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Contraction property of Fock type space of log-subharmonic functions in $\mathbb{R}^m$

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ABSTRACT. We prove a contraction property of Fock type spaces $\mathcal{L}^p_\alpha$ of log-subharmonic functions in $\mathbb{R}^n$. To prove the result, we demonstrate a certain monotonic property of measures of the superlevel set of the function $u(x) = |f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}$, provided that f is a certain log-subharmonic function in $\mathbb{R}^m$. The result recover a contraction property of holomorphic functions in the Fock space $\mathcal{F}^p_\alpha$ proved by Carlen in [Car1991].

CONTENTS

1. Introduction

Let $m \geqslant 1$ and let $\mathbb{R}^m$ be the Euclidean space endowed with the Euclidean norm: $|x| = \sqrt{\langle x, x\rangle}$, where $\langle x, y\rangle = \sum_{i=1}^m x_i y_i$, and $x = (x_1, \dots, x_m), y = (y_1, \dots, y_n) \in \mathbb{R}^m$. If $\alpha > 0$ and $p > 0$ and $m = 2n$ is an even integer, we define the Fock space or Segal-Bargmann space $\mathcal{F}^p_\alpha$ (cf. [Bar62, Bar61, KZ2012]) of entire holomorphic functions f in $\mathbb{C}^n = \mathbb{R}^{2n}$ so that:

$$\|f\|^p_{p,\alpha} := c_{p,\alpha} \int_{\mathbb{R}^m} |f(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dA(x) < \infty,$$

where

$$c_{p,\alpha} = \left(\frac{\alpha p}{2\pi}\right)^{\frac{m}{2}}, \tag{1.1}$$

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and $dA(x)$ is Lebesgue measure on $\mathbb{R}^m$. Note that $c_{p,\alpha}e^{-\frac{\alpha}{2}p|x|^2}dA(x)$ is the Gaussian probabily measure in $\mathbb{R}^m$.

Assume now that $m \in \mathbb{N}$ is an arbitrary integer. We say that a real twice differentiable function f defined in a domain $\Omega \subset \mathbb{R}^m$ is subharmonic if $\Delta f(x) \geqslant 0$ for $x \in \Omega$. Here, Δ is the Laplacian. This definition can also be extended to not necessary double differentiable functions, by using the sub-mean value property ([HK1976]). We say that a mapping f is log-subharmonic, if $\log|f(x)|$ is subharmonic in $\Omega \setminus f^{-1}(0)$. We denote by $\mathcal{L}^p_\alpha$ the space of complex-valued, real-analytic functions whose absolute value is a log-subharmonic function, defined in $\mathbb{R}^m$, with a finite $\|f\|_{p,\alpha}$ norm as defined in (1). Here, m is an arbitrary positive integer. Observe that for $m = 2n$ we have $\mathcal{F}^p_\alpha \subset \mathcal{L}^p_\alpha$: If f is holomorphic in Ω, then $|f(z)|$ is log-subharmonic. Indeed

$$\Delta \log|f(z)| = \sum_{k=1}^{n} \Delta_{z_k} \log|f(z)| = 0,$$

where $z = (z_1, \dots, z_n)$, and

$$\Delta_{z_k} = \frac{\partial^2}{(\partial_{x_k})^2} + \frac{\partial^2}{(\partial_{y_k})^2},$$

$z_k = x_k + iy_k$ for $k = 1, \dots, n$ and $z \in \Omega \setminus f^{-1}(0)$.

2. Motivation and main results

Carlen, in his paper [Car1991] proved the following result:

Theorem 2.1. *If $0 < p < q < \infty$, then $\mathcal{F}^p_\alpha(\mathbb{C}^n) \subset \mathcal{F}^q_\alpha(\mathbb{C}^n)$ and the inclusion is proper and continuous. Moreover*

$$\|f\|_{q,\alpha} \leq \|f\|_{p,\alpha}.$$

Theorem 2.1 is applied in [Car1991] to the coherent state transform in a new proof of Wehrl's entropy conjecture [LIEB1978]. In this paper, among other results, we recover Theorem 2.1 and provide a proof that works for a more general class of mappings, namely real analytic complex mappings whose absolute value is a log-subharmonic function in $\mathbb{R}^m$ and belongs to the Fock-type space $\mathcal{L}^p_\alpha$.

Let f be a real analytic complex-valued function defined in the Euclidean space $\mathbb{R}^m$, such that $v = |f|$ is a log-subharmonic function in $\mathbb{R}^m$ and such that $u(x) = v(x)^p e^{-\alpha p/2|x|^2}$ is bounded and goes to 0 uniformly as $|x| \to \infty$. Then the superlevel sets $A_t = \{x : u(x) > t\}$ for $t > 0$ are compactly embedded in $\mathbb{R}^m$ and thus have finite Lebesgue measure $\mu(t) = |A_t|$.

Those are the main results:

Theorem 2.2. *Let $\alpha > 0$ and $p > 0$ and assume that f is a real analytic complex valued function such that $v = |f| : \mathbb{R}^m \to [0, +\infty)$ is a log-subharmonic function. Assume further that the function $u(x) = |f(x)|^p e^{-\frac{\alpha p}{2}|x|^2}$ is bounded and*

$u(x)$ tends to 0 uniformly as $|x| \to \infty$. Then the function

$$g(t) = t \exp\left[\frac{\alpha p(\Gamma(m/2))^{2/m}}{2\pi}\mu^{2/m}(t)\right],$$

is decreasing on the interval $(0, t_\circ)$, where $t_\circ = \max_{x\in\mathbb{R}^m} u(x)$.

If $f(x) \equiv 1$, the function g turns out to be constant and this is an important property of g.

The proof of this theorem is mostly based on the methods developed by Nicola and Tilli in [NT2022] (see also the subsequent papers where similar methods are used: [KU2022], [KA2024], [RT2023], [KNOT2022], and [Fr2023]).

By using Theorem 2.2, we will prove the following theorem:

Theorem 2.3. *Let $p > 0$ and $\alpha > 0$. Let $G : [0, \infty) \to \mathbb{R}$ be a convex function. Then the maximum value of*

$$\int_{\mathbb{R}^m} G(|f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}) dA(x) \tag{2.1}$$

is attained for

$$f_a(x) = e^{\alpha\langle a,x\rangle - \frac{\alpha}{2}|a|^2},$$

where $a \in \mathbb{C}^n$ is arbitrary, subject to the condition that $f \in \mathcal{L}^p_\alpha$ and $\|f\|_{p,\alpha} = 1$.

Applying Theorem 2.3 to the convex and increasing function $G(t) = t^{q/p}$, we get the extension of theorem [Car1991, Theorem 2] by proving:

Theorem 2.4. *For all $0 < p < q < \infty$ and $0 < \alpha$ and for $f \in \mathcal{L}^p_\alpha(\mathbb{R}^m)$, we have $f \in \mathcal{L}^q_\alpha(\mathbb{R}^m)$ and*

$$\|f\|_{q,\alpha} \le \|f\|_{p,\alpha}$$

with equality for $f_a(x) = e^{\alpha\langle a,x\rangle - \frac{\alpha}{2}|a|^2}$, where $a \in \mathbb{R}^m$ is arbitrary.

Proof of Theorem 2.4. For $\|f\|_{p,\alpha} = N$, $\|f/N\|_{p,\alpha} = 1$ and from Theorem 2.3 we have

$$\int_{\mathbb{R}^m} |f(x)/N|^q e^{-\frac{\alpha}{2}q|x|^2} dA(x) \le \int_{\mathbb{R}^m} e^{-\frac{\alpha}{2}q|x|^2} dA(x) = 1/c_{q,\alpha}.$$

Thus,

$$c_{q,\alpha} \int_{\mathbb{R}^m} |f(x)|^q e^{-\frac{\alpha}{2}q|x|^2} dA(x) \le N^q,$$

or what is the same

$$\|f\|_{q,\alpha} \le \|f\|_{p,\alpha}.$$

The equality statement follows from the equality statement of Theorem 2.4, but can be proved by using the same approach as in the monograph of Zhu [KZ2012, Lemma 2.33]. □

Remark 2.5. *The last theorem is an extension of Theorem 2.1. Moreover, its proof is different from the proof in [Car1991] and seems to be simpler. We refer to the paper [GKL2010] for some related inequalities for log-subharmonic functions in* $\mathbb{R}^n$.

Theorem 2.4 is a counterpart of a similar contraction property of Bergman spaces $\mathbf{B}^p_\alpha$ ([HKZ2000, p. 2]), proved by Kulikov in [KU2022] for holomorphic functions in the unit disk and for $\mathcal{M}-$log-subharmonic functions in the unit ball in $\mathbb{R}^n$ by the author in [KA2024]. It is known that

$$\mathbf{B}^p_\alpha \subset \mathbf{B}^q_\beta, \quad \frac{p}{\alpha} = \frac{q}{\beta} = r, \quad p < q.$$

For $n = 2$, it was asked whether these embeddings are contractions; that is, whether the norm $\|f\|_{\mathbf{B}^{r\alpha}_\alpha}$ is decreasing in α. In the case of Bergman spaces, this question was asked by Lieb and Solovej [LiSo2021]. They proved that such contractivity implies their Wehrl-type entropy conjecture for the $SU(1,1)$ group. In the case of contractions from the Hardy spaces to the Bergman spaces, it was asked by Pavlović in [MP2014] and by Brevig, Ortega-Cerdà, Seip, and Zhao [BOSZ2018] concerning the estimates for analytic functions. The mentioned contraction property proved by Kulikov confirmes these conjectures. An interesting application of Kulikov result has been given by Melentijević in [PM2023].

We end this paper with the construction of a new normed Fock type space:

Definition 2.6 (Fock limit space)**.** *Let f be a holomorphic function in $\mathbb{C}^n$. Then for $\alpha > 0$ we say $f \in \mathcal{F}_\alpha$ if $f \in \bigcap_{p>0} \mathcal{F}^p_\alpha$. Then we define*

$$\|f\|_\alpha := \inf_{p>0} \|f\|_{p,\alpha}.$$

For $\alpha > 0$ define as in [KZ2012, eq. 2.2] the following Banach norm

$$\|f\|_{\infty,\alpha} := \operatorname{esssup}\{|f(z)|e^{-\frac{\alpha}{2}|z|^2}, z \in \mathbb{C}^n\}.$$

Then, we prove

Theorem 2.7. *For every $\alpha > 0$ we have*

$$\|f\|_\alpha = \|f\|_{\infty,\alpha}.$$

In particular $(\mathcal{F}_\alpha, \|\cdot\|_\alpha)$ is a normed subspace of Banach space $\mathcal{F}^\infty_\alpha$.

3. Proof of Theorem 2.2

Proof of Theorem 2.2. We start with the formula

$$\mu(t) = |A_t| = \int_{A_t} dx = \int_t^{\max u} \int_{|u(x)|=\kappa} d\mathcal{H}^{m-1}(x) d\kappa.$$

Then we get

$$-\mu'(t) = \int_{u=t} |\nabla u|^{-1} d\mathcal{H}^{m-1}(x) \tag{3.1}$$

along with the claim that $\{x : u(x) = t\} = \partial A_t$ and that this set is a smooth hypersurface for almost all $t \in (0, t_\circ)$. Here, $dS = d\mathcal{H}^{m-1}$ is $m-1$ dimensional Hausdorff measure. These assertions follow the proof of [NT2022, Lemma 3.2]. We point out that, since u is real analytic, then it is a well-known fact from measure theory that the level set $\{x : u(x) = t\}$ has a zero measure ([MI2020]), and this is equivalent to the fact that the μ is continuous.

Following the approach from [NT2022], our next step is to apply the Cauchy–Schwarz inequality to the $m-1$ dimensional measure of ∂A_t:

$$|\partial A_t|^2 = \left(\int_{\partial A_t} dS\right)^2 \leq \int_{\partial A_t} |\nabla u|^{-1} dS \int_{\partial A_t} |\nabla u| dS. \tag{3.2}$$

Let $\nu = \nu(x)$ be the outward unit normal to ∂A_t at a point x. Note that, ∇u is parallel to ν, but directed in the opposite direction. Thus, we have $|\nabla u| = -\langle \nabla u, \nu\rangle$. Also, we note that since for $x \in \partial A_t$ we have $u(x) = t$, we obtain for $x \in \partial A_t$ that

$$\frac{|\nabla u(x)|}{t} = \frac{|\nabla u(x)|}{u} = \left\langle \nabla \log u(x), \nu \right\rangle.$$

Now the second integral on the right-hand side of (3.2) can be evaluated by Gauss's divergence theorem:

$$\begin{aligned}\int_{\partial A_t} |\nabla u||dS| &= -t \int_{A_t} \operatorname{div}\left(\nabla \log u(x)\right) dA(x)\\ &= -t \int_{A_t} \Delta \log u(x) dA(x).\end{aligned}$$

Now we plug $u = |f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}$, and calculate

$$-t\Delta \log(|f(x)|^p e^{-\frac{\alpha}{2}p|x|^2}) = -(pt\Delta \log v - t\frac{\alpha}{2}p\Delta|x|^2) \leq 0 + mt\alpha p.$$

By using (3.1) and (3.2), we obtain

$$\begin{aligned}|\partial A_t|^2 &\leq (-\mu'(t)) \int_{\partial A_t} |\nabla u| dS.\\ &\leq -mt\alpha p\mu'(t)\mu(t).\end{aligned}$$

Now we use the isoperimetric inequality for the space:

$$|\partial A_t|^2 \geqslant \pi m^2 |A_t|^{\frac{2(m-1)}{m}} (\Gamma(m/2))^{-\frac{2}{m}},$$

which implies that

$$mt\alpha p\mu'(t)\mu(t) + m^2\pi\mu(t)^{\frac{2(m-1)}{m}} (\Gamma(m/2))^{-\frac{2}{m}} \leq 0 \tag{3.3}$$

with equality in (3.3) if and only if $v(x) = e^{\alpha\langle x,a\rangle - \frac{\alpha}{2}|a|^2}$ because in that case A_t is a ball centered at a. So,

$$M(t) := \alpha p\mu'(t)\mu(t)^{\frac{2-m}{m}} + \frac{m\pi\left(\Gamma(m/2)\right)^{-\frac{2}{m}}}{t} \leq 0. \tag{3.4}$$

Since $\mu(t^\circ) = 0$, we obtain that

$$G(t) = \int_{t_\circ}^{t} M(t)dt = m\pi(\Gamma(m/2))^{-2/m} \log \frac{t}{t_\circ} + \frac{m}{2}\alpha p \mu^{\frac{2}{m}}(t)$$

is a non-increasing function for $0 \le t < t_\circ$.

In the case $v(x) \equiv e^{\alpha\langle a,x\rangle - \frac{\alpha}{2}|a|^2}$, $t_\circ = 1$ and $\mu(t_\circ) = 0$. Moreover,

$$g(t) := \exp(G(t)) = t \exp\left[\frac{\alpha p(\Gamma(m/2))^{2/m}}{2\pi}\mu^{2/m}(t)\right]$$

is non-increasing for $0 \le t < t_\circ$.

□

Remark 3.1. *Note that for the function $f(x) \equiv 1$ or*

$$f(x) = e^{-\frac{\alpha}{2}|a|^2} e^{\alpha\langle a,x\rangle},$$

for a fixed a, everywhere in the proof above we have equalities for all values of p and α. Moreover in this case the maximum of $u(x)$ is equal to 1 and achieved for $x = a$.

4. Proof of Theorem 2.3

We need the following lemma:

Lemma 4.1. [KA2024] *Assume that Φ, Ψ are positive increasing functions and g positive non-increasing such that*

$$\int_0^{t_\circ} \Phi(g(t)/t)\, dt = \int_0^{t_\circ} \Phi(1/t)\, dt = c.$$

Then

$$\int_0^{t_\circ} \Phi(g(t)/t)\Psi(t)dt \le \int_0^{t_\circ} \Phi(1/t)\Psi(t)dt.$$

As in [KU2022, KA2024] where is treated Bergman version of this theorem, we restrict ourselves to the only nontrivial case $\lim_{t\to 0^+} G(t) = 0$. Let $\mu(t) = \mu(\{x : u(x) > t\})$ be the Lebesgue measure in $\mathbb{R}^m$, where $u(x) = |f(x)|^p e^{-\frac{\alpha p}{2}|x|^2}$. Applying Theorem 2.2 to f, we get that the function

$$g(t) = t \exp\left[\frac{\alpha(\Gamma(m/2))^{2/m}}{2\pi}\mu^{2/m}(t)\right],$$

is decreasing on $(0, t_\circ)$ with $t_\circ = \max_{x\in\mathbb{R}^m} u(x)$. Proposition 5.1 below ensures the existence of $t_\circ$.

For $f \equiv 1$, g is a constant function equal to 1.

Then,

$$\mu(t) = \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log \frac{g(t)}{t}\right)^{\frac{m}{2}}.$$

We assume that $\|f\|_{p,\alpha} = 1$, that is

$$I_1 = c_{p,\alpha} \int_0^{t_\circ} \mu(t)dt = c_{p,\alpha} \int_0^{t_\circ} \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log \frac{g(t)}{t} \right)^{m/2} dt = 1.$$

Now the integral in (2.1) can be rewritten as

$$I_2 = c_{p,\alpha} \int_0^{t_\circ} \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log \frac{g(t)}{t} \right)^{m/2} G'(t)dt.$$

Then, by Lemma 4.1, by taking $\Phi(s) = c_{p,\alpha} \left(\frac{2\pi}{\alpha(\Gamma(m/2))^{2/m}} \log s \right)^{\frac{m}{2}}$ and $\Psi(t) = G'(t)$, the maximum of I_2 under $I_1 = 1$ is attained for $g \equiv 1$.

5. Additional properties of Fock space and proof of Theorem 2.7

Now we prove the following proposition used in the proof of our main result.

Proposition 5.1. *Assume that f is a real-analytic log-subharmonic function in $\mathbb{R}^m$ belonging to the Fock type space. Then for every x,*

$$|f(x)|^p e^{-\frac{\alpha p}{2}|x|^2} \leq c_{p,\alpha} \int_{\mathbb{R}^m} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2} dA(y). \tag{5.1}$$

Moreover,

$$\lim_{|x|\to\infty} |f(x)| e^{-\frac{\alpha}{2}|x|^2} = 0. \tag{5.2}$$

Notice that (5.1) extends [KZ2012, Theorem 2.7] and the relation (5.2) extends corresponding relation in [KZ2012, p. 38].

Proof. Let $g(y) = |f(x+y)|^p e^{-\alpha p\langle (y+x),x\rangle}$. Now use the mean value property to the log-subharmonic function g (it is also subharmonic).

$$|g(0)| \leq c_{p,\alpha} \int_{\mathbb{R}^m} |g(y)| e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Then, we have

$$g(0) = |f(x)|^p e^{-\alpha p|x|^2} \leq c_{p,\alpha} \int_{\mathbb{R}^m} f^p(y+x) e^{-\frac{\alpha}{2} p\langle (x+y),x\rangle} e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Therefore,

$$|f(x)|^p e^{-\alpha p|x|^2} \leq c_{p,\alpha} \int_{\mathbb{R}^m} f^p(y) e^{-\alpha p\langle y,x\rangle} e^{-\frac{\alpha p}{2}|y-x|^2} dA(y).$$

So,

$$|f(x)|^p e^{-\frac{\alpha p}{2}|x|^2} \leq c_{p,\alpha} \int_{\mathbb{R}^m} f^p(y) e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Now, to prove (5.2), we use the following inequality, which is also a consequence of the sub-mean value property of subharmonic functions. Let $B_1(x) = \{y \in \mathbb{R}^m : |y - x| < 1\}$. Then for every subharmonic function g, we have

$$|g(0)| \le \frac{n}{\omega_n} \int_{B_1(0)} |g(y)| dA(y).$$

Thus,

$$|g(0)|e^{-\frac{\alpha p}{2}} \le \frac{n}{\omega_n} \int_{B_1(0)} |g(y)|e^{-\frac{\alpha p}{2}|y|^2} dA(y). \tag{5.3}$$

By applying the previous inequality for $g(y) = |f(x+y)|^p e^{-\alpha p\langle (y+x),x\rangle}$, we obtain from (5.3) that

$$\begin{aligned} |f(x)|^p e^{-\alpha p|x|^2} e^{-\frac{\alpha p}{2}} &\le \frac{n}{\omega_n} \int_{B_1(0)} |f(x+y)|^p e^{-\alpha p\langle (y+x),x\rangle} e^{-\frac{\alpha p}{2}|y|^2} dA(y) \\ &= \frac{n}{\omega_n} \int_{B_1(x)} |f(y)|^p e^{-\alpha p\langle y,x\rangle} e^{-\frac{\alpha p}{2}|y-x|^2} dA(y) \\ &= \frac{n}{\omega_n} e^{-\frac{\alpha p}{2}|x|^2} \int_{B_1(x)} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2} dA(y). \end{aligned}$$

Thus,

$$|f(x)|^p e^{-\frac{\alpha p}{2}|x|^2} e^{-\frac{\alpha p}{2}} \le \frac{n}{\omega_n} \int_{B_1(x)} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2} dA(y).$$

Since $f \in \mathcal{L}^p_\alpha$, it follows that

$$\lim_{|x|\to\infty} \frac{n}{\omega_n} \int_{B_1(x)} |f(y)|^p e^{-\frac{\alpha p}{2}|y|^2} dA(y) = 0.$$

This implies (5.2). □

It follows from the following lemma that $\|f\|_\alpha$ is a norm on $\mathcal{F}_\alpha$. Theorem 2.7 is a direct application of the following lemma

Lemma 5.2. *a) If $f, g \in \mathcal{F}_\alpha$, then $\|f + g\|_\alpha \le \|f\|_\alpha + \|g\|_\alpha$.*

b) For every $\alpha > 0$ and $f \in \mathcal{F}_\alpha$ and $x \in \mathbb{C}^m$ we have $|f(x)|e^{-\frac{\alpha}{2}|x|^2} \le \|f\|_\alpha$.

c) For every $\alpha > 0$ and $f \in \mathcal{F}_\alpha$, $\|f\|_\alpha = \sup_{x\in\mathbb{C}^n}\left(|f(x)|e^{-\frac{\alpha}{2}|x|^2}\right)$.

Proof. Let us restrict ourselves to the case $n = 1$. The general case is a trivial modification of this case.

a) Let $f, g \in \mathcal{F}_\alpha$. Then for every $\alpha > 0$, $f, g \in \mathcal{F}^p_\alpha$ and by the triangle inequality for the norm in $\mathcal{F}^p_\alpha$ we obtain

$$\begin{aligned} \|f + g\|_\alpha &= \lim_{p\to\infty} \|f + g\|_{p,\alpha} \\ &\le \lim_{p\to\infty} \|f\|_{p,\alpha} + \lim_{p\to\infty} \|g\|_{p,\alpha} \\ &= \|f\|_\alpha + \|g\|_\alpha. \end{aligned}$$

b) This follows from Proposition 5.1.
c) It follows from (5.1) that

$$|f(x)|e^{-\frac{\alpha}{2}|x|^2} \leq \|f\|_{p,\alpha}.$$

By letting $p \to \infty$ we obtain

$$|f(x)|e^{-\frac{\alpha}{2}|x|^2} \leq \|f\|_{\alpha}.$$

Thus,

$$\text{ess sup}\,|f(x)|e^{-\frac{\alpha}{2}|x|^2} \leq \|f\|_{\alpha}.$$

To prove the converse, fix an $R > 0$ and assume first that $f = P$ is a polynomial. Then

$$\|P\|_{p,\alpha}^p = \int_{|x|\leq R} |P(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dx + \int_{|x|>R} |P(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dx.$$

Moreover, for sufficiently large R

$$I(R) := \int_{|x|>R} |P(x)|^p e^{-\frac{\alpha}{2}p|x|^2} dx \leq c_P \int_{|x|>R} |z|^{n_P p} e^{-\frac{\alpha}{2}p|x|^2} dx$$

and the last expression is smaller than $\|F\|_{\infty,\alpha}^p$. In fact, the last expression tends to zero as $R \to \infty$. Therefore,

$$\|P\|_{p,\alpha} \leq (\|P\|_{\infty,\alpha}^p R^n \omega_n + \|P\|_{\infty,\alpha}^p)^{1/p},$$

where ω_n is the meausre of the unit sphere. Thus,

$$\|P\|_{\alpha} = \lim_{p\to\infty} \|P\|_{p,\alpha} \leq \|P\|_{\infty,\alpha}.$$

Thus, if f is a polynomial, then

$$\|f\|_{\alpha} = \|f\|_{\infty,\alpha}. \tag{5.4}$$

Further, if f is not a polynomial and $\epsilon > 0$ is arbitrary, then for $p = 2$, there exists a polynomial P so that $\|P - f\|_{p,\alpha} < \epsilon$. Moreover,

$$\|f\|_{\alpha} \leq \|P\|_{\alpha} + \|f - P\|_{\alpha} = \|P\|_{\infty,\alpha} + \|f - P\|_{\alpha} \leq \|P\|_{\alpha} + \epsilon.$$

Since ϵ is arbitrary, we conclude that (5.4) hold for every function $f \in \mathcal{F}_{\alpha}$.
□

Remark 5.3. *One can ask, given a holomorphic function f, when this*

$$\lim_{p\to 0} \|f\|_{\alpha,p}$$

exists. The answer is that limit is infinity except for the case when $f \equiv const$, so we cannot produce a Hardy type space for holomorphic mappings in $\mathbb{C}^n$.

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