j - \alpha_j + 1), \quad \alpha_j \geqslant 2, \ j = 1, \dots, n.$$

We denote $\alpha \leqslant \beta$ if $\alpha_j \leqslant \beta_j$ for every $j = 1, \dots, n$. For $\alpha \leqslant \beta$, we let $\binom{\beta}{\alpha} = \prod_{j=1}^n \binom{\beta_j}{\alpha_j}$.

For a net $\{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ and $j = 1, \dots, n$, let $\triangle_j$ denote the *forward difference operator* given by

$$\triangle_j a_\alpha = a_{\alpha + \varepsilon_j} - a_\alpha, \quad \alpha \in \mathbb{Z}_+^n,$$

where ε_j stands for the n-tuple with jth entry equal to 1 and 0 elsewhere. Note that for any $i, j \in \{1, 2, \dots, n\}$, $\triangle_i \triangle_j = \triangle_j \triangle_i$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, let $\triangle^\alpha$ denote the operator $\prod_{j=1}^n \triangle_j^{\alpha_j}$. For a polynomial p in one variable, let $\deg p$ denote the degree of p. A polynomial p of two variables is said to be of *bi-degree* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ if for each $j = 1, 2$, α_j is the largest integer for which $\partial_j^{\alpha_j} p \neq 0$.

We now recall the definition of joint complete monotonicity of a net. A net $\mathfrak{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is said to be *joint completely monotone* if

$$(-1)^{|\beta|} \triangle^\beta a_\alpha \geqslant 0, \quad \alpha, \beta \in \mathbb{Z}_+^n.$$

When $n = 1$, we simply refer to $\mathfrak{a}$ as a *completely monotone* sequence. We say $\mathfrak{a}$ is a *separate completely monotone* if for every $j \in \{1, \dots, n\}$, $k \in \mathbb{Z}_+$,

$$(-1)^k \triangle_j^k a_\alpha \geqslant 0, \quad \alpha \in \mathbb{Z}_+^n.$$

For a detailed account of complete monotonicity in one and several variables, the reader is referred to [11, 12, 13, 21].

Remark 1.1. It is readily seen that a joint completely monotone net is separate completely monotone. Also, if ϕ is completely monotone function on $\mathbb{Z}_+^n$, then for any $\beta \in \mathbb{Z}_+^n$, the function $\alpha \mapsto \phi(\alpha + \beta)$ is also completely monotone on $\mathbb{Z}_+^n$.

We now recall a solution to the multi-dimensional Hausdorff moment problem. A net $\mathfrak{a} = \{a_\alpha\}_{\alpha \in \mathbb{Z}_+^n}$ is joint completely monotone if and only if it is a *Hausdorff moment net*, that is, if there exists a positive Radon measure μ concentrated on $[0, 1]^n$ such that

$$a_\alpha = \int_{[0,1]^n} t^\alpha \mu(dt), \quad \alpha \in \mathbb{Z}_+^n.$$

(see [12, Proposition 4.6.11]). If such a measure μ exists, then it is unique. This is an outcome of the n-dimensional Weierstrass theorem and the Riesz representation theorem (see [23, Theorem 2.14] and [25, Lemma 4.11.3]). We refer to μ as the *representing measure* of $\mathfrak{a}$.

We now recall some operator-theoretic prerequisite. Let n be a positive integer. A operator tuple $T = (T_1, \dots, T_n)$ on a complex separable Hilbert space H is said to be *commuting n-tuple* if $T_1, \dots, T_n$ are bounded linear operator on H and $T_iT_j = T_jT_i$ for every $1 \leqslant i \neq j \leqslant n$. A commuting n-tuple T is said to be a *toral expansion* (resp. a *toral contraction*) if $T_j^*T_j \geqslant I$ (resp. $T_j^*T_j \leqslant I$) for every $j \in \{1, \dots, n\}$. We say that a commuting n-tuple $T = (T_1, \dots, T_n)$ is *jointly subnormal* if there exist a Hilbert space K containing H and a commuting n-tuple N of normal operators $N_1, \dots, N_n$ on K such that

$$T_j = N_j|_H, \quad j = 1, \dots, n.$$

Let m be a positive integer. Following [1, 8, 22], we say that a commuting n-tuple T is said to be a *toral m-isometry* if

$$\sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ 0 \leqslant \alpha \leqslant \beta}} (-1)^{|\alpha|} \binom{\beta}{\alpha} T^{*\alpha} T^{\alpha} = 0, \quad \beta \in \mathbb{Z}_+^n, \ |\beta| = m,$$

where T^α denotes the bounded linear operator $\prod_{j=1}^n T_j^{\alpha_j}$ and $T^{*\alpha}$ stands for the Hilbert space adjoint of T^α. The reader is referred to [1, 2, 8, 9, 14, 17, 24] for the basic theory of toral m-isometries.

Assume that $T_j^*T_j$ is invertible for every $j = 1, \dots, n$. Following [14, 24], we refer to the n-tuple $T^{\mathfrak{t}} := (T_1^{\mathfrak{t}}, \cdots, T_n^{\mathfrak{t}})$ as the *operator tuple torally Cauchy dual* to T where $T_j^{\mathfrak{t}} := T_j(T_j^*T_j)^{-1}$, for $j = 1, \cdots, n$. Note that *toral m-isometric tuple* $T = (T_1, \dots, T_n)$ is a *separate m-isometric tuple*, that is, $T_1, \dots, T_n$ are m-isometries. By [2, Lemma 1.21], T_j is left invertible for $1 \leqslant j \leqslant n$. Hence, the toral Cauchy dual of a toral m-isometric n-tuple exists.

Let $\mathscr{H}$ be a Hilbert space with orthonormal basis $\mathscr{E} = \{e_\alpha : \alpha \in \mathbb{Z}_+^n\}$. Let $\mathbf{w} = \{w_\alpha^{(j)} : j = 1, \dots, n, \ \alpha \in \mathbb{Z}_+^n\}$ be a collection of complex numbers. For $j = 1, \dots, n$ and any $\alpha \in \mathbb{Z}_+^n$, define $\mathscr{W} = (\mathscr{W}_1, \dots, \mathscr{W}_n)$ by $\mathscr{W}_j e_\alpha = w_\alpha^{(j)} e_{\alpha+\varepsilon_j}$, where ε_j is a vector with a 1 in the jth position and zeros elsewhere. Note that by extending it linearly on $\mathscr{E}$, $\mathscr{W}_1, \dots, \mathscr{W}_n$ define bounded operator on $\mathscr{H}$ if and only if $\sup_{\alpha \in \mathbb{Z}_+^n} |w_\alpha^{(j)}| < \infty$ for every $j = 1, \dots, n$. Also for any $i, j \in \{1, \dots, n\}$, $\mathscr{W}_i$ and $\mathscr{W}_j$ commute if and only if

$$w_\alpha^{(i)} w_{\alpha+\varepsilon_i}^{(j)} = w_\alpha^{(j)} w_{\alpha+\varepsilon_j}^{(i)}, \quad \alpha \in \mathbb{Z}_+^n.$$

Let $\mathscr{W}$ be a commuting weighted n-shift. Note that for any $\beta \in \mathbb{Z}_+^n$, there exists a positive scalar $m(\beta)$ such that

$$\mathscr{W}^\beta e_0 = m(\beta) e_\beta. \tag{1}$$

For more information on the basic theory of weighted multi-shifts, the reader is referred to [9, 19, 20].

In what follows, we assume that $\mathbf{w}$ forms a bounded subset of positive real numbers and $\mathscr{W}$ is a commuting n-tuple. We will denote the weighted n-shift $\mathscr{W}$ with weight multi-sequence $\mathbf{w}$ by $\mathscr{W} : w_\alpha^{(j)}$.

Let $\mathscr{W} : \{w_\alpha^{(j)}\}$ be a weighted n-shift such that $\mathscr{W}_j^*\mathscr{W}_j$ is invertible for each $j = 1, \dots, n$. The operator tuple $\mathscr{W}^{\mathfrak{t}}$ torally Cauchy dual to the weighted n-shift $\mathscr{W}$, satisfies the following relation:

$$\mathscr{W}_j^{\mathfrak{t}} e_\alpha = \frac{1}{w_\alpha^{(j)}} e_{\alpha+\varepsilon_j}, \quad j = 1, \dots, n. \tag{2}$$

It is now easy to see that:

$$\|(\mathscr{W}^{\mathfrak{t}})^\alpha e_0\|^2 = \frac{1}{\|\mathscr{W}^\alpha e_0\|^2}, \quad \alpha \in \mathbb{Z}_+^n. \tag{3}$$

To state the main result, we find it convenient to introduce the following notation: For $i, j \in \{0, 1, 2\}$,

$$\rho_{ij} = \triangle_1^i \triangle_2^j (\|\mathscr{W}^\alpha e_0\|^2)|_{\alpha=0}, \ \rho_1 = 2\rho_{10} - \rho_{20}, \ \rho_2 = 2\rho_{01} - \rho_{02}. \tag{4}$$

We are now ready to state the main result of this paper. For the sake of completeness, we include the separate 2-isometry case as part (a) below (see [6, Theorem 4.9]).

Theorem 1.2. *Let $\mathscr{W} : \{w_\alpha^{(j)}\}$ be a torally expansive toral 3-isometric weighted 2-shift and let $\mathscr{W}^{\mathfrak{t}}$ be the operator tuple torally Cauchy dual to $\mathscr{W}$. Let ρ_1, ρ_2 and $\rho_{ij}, i, j \in \{0, 1, 2\}$ be as given in (4). The following statements holds:*

(a) *Assume that $\mathscr{W}$ is a separate 2-isometry. Then the operator tuple $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if $\rho_{11} \leqslant \rho_{10}\rho_{01}$.*

(b) *Assume that $\mathscr{W}_1$ is not a 2-isometry. Then the operator tuple $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if $\rho_1 > 0, \rho_1^2 \geqslant 8\rho_{20}$, and exactly any one of the following holds:*
   (i) $\rho_{11} = 0, \rho_{01} = 0, \rho_{02} = 0,$
   (ii) $\rho_{11} > 0, \rho_2 > 0, \rho_{11}^2 \geqslant \rho_{20}\rho_{02},$

$$(\rho_{20}\rho_2 - \rho_{11}\rho_1)^2 \leqslant (4\rho_{11}^2 - \rho_{20}\rho_{02})(\frac{\rho_1^2}{4} - 2\rho_{20}).$$

(c) *Assume that $\mathscr{W}_2$ is not a 2-isometry. Then the operator tuple $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if $\rho_2 > 0, \rho_2^2 \geqslant 8\rho_{02}$, and exactly any one of the following holds:*
   (i) $\rho_{11} = 0, \rho_{10} = 0, \rho_{20} = 0,$
   (ii) $\rho_{11} > 0, \rho_1 > 0, \rho_{11}^2 \geqslant \rho_{20}\rho_{02},$

$$(\rho_{02}\rho_1 - \rho_{11}\rho_2)^2 \leqslant (4\rho_{11}^2 - \rho_{02}\rho_{20})(\frac{\rho_2^2}{4} - 2\rho_{02}).$$

Plan of the paper. In Section 2, we consider polynomial $p : \mathbb{R}_+^2 \to (0, \infty)$ of the form $p(x, y) = b_0(x+b_1)(x+b_2)+a_0(x+a_1)y$, where $a_0, a_1, b_0, b_1, b_2 \in \mathbb{R}$, with $b_1 \leqslant b_2$ and $a_0, a_1 \neq 0$. We describe all polynomials p for which $1/p$ is a joint completely monotone net (see Theorem 2.1). As a consequence of Theorem 2.1, we obtain some necessary conditions for the polynomial $q(x, y) = b_0(x+b_1)(x+b_2)+(a_1x+a_2)y$, whose reciprocal is a joint completely monotone net (see Corollary 2.3). In Section 3, we consider the polynomial $p(x, y) = a(x) + b(x)y + y^2$, where $a(x) = a_0(x + a_1)(x + a_2), b(x) = b_0(x + b_1)$, $a_0, a_1, a_2, b_0, b_1 \in \mathbb{R}$ with $a_1 \leqslant a_2$. Under the assumption $p(m, n) > 0$, we characterize the joint complete monotonicity of $\{1/p(m, n)\}_{m,n\in\mathbb{Z}_+}$ (see Theorem 3.1). Proof of this theorem is fairly long and requires several lemmas (see Lemmas 3.2-3.4). In Section 4, we provide a solution to the Cauchy dual subnormality problem for torally expansive toral 3-isometric weighted 2-shifts, which completes the proof of Theorem 1.2. Note that the proof of Theorem 1.2 relies on Theorems 2.1 and 3.1 and a characterization of toral 3-isometries (see Proposition 4.1).

2. A special case of bi-degree (2,1)

In this section, we present a proof of Theorem 2.1. The proof of the sufficiency part of this theorem is obtained in [6, Theorem 3.6]. Here, we obtain a proof of the necessity part.

Recall that for a positive real number ν, the *Bessel function $J_\nu(z)$ of the first kind of order ν* is given by

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu \sum_{k=0}^{\infty} \Big(\frac{-z^2}{4}\Big)^k \frac{1}{k!\Gamma(\nu + k + 1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0],$$

where Γ denotes the Gamma function.

Theorem 2.1 (Special case of bi-degree $(2, 1)$). *Let $p : \mathbb{R}_+^2 \to (0, \infty)$ be a polynomial given by $p(x, y) = b(x) + a(x)y$, where $a(x) = a_0(x + a_1)$ and $b(x) = b_0(x+b_1)(x+b_2)$, $a_0, a_1, b_0, b_1, b_2 \in \mathbb{R}$, with $b_1 \leqslant b_2$ and $a_0, a_1 \neq 0$. Then the net $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is joint completely monotone if and only if $b_1 \leqslant a_1 \leqslant b_2$.*

Proof. Since range of p is contained in $(0, \infty)$ and $a_0, a_1 \neq 0$, an elementary checking shows that $a_0, a_1, a_2, b_0, b_1 > 0$ (see discussion prior to [6, Proposition 3.2]). It was implicitly recorded in the proof of [6, Theorem 3.6] that for $m, n \in \mathbb{Z}_+$

$$\begin{aligned} \frac{1}{p(m,n)} \quad = \quad & \int_{[0,1]^2} t^n s_1^m \, \frac{(s_1/t^{c_0})^{a_1-1} t^{c_0(b_1+b_2-a_1)-1}}{a_0 t^{c_0}} \\ & \sum_{k=0}^{\infty} \frac{(-c_0c_1 \log (s_1/t^{c_0}) \log t)^k}{k!^2} \mathbb{1}_{[0,t^{c_0}]}(s_1) ds_1 dt, \end{aligned}$$

where $c_0 = b_0/a_0 > 0$ and $c_1 = (a_1 - b_2)(a_1 - b_1)$. So the weight function for the net $\left\{\frac{1}{p(m,n)}\right\}_{(m,n)\in\mathbb{Z}_+^2}$ is

$$w(s,t) = \frac{(s/t^{c_0})^{a_1-1} t^{c_0(b_1+b_2-a_1)-1}}{a_0 t^{c_0}} \sum_{k=0}^{\infty} \frac{(-c_2 \log(s/t^{c_0}) \log t)^k}{k!^2} \mathbb{1}_{[0,t^{c_0}]}(s),$$

where $s, t \in (0,1)$ and $c_2 = c_0 c_1$. Sufficiency part follows from [6, Theorem 3.6]. To prove the necessity part, assume that $a_1 \notin [b_1, b_2]$. We will show that $w(s,t) < 0$ on some open set contained in $(0,1)^2$. Since by the pasting lemma, $w(s,t)$ is continuous on $(0,1)^2$, it only require to show that $w(s,t) < 0$ for some $s, t \in (0,1)$. It now suffices to check that

$$\sum_{k=0}^{\infty} \frac{(-c_2 \log(s/t^{c_0}) \log t)^k}{k!^2} \mathbb{1}_{[0,t^{c_0}]}(s) < 0,$$

for some $s, t \in (0,1)$. Observe that $c_2 = c_0 c_1 > 0$. Take $t_0 = 1/2$ and $s_0 = \frac{e^{-\frac{5}{c_2 \log(2)}}}{2^{c_0}} < \frac{1}{2^{c_0}}$. It is easy to see that

$$\sum_{k=0}^{\infty} \frac{(-c_2 \log(s_0/t_0^{c_0}) \log t_0)^k}{k!^2} \mathbb{1}_{[0,t_0{}^{c_0}]}(s_0) = \sum_{k=0}^{\infty} \frac{(-5)^k}{k!^2} = J_0(2\sqrt{5}) \approx -0.3268,$$

where $J_0(x)$ is the Bessel function of the first kind of order 0. This, together with the continuity of $w(s,t)$ on $(0,1)^2$, implies that $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is not a joint completely monotone net. Therefore, we have $b_1 \leqslant a_1 \leqslant b_2$. This completes the proof. □

Next lemma is stated for frequent use (for a variant, see [5, Lemma 3.1]).

Lemma 2.2. *Let p be a polynomial of degree 2 given by $p(x) = a + bx + cx^2$, where $a, b, c \in \mathbb{R}$ such that $p(n) \neq 0, n \in \mathbb{Z}_+$. Then the sequence $\left\{\frac{1}{p(n)}\right\}_{n\in\mathbb{Z}_+}$ is completely monotone if and only if a, b, c are positive real numbers and p is reducible over $\mathbb{R}$.*

Proof. To see the proof of the necessity part, assume that $\left\{\frac{1}{p(n)}\right\}_{n\in\mathbb{Z}_+}$ is a completely monotone sequence. Note that $p(n) > 0, n \in \mathbb{Z}_+$, and hence $a > 0, c > 0$. An application of [4, Theorem 1.5] shows that

$$\text{the roots of } p \text{ lies in } \{z \in \mathbb{C} : \Re(z) < 0\}. \tag{5}$$

Let, if possible, p be irreducible. Since $p(0) > 0$, we must have $p(x) > 0$ for all $x \in \mathbb{R}$. An application of [3, Propositions 4.3] together with (5) shows that $\{1/p(n)\}_{n\in\mathbb{Z}_+}$ is not completely monotone. This contradiction shows that p is reducible over $\mathbb{R}$. Thus p has negative real roots, say, α_1 and α_2. Since $b = -c(\alpha_1 + \alpha_2)$, b is positive. For the proof of the sufficiency part, note that $\{1/p(n)\}_{n\in\mathbb{Z}_+}$

is the product of two completely monotone sequences and hence the sequence $\{1/p(n)\}_{n\in\mathbb{Z}_+}$ is completely monotone. $\square$

With Lemma 2.2, we can now obtain the following corollary.

Corollary 2.3. *Let q be a polynomial given by $q(x,y) = b(x) + a(x)y$, where $b(x) = b_0(x+b_1)(x+b_2)$, $a(x) = a_1x + a_2$, $a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$ with $b_1 \leqslant b_2$ such that $q(m,n) \neq 0, m,n \in \mathbb{Z}_+$. Then the following holds:*

(i) *if $q(m,n) > 0, m,n \in \mathbb{Z}_+$, then $a_1, a_2 \geqslant 0$,*

(ii) *if $\left\{\frac{1}{q(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is a joint complete monotone net then $b_0, b_1, b_2 > 0$ and $a_1, a_2 \geqslant 0$. Moreover, a_1 and a_2 are zero or positive real numbers simultaneously.*

Proof. Assume that $q(m,n) > 0, m,n \in \mathbb{Z}_+$. Let if possible $a_2 < 0$. Choose a large value $n_0 \in \mathbb{Z}_+$ such that $q(0,n_0) < 0$. This contradicts the assumption. Hence $a_2 \geqslant 0$. A similar argument can be used to see $a_1 \geqslant 0$. This completes the proof of (i). Assume that the net $\left\{\frac{1}{q(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is joint completely monotone. Thus, it is separate completely monotone. This implies $\left\{\frac{1}{q(m,0)}\right\}_{m\in\mathbb{Z}_+}$ is a completely monotone sequence. It now follows from $q(m,0) \neq 0, m \in \mathbb{Z}_+$, and Lemma 2.2, that $b_0, b_1, b_2 > 0$. Note that $q(m,n) > 0, m,n \in \mathbb{Z}_+$. By (i), we have $a_1, a_2 \geqslant 0$. We now consider two cases here:

Case 1. $a_1 = 0$. Let, if possible, $a_2 > 0$. In this case for large values of $n_0 \in \mathbb{Z}_+$, $q(.,n_0)$ is irreducible which contradicts the complete monotonicity of $\left\{\frac{1}{q(m,n_0)}\right\}_{m\in\mathbb{Z}_+}$. Hence $a_2 = 0$.

Case 2. $a_1 > 0$. Let, if possible, $a_2 = 0$. By Remark 1.1, for $k \in \mathbb{Z}_+$, $\left\{\frac{1}{q(m+k,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is a joint completely monotone net. By Theorem 2.1, $b_1 + k \leqslant k \leqslant b_2 + k$. This yields $b_1 \leqslant 0$, which is a contradiction. Hence, $a_2 > 0$. This completes the proof. $\square$

3. A special case of bi-degree (2,2)

In this section, we consider a class of polynomials of bi-degree $(2,2)$ and characterize the joint complete monotonicity of their reciprocals.

Theorem 3.1 (Special case of bi-degree $(2,2)$). *Let p be a polynomial given by $p(x,y) = a(x) + b(x)y + y^2$, where $a(x) = a_0(x+a_1)(x+a_2)$, $b(x) = b_0(x+b_1)$, $a_0, a_1, a_2, b_0, b_1 \in \mathbb{R}$ with $a_1 \leqslant a_2$. Assume that $p(m,n) > 0$ for every $m,n \in \mathbb{Z}_+$. Then $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is a joint completely monotone net if and only if $a_1, a_2, b_0, b_1 > 0$, $b_0^2 \geqslant 4a_0$,*

$$a_0(a_2 - a_1)^2 \leqslant b_0^2(b_1 - a_1)(a_2 - b_1). \tag{6}$$

The following lemma plays an important role in solving CDSP for torally expansive toral 3-isometric weighted 2-shifts.

Lemma 3.2. *Let p be a polynomial given by $p(x,y) = a(x) + b(x)y + y^2$, where $a(x) = a_0(x + a_1)(x + a_2)$, $b(x) = b_1 x + b_2$, $a_0, a_1, a_2, b_1, b_2 \in \mathbb{R}$ with $a_1 \leqslant a_2$ such that $p(m,n) \neq 0, m, n \in \mathbb{Z}_+$. Assume that $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is a joint complete monotone net. Then $a_0, a_1, a_2, b_1, b_2 > 0$.*

Proof. A similar argument as used in the proof of Corollary 2.3 shows that a_0, a_1 and a_2 are strictly positive real numbers. By symmetry, one can see that $b_2 > 0$. We now consider the following cases.

Case 1. $b_1 = 0$. Note that for large values of $m_0 \in \mathbb{Z}_+$, $p(m_0, .)$ is irreducible and in view of Lemma 2.2, this contradicts the complete monotonicity of $\left\{\frac{1}{p(m_0,n)}\right\}_{n\in\mathbb{Z}_+}$.

Case 2. $b_1 < 0$. Choose $m_0 \in \mathbb{Z}_+$ such that $b_1 m_0 + b_2 < 0$. Since $\left\{\frac{1}{p(m_0,n)}\right\}_{n\in\mathbb{Z}_+}$ is a complete monotone sequence, this contradicts Lemma 2.2.

Hence, $b_1 > 0$. This completes the proof. □

The following lemma provides necessary conditions for a class of polynomials in two variables whose reciprocal is joint completely monotone.

Lemma 3.3. *Let p be a polynomial in two variables given by $p(x,y) = q(x) + r(x)y + s(x)y^2$, where q, r and s are polynomials in one variable. Assume that $p(m,n) \neq 0$ for every $m, n \in \mathbb{Z}_+$. If the net $\{1/p(m,n)\}_{m,n\in\mathbb{Z}_+}$ is joint completely monotone, then*

$$4q(m)s(m) \leqslant r^2(m), \quad m \in \mathbb{Z}_+, \tag{7}$$

$$\deg(q) + \deg(s) \leqslant 2\deg(r). \tag{8}$$

Proof. Assume that the net $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is joint completely monotone. As noted earlier, $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is separate completely monotone. Therefore, by Lemma 2.2, for any $m \in \mathbb{Z}_+$, the roots of $p(m, .)$ are real numbers. Thus, we can apply the formula for the roots of a quadratic equation to obtain (7). Note that (7) yields (8). □

We need the following in the proof of the necessity part of Theorem 3.1.

Lemma 3.4. *Let p be a polynomial in two variables given by $p(x,y) = a(x) + b(x)y + y^2$, where a and b are polynomials in one variable. Assume that $p(m,n) \neq 0$ and $b^2(m) \neq 4a(m)$ for every $m, n \in \mathbb{Z}_+$. Let $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ be a joint completely monotone net. Then, for any positive real numbers α and β, the sequence $\left\{\frac{1}{p(m,\alpha m+\beta)}\right\}_{m\in\mathbb{Z}_+}$ is completely monotone.*

Proof. By Lemma 3.3 and the assumption that $b^2(m) \neq 4a(m)$ for every $m \in \mathbb{Z}_+$,

$$b^2(m) - 4a(m) > 0, \quad m \in \mathbb{Z}_+.$$

Also, for $m \in \mathbb{Z}_+$ and $y \in \mathbb{R}_+$,

$$p(m, y) = (y + r_1(m))(y + r_2(m)),$$

where r_1 and r_2 are given by

$$r_1(m) = \frac{b(m) + \sqrt{b^2(m) - 4a(m)}}{2}, \quad r_2(m) = \frac{b(m) - \sqrt{b^2(m) - 4a(m)}}{2}.$$

Note that for every $m \in \mathbb{Z}_+$ and $y \in \mathbb{R}_+$,

$$\begin{aligned} \frac{1}{p(m,y)} &= \frac{1}{(y + r_1(m))(y + r_2(m))} \\ &= \frac{1}{r_2(m) - r_1(m)} \left(\frac{1}{y + r_1(m)} - \frac{1}{y + r_2(m)} \right) \\ &= \int_{[0,1]} t^y \left(\frac{t^{r_1(m)-1} - t^{r_2(m)-1}}{r_2(m) - r_1(m)} \right) dt. \end{aligned}$$

Therefore,

$$\frac{1}{p(m,y)} = \int_{[0,1]} t^y w_m(t) dt, \quad m \in \mathbb{Z}_+, \ y \in \mathbb{R}_+, \tag{9}$$

where w_m is given by

$$w_m(t) = \frac{t^{r_1(m)-1} - t^{r_2(m)-1}}{r_2(m) - r_1(m)}, \quad t \in (0,1), m \in \mathbb{Z}_+.$$

Since the net $\left\{\frac{1}{p(m,n)}\right\}_{m,n \in \mathbb{Z}_+}$ is joint completely monotone, by (9)

$$(-1)^i \triangle_1^i \frac{1}{p(m,n)} = \int_{[0,1]} t^n (-1)^i \triangle_1^i w_m(t) dt \geqslant 0, \quad i, m, n \in \mathbb{Z}_+.$$

This, together with the complete monotonicity of $\left\{(-1)^i \triangle_1^i \frac{1}{p(m,n)}\right\}_{n \in \mathbb{Z}_+}$, $i, m \in \mathbb{Z}_+$, implies that for every $t \in (0,1)$,

$$(-1)^i \triangle_1^i w_m(t) \geqslant 0, \quad i, m \in \mathbb{Z}_+.$$

Therefore, for each $t \in (0,1)$, $\{w_m(t)\}_{m \in \mathbb{Z}_+}$ is completely monotone. Let α and β be positive real numbers. Note that for every $t \in (0,1)$, $\{t^{\alpha m + \beta}\}_{m \in \mathbb{Z}_+}$ is a completely monotone sequence. By [12, Lemma 8.2.1(v)] , for every $t \in (0,1)$, we have

$$(-1)^i \triangle^i t^{\alpha m + \beta} w_m(t) \geqslant 0, \quad m \in \mathbb{Z}_+.$$

This, combined with (9), yields

$$(-1)^i \triangle^i \frac{1}{p(m, \alpha m + \beta)} = \int_{[0,1]} (-1)^i \triangle^i t^{\alpha m+\beta} w_m(t)dt \geqslant 0, \quad m \in \mathbb{Z}_+.$$

This shows that $\left\{\frac{1}{p(m,\alpha m+\beta)}\right\}_{m\in\mathbb{Z}_+}$ is a completely monotone sequence. □

Proof of Theorem 3.1. Assume that the net $\left\{\frac{1}{p(m,n)}\right\}_{m,n\in\mathbb{Z}_+}$ is joint completely monotone. A routine calculation shows that

$$\begin{aligned} & b^2(m) - 4a(m) \\ = \ & (b_0^2 - 4a_0)m^2 + (2b_0^2 b_1 - 4a_0(a_1 + a_2))m + b_0^2 b_1^2 - 4a_0 a_1 a_2, \end{aligned} \tag{10}$$

which, by (7) (applied to $q = a$, $r = b$ and $s = 1$), is nonnegative for every $m \in \mathbb{Z}_+$. It follows that $b_0^2 - 4a_0 \geqslant 0$ and $b_0^2 b_1^2 - 4a_0 a_1 \geqslant 0$. By Lemma 3.2, $a_0, a_1, a_2, b_0, b_1 > 0$.

Before we prove the necessity part, consider the polynomials given by

$$f(m) = \frac{b(m)}{2}, \ g(m) = \frac{\sqrt{b^2(m) - 4a(m)}}{2}, \quad m \in \mathbb{Z}_+, \tag{11}$$

(g is real-valued since $b^2 \geqslant 4a$) and note that

$$\frac{1}{p(m,n)} = \frac{1}{(n + f(m))^2 - g^2(m)}, \quad m, n \in \mathbb{Z}_+. \tag{12}$$

We will divide the verification of (6) into the following cases.

Case 1. $\deg b^2 - 4a \leqslant 1$. If $\deg b^2 - 4a = 0$, then by (10), $b_0^2 = 4a_0$ and $2b_1 = a_1 + a_2$, and hence (6) holds. If possible, then assume that $b^2(m) - 4a(m)$ is a linear polynomial. By (10), $b_0^2 = 4a_0$, and hence for every $m \in \mathbb{Z}_+$,

$$\begin{aligned} b^2(m) - 4a(m) &= b_0^2\big(2b_1 - (a_1 + a_2)\big)m + b_0^2(b_1^2 - a_1 a_2) \\ &= c_0 m + c_1, \end{aligned} \tag{13}$$

where $c_0 = b_0^2\big(2b_1 - (a_1 + a_2)\big)$ and $c_1 = b_0^2(b_1^2 - a_1 a_2)$. Since $b^2(m) - 4a(m)$ is a nonnegative linear polynomial (see (7)), we have

$$a_1 + a_2 < 2b_1. \tag{14}$$

A routine calculation using (11) and (13) shows that for $m, n \in \mathbb{Z}_+$,

$$\begin{aligned} p(m,n) &\overset{(12)}{=} (n + f(m))^2 - g^2(m) \\ &= \Big(\frac{b_0 m}{2} + \frac{\frac{b_0^2 b_1}{2} + b_0 n - \frac{c_0}{4}}{b_0}\Big)^2 + b_0\Big(b_1 - \frac{(a_1 + a_2)}{2}\Big)n \\ &+ \frac{b_0^2 b_1^2}{4} - \frac{\Big(\frac{b_0^2 b_1^2}{2} - \frac{c_0}{4}\Big)^2}{b_0^2}. \end{aligned}$$

Since $a_1 + a_2 < 2b_1$ (see (14)) and $b_0 > 0$, we note that there exists $n_0 \in \mathbb{Z}_+$ such that

$$\frac{\left(\frac{b_0^2 b_1^2}{2} - \frac{c_0}{4}\right)^2}{b_0^2} - \frac{b_0^2 b_1^2}{4} < b_0\Big(b_1 - \frac{(a_1+a_2)}{2}\Big)n, \quad n \geqslant n_0.$$

It follows that $p(m, n_0)$ is irreducible in m, and hence $\left\{\frac{1}{p(m,n)}\right\}_{(m,n)\in\mathbb{Z}_+^2}$ is not separate completely monotone. Hence, $\deg b^2 - 4a = 0$, which completes the proof in this case.

Case 2. $b^2(x) - 4a(x)$ is a quadratic polynomial. Note that by (7) and (10), $b_0^2 - 4a_0 > 0$. It is easy to see using (11) that for every $m \in \mathbb{Z}_+$,

$$g^2(m) \;=\; (c_0 m + c_1)^2 + c_2,$$

where c_0, c_1, c_2 are given by

$$\begin{aligned} c_0 &= \frac{\sqrt{b_0^2 - 4a_0}}{2}, \quad c_1 = \frac{b_0^2 b_1 - 2a_0(a_1+a_2)}{2\sqrt{b_0^2-4a_0}}, \\ c_2 &= -\frac{a_0(a_2-a_1)^2 - b_0^2(b_1-a_1)(a_2-b_1)}{b_0^2 - 4a_0}. \end{aligned} \tag{15}$$

Now, we choose a very large $\alpha_0 \in \mathbb{Z}_+$ such that $c_0\alpha_0 + c_1 > 0$ and $b^2(m+\alpha_0) \neq 4a(m+\alpha_0)$ for every $m \in \mathbb{Z}_+$. We also choose a very large natural number, say $N_0 > 1$, such that

$$l_1 := N_0 c_0 - \frac{b_0}{2} > 0, \quad l_2 := N_0(c_0\alpha_0 + c_1) - \frac{b_0\alpha_0}{2} - \frac{b_0 b_1}{2} > 0. \tag{16}$$

Take $n = l_1 m + l_2$ and consider

$$\begin{aligned} & \frac{1}{p(m+\alpha_0, l_1 m + l_2)} \\ \overset{(12)}{=} \; & \frac{1}{(l_1 m + l_2 + f(m+\alpha_0))^2 - g^2(m+\alpha_0)} \\ \overset{(11),(16)}{=} \; & \frac{1}{(N_0 c_0 m + N_0(c_0\alpha_0 + c_1))^2 - (c_0 m + c_0\alpha_0 + c_1)^2 - c_2} \\ = \; & \frac{1}{(N_0^2 - 1)(c_0 m + c_0\alpha_0 + c_1)^2 - c_2}. \end{aligned}$$

Assume that (6) does not hold. By (15), we obtain $c_2 < 0$. Therefore, the polynomial $(N_0^2 - 1)(c_0 m + c_0\alpha_0 + c_1)^2 - c_2$ is irreducible in m. One may see, using Lemma 2.2 that the sequence $\left\{\frac{1}{p(m+\alpha_0, l_1 m + l_2)}\right\}_{m\in\mathbb{Z}_+}$ is not completely monotone. This is not possible in view of Lemma 3.4 and Remark 1.1.

This completes the proof of the necessity part.

We will divide the verification of the sufficiency part into several cases.

Case 1. $b^2(m) - 4a(m)$ is a constant. By (10), we have $2b_1 = a_1 + a_2$ and $b_0^2 = 4a_0$. This implies for every $m \in \mathbb{Z}_+$,

$$b^2(m) - 4a(m) \overset{(10)}{=} b_0^2 b_1^2 - 4a_0 a_1 a_2 = a_0(a_1 - a_2)^2 \geqslant 0.$$

It follows that for $m, n \in \mathbb{Z}_+$,

$$p(m,n) \overset{(12)}{=}$$

$$\Big(n + \frac{b_0}{2}m + \frac{b_0 b_1}{2} + \frac{\sqrt{b_0^2 b_1^2 - 4a_0 a_1 a_2}}{2}\Big)\Big(n + \frac{b_0}{2}m + \frac{b_0 b_1}{2} - \frac{\sqrt{b_0^2 b_1^2 - 4a_0 a_1 a_2}}{2}\Big).$$

Clearly, $\frac{b_0 b_1}{2} + \frac{\sqrt{b_0^2 b_1^2 - 4a_0 a_1 a_2}}{2} \geqslant 0$ and $\frac{b_0 b_1}{2} - \frac{\sqrt{b_0^2 b_1^2 - 4a_0 a_1 a_2}}{2} \geqslant 0$. In this case, $\{1/p(m,n)\}_{m,n \in \mathbb{Z}_+}$ is a joint completely monotone net since it is the product of two joint completely monotone net (see [12, Lemma 8.2.1(v)]).

Case 2. $b^2(m) - 4a(m)$ is a linear polynomial. Note that from (6),

$$a_0(a_2 - a_1)^2 \leqslant b_0^2(b_1 - a_1)(a_2 - b_1).$$

Since $b^2(m) - 4a(m)$ is a linear polynomial, we have $4a_0 = b_0^2$, and hence

$$(a_2 - b_1 + b_1 - a_1)^2 \leqslant 4(b_1 - a_1)(a_2 - b_1).$$

It now follows that

$$(a_2 - b_1)^2 + (b_1 - a_1)^2 + 2(a_2 - b_1)(b_1 - a_1) \leqslant 4(b_1 - a_1)(a_2 - b_1),$$

which clearly yields $(a_2 - 2b_1 + a_1)^2 \leqslant 0$, or equivalently, $a_1 + a_2 = 2b_1$. Thus, this case reduces to that of (1). Therefore, the net $\left\{\frac{1}{p(m,n)}\right\}_{m,n \in \mathbb{Z}_+}$ is joint completely monotone.

Case 3. $b^2(x) - 4a(x)$ is a quadratic polynomial. Note that $b_0^2 > 4a_0$. For every $m \in \mathbb{Z}_+$, we already noted that $g^2(m) = (c_0 m + c_1)^2 + c_2$, where c_0, c_1, c_2 are given by

$$\begin{aligned} c_0 &= \frac{\sqrt{b_0^2 - 4a_0}}{2}, \quad c_1 = \frac{b_0^2 b_1 - 2a_0(a_1 + a_2)}{2\sqrt{b_0^2 - 4a_0}}, \\ c_2 &= -\frac{a_0(a_2 - a_1)^2 - b_0^2(b_1 - a_1)(a_2 - b_1)}{b_0^2 - 4a_0}. \end{aligned} \tag{17}$$

Also note that by (12), for $m, n \in \mathbb{Z}_+$,

$$\begin{aligned} \frac{1}{p(m,n)} &= \frac{1}{(n+b(m)/2)^2 - ((c_0 m + c_1)^2 + c_2)} \\ &= \frac{1}{(n+\frac{b(m)}{2} + c_0 m + c_1)(n + \frac{b(m)}{2} - c_0 m - c_1) - c_2} \\ &= \frac{1}{p_1(m,n)p_2(m,n) - c_2}, \end{aligned}$$

where p_1 and p_2 are given by

$$\begin{aligned} p_1(m,n) &:= n + (b_0/2 + c_0)m + (b_0 b_1/2 + c_1), \quad m,n \in \mathbb{Z}_+, \\ p_2(m,n) &:= n + (b_0/2 - c_0)m + (b_0 b_1/2 - c_1), \quad m,n \in \mathbb{Z}_+. \end{aligned}$$

By (6) and (17), $c_2 \geqslant 0$. If $c_1 \geqslant 0$, then $b_0 b_1/2 + c_1 \geqslant 0$, and since

$$(b_0 b_1/2 + c_1)(b_0 b_1/2 - c_1) = p_1(0,0)p_2(0,0) > c_2 \geqslant 0,$$

we must have $b_0 b_1/2 - c_1 > 0$. Similarly, if $c_1 < 0$, then $b_0 b_1/2 - c_1 > 0$, and hence $b_0 b_1/2 + c_1 > 0$. Thus, for any real value of c_1, $\{1/p_1(m,n)\}_{m,n\in\mathbb{Z}_+}$ and $\{1/p_2(m,n)\}_{m,n\in\mathbb{Z}_+}$ are joint completely monotone nets (see [6, Theorem 3.1]). Note that for $m, n \in \mathbb{Z}_+$, $p(m,n) = p_1(m,n)p_2(m,n) - c_2 > 0$. Thus, we have

$$\frac{c_2}{p_1(m,n)p_2(m,n)} < 1, \quad m,n \in \mathbb{Z}_+. \tag{18}$$

Therefore, for all $m, n \in \mathbb{Z}_+$,

$$\begin{aligned} \frac{1}{p_1(m,n)p_2(m,n) - c_2} &= \frac{1}{p_1(m,n)p_2(m,n)}\Big(\frac{1}{1 - \frac{c_2}{p_1(m,n)p_2(m,n)}}\Big) \\ &\overset{(18)}{=} \sum_{k=0}^{\infty} \frac{c_2^k}{(p_1(m,n)p_2(m,n))^{k+1}}. \end{aligned}$$

Since, for each $k \in \mathbb{Z}_+$, $\left\{\frac{c_2^k}{(p_1(m,n)p_2(m,n))^{k+1}}\right\}_{m,n\in\mathbb{Z}_+}$ is a joint completely monotone net, the finite sum $\left\{\sum_{k=0}^{\ell} \frac{c_2^k}{(p_1(m,n)p_2(m,n))^{k+1}}\right\}_{m,n\in\mathbb{Z}_+}$, where $\ell \in \mathbb{Z}_+$, is also joint completely monotone. Since the limit of the joint completely monotone net is joint completely monotone (see [12, p. 130]), we conclude that the net $\left\{\sum_{k=0}^{\infty} \frac{c_2^k}{(p_1(m,n)p_2(m,n))^{k+1}}\right\}_{m,n\in\mathbb{Z}_+}$ is joint completely monotone. This completes the proof of the sufficiency part. □

4. The Cauchy dual subnormality problem

In this section, we present a proof of the Theorem 1.2. We begin with the following proposition which is a consequence of [6, Proposition 4.6].

Proposition 4.1. *For a weighted 2-shift $\mathscr{W} : \{w_\alpha^{(j)}\}$, the following statements are valid:*

(i) *$\mathscr{W}$ is a toral 3-isometry if and only if for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,*

$$\|\mathscr{W}^\alpha e_0\|^2 = 1 + a_1\alpha_1 + a_2\,\alpha_1^2 + (b_1\alpha_1 + b_2)\alpha_2 + c_1\,\alpha_2^2,$$

where a_1, a_2, b_1, b_2 and c_1 are as follows:

$$a_1 = \rho_{10} - \frac{\rho_{20}}{2},\ a_2 = \frac{\rho_{20}}{2},\ b_1 = \rho_{11},\ b_2 = \rho_{01} - \frac{\rho_{02}}{2},\ c_1 = \frac{\rho_{02}}{2}. \tag{19}$$

(ii) *$\mathscr{W}$ is a toral 3-isometry with $\mathscr{W}_2$ being a 2-isometry if and only if for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,*

$$\|\mathscr{W}^\alpha e_0\|^2 = 1 + a\,\alpha_1 + b\,\alpha_1^2 + (c + d\,\alpha_1\,)\,\alpha_2,$$

where a, b, c and d are as follows:

$$a = \rho_{10} - \frac{\rho_{20}}{2}, \quad b = \frac{\rho_{20}}{2}, \quad c = \rho_{01}, \quad d = \rho_{11}.$$

Proof. This follows from [6, Proposition 4.6] (the case of $m = 3$). □

The following proposition reveals a relation between joint subnormality of the toral Cauchy dual and joint complete monotonicity.

Proposition 4.2. *Let $\mathscr{W}$ be a torally expansive weighted n-shift and let $\mathscr{W}^{\mathfrak{t}}$ be its toral Cauchy dual. Then $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if $\left\{\frac{1}{\|\mathscr{W}^\alpha e_0\|^2}\right\}_{\alpha\in\mathbb{Z}_+^n}$ is a joint completely monotone net.*

Proof. Recall that for a torally contractive weighted n-shift U, the following holds:

$$U \text{ is jointly subnormal if and only if } \{\|U^\alpha e_0\|^2\}_{\alpha\in\mathbb{Z}_+^n} \text{ is a joint completely monotone net.} \tag{20}$$

This fact can be deduced from [7, Theorem 4.4], together with (1) (see also the discussion prior to [8, Eqn (E)]). By (2) and the discussion following it, $\mathscr{W}^{\mathfrak{t}}$ is a commuting weighted n-shift. Since $\mathscr{W}$ is torally expansive, routine calculations show that $\mathscr{W}^t$ is torally contractive. This, combined with (3) and (20), completes the proof of the proposition. □

We now present a solution to the CDSP for torally expansive toral 3-isometric weighted 2-shifts.

Proof of Theorem 1.2. By Proposition 4.1(i), for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,

$$\|\mathscr{W}^\alpha e_0\|^2 = 1 + a_1\alpha_1 + a_2\alpha_1^2 + (b_1\alpha_1 + b_2)\alpha_2 + c_1\alpha_2^2, \tag{21}$$

where a_1, a_2, b_1, b_2 and c_1 are given by

$$a_1 = \rho_{10} - \frac{\rho_{20}}{2}, \quad a_2 = \frac{\rho_{20}}{2}, \quad b_1 = \rho_{11}, \quad b_2 = \rho_{01} - \frac{\rho_{02}}{2}, \quad c_1 = \frac{\rho_{02}}{2}.$$

It is clear from (21) that $a_2 \geqslant 0$ and $c_1 \geqslant 0$. Let $p(x,y) = 1 + a_1 x + a_2 x^2 + (b_1 x + b_2)y + c_1 y^2$. By (3), $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$,

$$\|(\mathscr{W}^{\mathfrak{t}})^{\alpha} e_0\|^2 = \frac{1}{p(\alpha_1, \alpha_2)}.$$

We record that for every $\alpha_1, \alpha_2 \in \mathbb{Z}_+$, $p(\alpha_1, \alpha_2) > 0$.

(a) This has been proved in [6, Theorem 4.9].

(b) Since $\mathscr{W}_1$ is not a 2-isometry, $a_2 > 0$. We divide the proof in two cases.

Case 1. $\mathscr{W}_2$ is a 2-isometry. Note that $c_1 = 0$. By Corollary 2.3(i), $b_1, b_2 \geqslant 0$. For the necessity part, assume that $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal. By Proposition 4.2, $\left\{\frac{1}{p(\alpha_1,\alpha_2)}\right\}_{\alpha_1,\alpha_2 \in \mathbb{Z}_+}$ is a joint completely monotone net. Hence, it is separate completely monotone. Applying Lemma 2.2 to the polynomial $p(x,0)$, we obtain $a_1^2 \geqslant 4a_2$ and $a_1 > 0$. This yields $(2\rho_{10} - \rho_{20})^2 \geqslant 8\rho_{20}$ and $2\rho_{10} > \rho_{20}$. In view of Corollary 2.3(ii), either $b_1, b_2 = 0$ or $b_1, b_2 > 0$. If $b_1, b_2 = 0$, we are done. If $b_1, b_2 > 0$, we apply Theorem 2.1 to complete the proof of the necessity part.

To see the sufficiency part, assume that $(2\rho_{10} - \rho_{20})^2 \geqslant 8\rho_{20}$ and $2\rho_{10} > \rho_{20}$. This is equivalent to $a_1 > 0$ and $a_1^2 \geqslant 4a_2$. Since $a_1^2 \geqslant 4a_2$, we have

$$\begin{aligned} p(x,y) &= a_2\Big(x + \frac{a_1 - \sqrt{a_1^2 - 4a_2}}{2a_2}\Big)\Big(x + \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2a_2}\Big) \\ &\quad +(b_1 x + b_2)y. \end{aligned}$$

We consider the following two subcases to complete the proof of sufficiency part in this case.

First, $\rho_{11} = 0, \rho_{01} = 0, \rho_{02} = 0$ (equivalently, $b_1 = 0, b_2 = 0$). Note that $\left\{\frac{1}{p(\alpha_1,\alpha_2)}\right\}_{\alpha_1,\alpha_2 \in \mathbb{Z}_+}$ is the product of two completely monotone sequences and hence it is joint completely monotone. By Proposition 4.2, $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal.

Second, $\rho_{11} > 0, 2\rho_{01} > \rho_{02}$ and

$$(\rho_{20}\rho_2 - \rho_{11}\rho_1)^2 \leqslant (4\rho_{11}^2 - \rho_{20}\rho_{02})(\frac{\rho_1^2}{4} - 2\rho_{20})$$

where $\rho_1 = 2\rho_{10} - \rho_{20}$ and $\rho_2 = 2\rho_{01} - \rho_{02}$ (equivalently, $b_1 > 0, b_2 > 0$, and $(2a_2 b_2 - a_1 b_1)^2 \leqslant b_1^2(a_1^2 - 4a_2)$). A routine calculation yields that

$$\frac{a_1 - \sqrt{a_1^2 - 4a_2}}{2a_2} \leqslant \frac{b_2}{b_1} \leqslant \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2a_2}.$$

We now apply Theorem 2.1 and Proposition 4.2 to complete the proof of the sufficiency part in this case.

Case 2. $\mathscr{W}_2$ is not a 2-isometry. Note that $c_1 > 0$. By Proposition 4.2, $\mathscr{W}^{\mathfrak{t}}$ is jointly subnormal if and only if the net $\left\{\frac{1}{p(\alpha_1,\alpha_2)}\right\}_{\alpha_1,\alpha_2 \in \mathbb{Z}_+}$ is joint completely monotone or equivalently

$$\left\{\frac{1}{1/c_1 + (a_1/c_1)\alpha_1 + (a_2/c_1)\alpha_1^2 + ((b_1\alpha_1 + b_2)/c_1)\alpha_2 + \alpha_2^2}\right\}_{\alpha_1,\alpha_2\in\mathbb{Z}_+}$$

is a joint completely monotone net. The proof of the necessity part now follows from Lemmas 2.2, 3.2 and Theorem 3.1. Sufficiency follows from Theorem 3.1.

(c) This follows from part (b), by interchanging the role of $\mathscr{W}_1$ and $\mathscr{W}_2$. □

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