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Boundary behaviour of Neumann harmonic functions with Lebesuge, Hardy and BMO traces in the upper half-space

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ABSTRACT. This paper is concerned with the boundary value problem for the elliptic equation of Neumann type on the upper half-space $\mathbb{R}^n \times \mathbb{R}_+$

$$\begin{cases} -\partial_t^2 u(x,t) - \operatorname{div} A\nabla u(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \partial_{x_n} u(x',0,t) = 0, & x' \in \mathbb{R}^{n-1},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where the matrix $A = (a^{ij}(x))_{n\times n}$ is even with respect to the n-th variable and satisfies the ellipticity condition. By using the reflection method from Strauss's book [PDEs. An introduction, 2ed, 2008], we derive that the solution u to the above equation can be represented as the Poisson integral (with an additional perturbation) of the initial value u_0. As applications, the real-variable characterizations of Neumann harmonic functions with Lebesuge, Hardy and BMO traces are established, respectively, via the gluing technology. Finally, the boundary value problem for the parabolic equation is also considered.

CONTENTS

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1. Introduction

In mathematics, mathematical physics and the theory of stochastic processes, a harmonic function is a twice continuously differentiable function $f : \Omega \to \mathbb{R}$ (Ω is an open subset of $\mathbb{R}^n$) which satisfies Laplace's equation, i.e.,

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Harmonic functions that arise in physics are determined by their singularities and boundary conditions (such as Dirichlet/Neumann boundary condition).

The Dirichlet problem for Laplace's equation consists of finding a harmonic function f on some domain Ω such that it takes prescribed values on the boundary of Ω. Whereas, the Neumann boundary condition specifies not the function f itself on the boundary of Ω (not like the Dirichlet case), but its normal derivative. Physically, this corresponds to the construction of a potential for a vector field whose effect is known at the boundary of Ω alone.

In his book [38], Strauss handled the boundary value problem for the heat equation of Neumann type on the half-line $(0, \infty)$

$$\begin{cases} \partial_t w(x,t) - \partial_x^2(x,t) = 0, & 0 < x, t < \infty, \\ \partial_x w(0,t) = 0, & 0 < t < \infty, \\ w(x,0) = \phi(x), & 0 < x < \infty. \end{cases}$$

By using the reflection method, he end up this problem with an explicit formula for $w(x,t)$. It is

$$w(x,t) = \int_0^\infty \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{|x-y|^2}{4t} \right) + \exp\left(-\frac{|x+y|^2}{4t} \right) \right] \phi(y) dy.$$

For the n-dimension case, we refer the readers to [1, 4, 6, 13, 29] for more details about this subject.

Inspired by [38], we consider the similar problem for the elliptic equation on $\mathbb{R}^n \times \mathbb{R}_+$. We derive that a Neumann harmonic function $u(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ can be represented as the Poisson integral (with an additional perturbation) of the initial value $u(x,0)$; see Section 2 for more details. This is similar to the classical case. As applications, the real-variable characterizations of Neumann harmonic functions with the Lebesgue, Hardy and BMO traces are considered, respectively. Precisely, when $1 < p < \infty$, we prove that, the initial value is in $L^p(\mathbb{R}^n)$ if and only if the solution to the elliptic equation of Neumann type is in $L^p(\mathbb{R}^n)$ uniformly in the time variable; see Theorem 4.1 below. When the Neumann boundary condition is removed, this result can reduce to the classical one; see [37, Theorems 2.1 & 2.5] or Appendix A below. However, our method is totally different from the that used in [37]. With the help of some new observations from [40], we split an L^p-function into two even function and glue them together in a sense. Then the Neumann problem can reduce to the classical case; see the proof of Theorem 4.1 for more details. For the endpoint case $p = 1$ or $p = \infty$, it is well known that the Hardy space $H^1(\mathbb{R}^n)$ or BMO space BMO$(\mathbb{R}^n)$ shares similar properties with the Lebesgue space $L^1(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$, and it often serves as a substitute for it. For example, the classical singular integrals do not map $L^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and do not map $L^\infty(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$, but $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ and $L^\infty(\mathbb{R}^n)$ to BMO$(\mathbb{R}^n)$; see [16]. And, in many instances, interpolation H^1-L^p or L^p-BMO works just as well L^1-L^p or L^p-L^∞. For the left endpoint case $p = 1$, we prove that, the initial value is in the Hardy space related to the Neumann problem if and only if the maximal function of the solution to the elliptic equation of Neumann type is in the Lebesgue space; see Theorem 5.1 below. For the other endpoint case $p = \infty$, there exists a natural and deep connection between the Carleson measure and the BMO function; indeed, certain types of measures defined in terms of functions are Carleson if and only if the underlying functions satisfy the BMO condition. Inspired by this and [11, 14, 18, 40], we prove that, the initial value is in the BMO space related to the Neumann problem if and only if the solution to the elliptic equation of Neumann type satisfies a certain Carleson condition; see Theorem 6.1 below. In fact, we consider the more general Morrey-Campanato space related to the Neumann problem as the initial value space. It is worth mentioning that Zhang-Yang [40] solved the Neumann problem for the heat equation on $\mathbb{R}^n \times \mathbb{R}_+$ with the BMO initial value. For more conclusions about the Dirichlet problem for the heat/Laplace equation, we refer the readers to [10, 12, 17, 18, 26, 31, 33, 34, 39] and references therein.

The present paper is built up as follows. In the next section, we first find the solution formula for the elliptic equation of Neumann type, and then provide some properties for the Poisson kernel related to the Neumann problem. In Section 3, we make use of the classical function classes to describe the function classes related to the Neumann problem. With the help of the conclusions in Section 3, we study the Neumann harmonic functions with Lebesgue, Hardy and BMO traces, respectively, in Sections 4-6. Some remarks are then presented in the final section.

Finally, we make some conventions on notation. For a function f defined on $\mathbb{R}^n$, we denote by $f_\pm$ the restriction of f to $\mathbb{R}^n_\pm$. For every $x = (x', x_n) \in \mathbb{R}^n$, set $\tilde{x} = (x', -x_n)$. Let $Q = Q(x_Q, r_Q)$ be the open cube centered at x_Q of sidelength $2r_Q$. Denote the reflection of Q across $\partial\mathbb{R}^n_+$ by $\widetilde{Q} = \{(x', x_n) \in \mathbb{R}^n : (x', -x_n) \in Q\}$. Let $Q_+ = Q \cap \mathbb{R}^n_+$ and $Q_- = Q \cap \mathbb{R}^n_-$. If both Q_+ and Q_- are not empty, we then define

$$\widehat{Q}_+ = \{(x', x_n) \in \mathbb{R}^n : x' \in Q \cap \mathbb{R}^{n-1}, 0 < x_n < 2r_Q\}$$

and

$$\widehat{Q}_- = \{(x', x_n) \in \mathbb{R}^n : x' \in Q \cap \mathbb{R}^{n-1}, -2r_Q < x_n < 0\}.$$

The letter C (or c) will stand for a positive constant that may vary from line to line but will remain independent of the main variables.

2. Neumann harmonic function & Poisson semigroup

2.1. Solution formula for the elliptic equation of Neumann type. Let

$$A = A(x) = (a^{ij}(x))_{n\times n}$$

be an $n\times n$ matrix of real symmetric, bounded measurable coefficients, defined on $\mathbb{R}^n$, and satisfy the ellipticity (or "accretivity") condition, namely, there exist a constant $\Lambda \geq 1$ such that

$$\Lambda^{-1}|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$$

for all $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n$. We define a second order elliptic operator

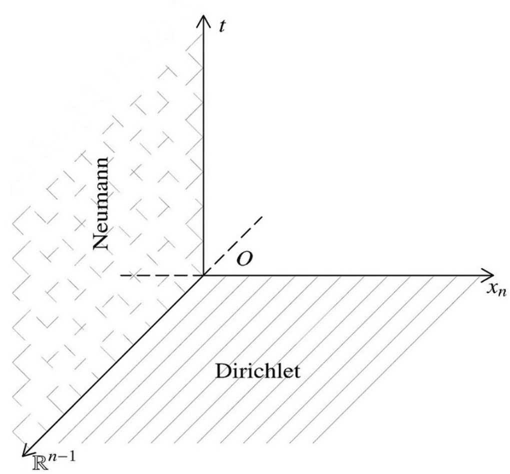
$$Lf = -\mathrm{div}(A\nabla f) = -\partial_{x_i}(a^{ij}\partial_{x_j} f),$$

which we interpret in the usual weak sense via a sesquilinear form.

Consider the elliptic equation of the form

$$\begin{cases} -\partial_t^2 u(x,t) + Lu(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ \partial_{x_n} u(x', 0, t) = 0, & x' \in \mathbb{R}^{n-1},\ t > 0, \\ u(x', x_n, 0) = f(x', x_n), & (x', x_n) \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

with mixed boundary value on $\mathbb{R}^n_+ \times \mathbb{R}_+$ as in the following Picture 1.



Picture 1

A solution $u(x,t)$ to (2.1) is called a harmonic function on $\mathbb{R}^n \times \mathbb{R}_+$ with Dirichlet condition at $\mathbb{R}^n \times \{0\}$ and Neumann condition at $\mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}_+$. If the Dirichlet boundary condition is removed, we shall call this solution u as the Neumann harmonic function on $\mathbb{R}^n \times \mathbb{R}_+$. Due to the Neumann condition, the matrix A is always assumed to be even with respect to the n-th variable. [1]

[1] In the case of the Dirichlet condition at $\mathbb{R}^{n-1} \times \{0\} \times \mathbb{R}_+$, the matrix A is also even; see Section 7 for more details.

We are looking for a solution formula for (2.1). In fact, we shall reduce this problem to the classical initial problem for the elliptic PDE. Our method relies on the idea of an even extension from [38, Chapter 3]. Any function $\psi(s)$ that satisfies $\psi(-s) = \psi(s)$ is called an even function. This just means that its graph $y = \psi(s)$ is symmetric with respect to the y-axis. Thus, the left and right halves of the graph are mirror images of each other. Moreover, since the Neumann boundary condition is imposed on the n-th variable x_n, the last row of the matrix

$$A = \begin{pmatrix} a^{11} & a^{12} & \cdots & a^{1n} \\ a^{21} & a^{22} & \cdots & a^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a^{n1} & a^{n2} & \cdots & a^{nn} \end{pmatrix}$$

is very important to us, and hence denote it by $\beta = (a^{n1}\ a^{n2}\ \cdots\ a^{nn})$.

Now we split the elliptic equation (2.1) into the positive part

$$\begin{cases} -\partial_t^2 u_+(x,t) + Lu_+(x,t) = 0, & x \in \mathbb{R}^n_+,\ t > 0, \\ \partial_{x_n} u_+(x',0,t) = 0, & x' \in \mathbb{R}^{n-1},\ t > 0, \\ u_+(x',x_n,0) = f_+(x',x_n), & x' \in \mathbb{R}^{n-1},\ x_n \geq 0, \end{cases} \tag{2.2}$$

and the negative part

$$\begin{cases} -\partial_t^2 u_-(x,t) + Lu_-(x,t) = 0, & x \in \mathbb{R}^n_-,\ t > 0, \\ \partial_{x_n} u_-(x',0,t) = 0, & x' \in \mathbb{R}^{n-1},\ t > 0, \\ u_-(x',x_n,0) = f_-(x',x_n), & x' \in \mathbb{R}^{n-1},\ x_n \leq 0. \end{cases} \tag{2.3}$$

Since initial datum f_+ of our problem (2.2) is defined only for $\mathbb{R}^n_+$, let $f_{+,e}$ be its unique even extension to the whole space $\mathbb{R}^n$, i,e.,

$$f_{+,e}(x) = \begin{cases} f_+(x), & x \in \mathbb{R}^n_+, \\ f_+(\tilde{x}), & x \in \mathbb{R}^n_-. \end{cases}$$

We shall call the following elliptic equation as the even extension of (2.2)

$$\begin{cases} -\partial_t^2 w(x,t) + Lw(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ w(x,0) = f_{+,e}(x), & x \in \mathbb{R}^n. \end{cases} \tag{2.4}$$

Due to the elliptic PDE theory, the solution to (2.4) is given via the Poisson formula

$$w(x,t) = \exp(-t\sqrt{L}) f_{+,e}(x) = \int_{\mathbb{R}^n} p_L(t,x,y) f_{+,e}(y)\,dy, \tag{2.5}$$

where $\exp(-t\sqrt{L})$ denotes the Poisson semigroup and $p_L(t,x,y)$ is its integral kernel. To solve the elliptic equation (2.2), we should introduce the following additional hypothesis on matrix A.

Definition 2.1. A matrix A on $\mathbb{R}^n$ is said to satisfy the admissible harmonic condition if, for each harmonic function $w(x,t)$ on $\mathbb{R}^n \times \mathbb{R}_+$, it holds

$$\int_{\mathbb{R}^n\times\mathbb{R}_+} \beta(\nabla w \partial_{x_n}\varphi + \nabla\varphi\partial_{x_n} w)dxdt = 2\int_{\mathbb{R}^n\times\mathbb{R}_+} a^{nn}\partial_{x_n} w \partial_{x_n}\varphi dxdt,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

Example 2.2. *Evidently, if $a^{n1} = \cdots a^{nn-1} = 0$, then*

$$A = \begin{pmatrix} a^{11} & \cdots & a^{1n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a^{n-11} & \cdots & a^{n-1n-1} & 0 \\ 0 & \cdots & 0 & a^{nn} \end{pmatrix}$$

is an admissible harmonic matrix.

For some technical reasons, the matrix A is always assumed to satisfy the admissible harmonic condition.

We claim the "restriction"

$$u_+(x', x_n, t) = w(x', x_n, t), \quad x_n \geq 0, \tag{2.6}$$

will be the unique solution of our problem (2.2). There is no difference at all between u_+ and w except that negative values of x_n are not considered when discussing u_+. Note first that the solution $w(x,t)$ to (2.4) must also be an even function with respect to x_n, i.e., one has

$$\widetilde{w}(x,t) = w(\tilde{x},t) = w(x', -x_n, t) = w(x', x_n, t) = w(x,t).$$

Indeed, on the one hand, it holds

$$\widetilde{w}(x,0) = w(\tilde{x},0) = f_{+,e}(\tilde{x},0) = f_{+,e}(x,0) = w(x,0).$$

One the other hand, the admissible harmonic matrix tells us that, for each smooth function φ on $\mathbb{R}^n \times \mathbb{R}_+$ with compact support,

$$\begin{aligned}
&\int_{\mathbb{R}^n\times\mathbb{R}_+} a^{ij}\partial_{x_i}\widetilde{w}\partial_{x_j}\varphi dxdt + \int_{\mathbb{R}^n\times\mathbb{R}_+} \partial_t\widetilde{w}\partial_t\varphi dxdt \\
&= \int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{i,j=1}^{n-1} a^{ij}\partial_{x_i}\widetilde{w}\partial_{x_j}\varphi dxdt + \int_{\mathbb{R}^n\times\mathbb{R}_+} a^{nn}\partial_{x_n}\widetilde{w}\partial_{x_n}\varphi dxdt \\
&\quad + \int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{i=1}^{n-1} a^{in}\partial_{x_i}\widetilde{w}\partial_{x_n}\varphi dxdt + \int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{j=1}^{n-1} a^{nj}\partial_{x_n}\widetilde{w}\partial_{x_j}\varphi dxdt \\
&\quad + \int_{\mathbb{R}^n\times\mathbb{R}_+} \partial_t\widetilde{w}\partial_t\varphi dxdt \\
&= \int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{i,j=1}^{n-1} a^{ij}\partial_{x_i} w\partial_{x_j}\widetilde{\varphi} dxdt + \int_{\mathbb{R}^n\times\mathbb{R}_+} a^{nn}\partial_{x_n} w\partial_{x_n}\widetilde{\varphi} dxdt
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{i=1}^{n-1} a^{in}\partial_{x_i}w\partial_{x_n}\widetilde{\varphi}dxdt - \int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{j=1}^{n-1} a^{nj}\partial_{x_n}w\partial_{x_j}\widetilde{\varphi}dxdt \\
&+ \int_{\mathbb{R}^n\times\mathbb{R}_+} \partial_t w\partial_t\widetilde{\varphi}dxdt \\
= &\int_{\mathbb{R}^n\times\mathbb{R}_+} a^{ij}\partial_{x_i}w\partial_{x_j}\widetilde{\varphi}dxdt + \int_{\mathbb{R}^n\times\mathbb{R}_+} \partial_t w\partial_t\widetilde{\varphi}dxdt \\
&- 2\int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{i=1}^{n-1} a^{in}\partial_{x_i}w\partial_{x_n}\widetilde{\varphi}dxdt - 2\int_{\mathbb{R}^n\times\mathbb{R}_+} \sum_{j=1}^{n-1} a^{nj}\partial_{x_n}w\partial_{x_j}\widetilde{\varphi}dxdt \\
= &-2\int_{\mathbb{R}^n\times\mathbb{R}_+} \left[\beta(\nabla w\partial_{x_n}\varphi + \nabla\varphi\partial_{x_n}w) - 2a^{nn}\partial_{x_n}w\partial_{x_n}\varphi\right] dxdt = 0,
\end{aligned}
$$

which yields $w(\tilde{x},t) = w(x,t)$. Then, differentiation shows that its derivative is an odd function. So automatically, its slope at the origin is zero: $\partial_{x_n}u_+(x',0,t) = \partial_{x_n}w(x',0,t) = 0$, i.e., the Neumann boundary condition is satisfied. Moreover, u_+ solves the PDE as well as the initial condition for $x_n \geq 0$, simply because it is equal to w for $x_n \geq 0$, and w satisfies the same PDE for all x_n and the same initial condition for $x_n \geq 0$.

The explicit formula for $u_+(x,t)$ is easily deduced from (2.5) and (2.6). On the one hand, by the Poisson formula we have

$$
\begin{aligned}
w(x,t) &= \int_{\mathbb{R}^n_+} p_L(t,x,y)f_+(y)dy + \int_{\mathbb{R}^n_-} p_L(t,x,y)f_+(\tilde{y})dy \\
&= \int_{\mathbb{R}^n_+} p_L(t,x,y)f_+(y)dy + \int_{\mathbb{R}^n_+} p_L(t,x,\tilde{y})f_+(y)dy \\
&= \int_{\mathbb{R}^n_+} [p_L(t,x,y) + p_L(t,x,\tilde{y})]f_+(y)dy.
\end{aligned}
$$

On the other hand, the fact $w(\tilde{x},t) = w(x,t)$ yields

$$p_L(t,\tilde{x},y) + p_L(t,\tilde{x},\tilde{y}) = p_L(t,x,y) + p_L(t,x,\tilde{y}) \tag{2.7}$$

which further implies

$$p_L(t,\tilde{y},x) + p_L(t,\tilde{y},\tilde{x}) = p_L(t,y,x) + p_L(t,y,\tilde{x}).$$

From this identity and the symmetry of the Poisson kernel (which is guaranteed by the self-adjoint of L), it follows

$$p_L(t,x,\tilde{y}) + p_L(t,\tilde{x},\tilde{y}) = p_L(t,x,y) + p_L(t,\tilde{x},y). \tag{2.8}$$

By comparing (2.7) and (2.8), we arrive at

$$p_L(t,x,\tilde{y}) = p_L(t,\tilde{x},y).$$

Therefore, for all $x \in \mathbb{R}^n_+$, it holds

$$u_+(x,t) = \int_{\mathbb{R}^n_+} [p_L(t,x,y) + p_L(t,x,\tilde{y})]\, f_+(y)dy$$
$$= \int_{\mathbb{R}^n_+} [p_L(t,x,y) + p_L(t,\tilde{x},y)] f_+(y)dy,$$

which is the complete solution formula for (2.2).

For the negative part, we can use the similar arguments above to obtain that, for any $x \in \mathbb{R}^n_-$,

$$u_-(x,t) = \int_{\mathbb{R}^n_-} [p_L(t,x,y) + p_L(t,x,\tilde{y})]\, f_-(y)dy$$
$$= \int_{\mathbb{R}^n_-} [p_L(t,x,y) + p_L(t,\tilde{x},y)] f_-(y)dy,$$

which is the complete solution formula for (2.3).

Finally, the solution to (2.1) can be reads as

$$u(x,t) = \int_{\mathbb{R}^n} [p_L(t,x,y) + p_L(t,\tilde{x},y)] f(y)dy$$

via the gluing the positive part and the negative part.

2.2. Poisson semigroup of Neumann type. Recall the Neumann boundary problem (2.2) and then denote this corresponding Neumann elliptic operator by L_{N_+}. Similarly we can define the Neumann elliptic operator L_{N_-} on $\mathbb{R}^n_-$.

The Neumann elliptic operators $L_{N_\pm}$ are positive-definite and self-adjoint operators. From the spectral theorem one can define the Poisson semigroups $\{\exp(-t\sqrt{L_{N_\pm}})\}_{t>0}$ generated by these operators $L_{N_\pm}$. Denote by $p_{L_{N_\pm}}(t,x,y)$ the corresponding integral kernels. Based on the arguments above, we see that

$$p_{L_{N_\pm}}(t,x,y) = p_L(t,x,y) + p_L(t,\tilde{x},y), \quad x, y \in \mathbb{R}^n_\pm.$$

Note that, for each function f on $\mathbb{R}^n_\pm$, we have

$$\exp(-t\sqrt{L}) f_e(x) = \begin{cases} \exp(-t\sqrt{L_{N_\pm}}) f(x), & x \in \mathbb{R}^n_\pm,\ t > 0, \\ \exp(-t\sqrt{L_{N_\pm}}) f(\tilde{x}), & x \in \mathbb{R}^n_\mp,\ t > 0. \end{cases} \tag{2.9}$$

Now let L_N be the uniquely determined unbounded operator acting on $L^2(\mathbb{R}^n)$ such that

$$(L_N f)_\pm = L_{N_\pm} f_\pm$$

for all $f : \mathbb{R}^n \to \mathbb{R}$ satisfying $f_\pm \in W^{1,2}(\mathbb{R}^n_\pm)$. Then L_N is a positive self-adjoint operator and there holds

$$(\exp(-t\sqrt{L_N}) f)_\pm = \exp(-t\sqrt{L_{N_\pm}}) f_\pm.$$

The Poisson kernel of $\exp(-t\sqrt{L_N})$, denoted by $p_{L_N}(t,x,y)$, is then given as

$$p_{L_N}(t,x,y) = [p_L(t,x,y) + p_L(t,\tilde{x},y)]\chi_{[0,\infty)}(x_n y_n).$$

On the other hand, the Poisson kernel $p_{L_N}(t,x,y)$ can be obtain through Bochner's subordination formula

$$p_{L_N}(t,x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty \left(\frac{t^2}{4s}\right)^{1/2} \exp\left(-\frac{t^2}{4s}\right) h_{L_N}(s,x,y)\frac{ds}{s},$$

where $h_{L_N}(s,x,y)$ is the heat kernel associated to L_N as

$$h_{L_N}(s,x,y) = [h_L(s,x,y) + h_L(s,\tilde{x},y)]\chi_{[0,\infty)}(x_n y_n).$$

Indeed, one has

$$\begin{aligned} p_{L_N}(t,x,y) &= \frac{1}{\sqrt{\pi}}\int_0^\infty \left(\frac{t^2}{4s}\right)^{1/2} \exp\left(-\frac{t^2}{4s}\right) h_L(s,x,y)\frac{ds}{s}\chi_{[0,\infty)}(x_n y_n) \\ &\quad + \frac{1}{\sqrt{\pi}}\int_0^\infty \left(\frac{t^2}{4s}\right)^{1/2} \exp\left(-\frac{t^2}{4s}\right) h_L(s,\tilde{x},y)\frac{ds}{s}\chi_{[0,\infty)}(x_n y_n) \\ &= [p_L(t,x,y) + p_L(t,\tilde{x},y)]\chi_{[0,\infty)}(x_n y_n). \end{aligned}$$

Let us note that:

(i) All the operators L_N and $L_{N_\pm}$ are self-adjoint and they generate bounded analytic positive semigroups acting on all L^p-spaces for $1 \le p \le \infty$.

(ii) Suppose that $p_{\mathcal{L}}(t,x,y)$ is the Poisson kernel corresponding to the semigroup generated by one of the operators $\mathcal{L}$ listed in (i). If the matrix A satisfies ellipticity condition, then the kernel $p_{\mathcal{L}}(t,x,y)$ and its time derivative admit the Poisson upper bound

$$|t^k \partial_t^k p_{\mathcal{L}}(t,x,y)| \le C(k)\frac{t}{(t+|x-y|)^{n+1}}, \quad k = 0,1,2,\cdots$$

for all $x, y \in \Omega$, where $\Omega = \mathbb{R}^n$ for L_N, and $\Omega = \mathbb{R}^n_\pm$ for $L_{N_\pm}$.

(iii) If $\mathcal{L}$ is one of the operators L_N and $L_{N_\pm}$, then $\mathcal{L}$ conserves probability, that is

$$\exp(-t\sqrt{\mathcal{L}})1 = 1.$$

3. Some function classes of Neumann type

In this section, we first introduce some function classes related to the Neumann problem, and then describe them by the classical spaces.

3.1. Morrey space. The Morrey space was first introduced by Morrey [32] to show that certain systems of PDEs had Hölder continuous solutions. We shall establish an equivalent characterization of Morrey space (Lebesgue as a special case), which can reduce our Neumann boundary problem to the classical one.

Definition 3.1. For $0 < p < \infty$ and $-1/p \le \alpha < 0$, a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in $L^{p,\alpha}(\mathbb{R}^n)$, the Morrey space, if

$$\|f\|_{L^{p,\alpha}(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^\alpha}\left(\frac{1}{|Q|}\int_Q |f(x)|^p dx\right)^{1/p} < \infty.$$

For the endpoint case $\alpha = -1/p$, the Morrey space $L^{p,-1/p}(\mathbb{R}^n)$ coincides with the classical Lebesgue space $L^p(\mathbb{R}^n)$.

Proposition 3.2. *For $0 < p < \infty$ and $-1/p \le \alpha < 0$, the Morrey space $L^{p,\alpha}(\mathbb{R}^n)$ can be described in the following way*

$$L^{p,\alpha}(\mathbb{R}^n) = \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : f_{\pm,e} \in L^{p,\alpha}(\mathbb{R}^n)\}.$$

Proof. Pick a function $f \in L^{p,\alpha}(\mathbb{R}^n)$. For each $Q \subset \mathbb{R}^n$, we have

$$\begin{aligned}
&\int_Q |f_{+,e}(x)|^p dx + \int_Q |f_{-,e}(x)|^p dx \\
&\quad = \int_{Q_+} |f(x)|^p dx + \int_{Q_-} |f(\tilde{x})|^p dx + \int_{Q_+} |f(\tilde{x})|^p dx + \int_{Q_-} |f(x)|^p dx \\
&\quad = \int_Q |f(x)|^p dx + \int_{\widetilde{Q}} |f(x)|^p dx \le 2|Q|^{1+p\alpha}\|f\|^p_{L^{p,\alpha}(\mathbb{R}^n)},
\end{aligned}$$

which yields $f_{\pm,e} \in L^{p,\alpha}(\mathbb{R}^n)$.

Conversely, assume $f_{\pm,e} \in L^{p,\alpha}(\mathbb{R}^n)$. For each $Q \subset \mathbb{R}^n$, it follows

$$\begin{aligned}
\int_Q |f(x)|^p dx &\le \int_{\widehat{Q}_+} |f_{+,e}(x)|^p dx + \int_{\widehat{Q}_-} |f_{-,e}(x)|^p dx \\
&\le |\widehat{Q}_+|^{1+p\alpha}\|f_{+,e}\|^p_{L^{p,\alpha}(\mathbb{R}^n)} + |\widehat{Q}_+|^{1+p\alpha}\|f_{-,e}\|^p_{L^{p,\alpha}(\mathbb{R}^n)} \\
&= |Q|^{1+p\alpha}\|f_{+,e}\|^p_{L^{p,\alpha}(\mathbb{R}^n)} + |Q|^{1+p\alpha}\|f_{-,e}\|^p_{L^{p,\alpha}(\mathbb{R}^n)},
\end{aligned}$$

which implies $f \in L^{p,\alpha}(\mathbb{R}^n)$. The proof is completed. □

3.2. Hardy space. The real-variable theory of Hardy space on the Euclidean space $\mathbb{R}^n$ was initiated by Stein-Weiss [36], and then systematically developed by Fefferman-Stein [14]. However, to solve the Neumann problem, we shall recall the Hardy space associated with operator; see [3, 2, 6, 8, 19, 29, 30] for examples.

Definition 3.3. A function $f \in L^1(\Omega)$ is said to be in $H^1_{\mathcal{L}}(\Omega)$, the Hardy space associated with $\mathcal{L}$, if

$$\|f\|_{H^1_{\mathcal{L}}(\Omega)} = \int_\Omega \sup_{t>0} |\exp(-t\sqrt{\mathcal{L}})f(x)|dx < \infty,$$

where $\Omega = \mathbb{R}^n$ for $\mathcal{L} = L_N$, and $\Omega = \mathbb{R}^n_\pm$ for $\mathcal{L} = L_{N_\pm}$.

Proposition 3.4. *The Neumann Hardy space $H^1_{L_N}(\mathbb{R}^n)$ can be described in the following way*

$$\begin{aligned} H^1_{L_N}(\mathbb{R}^n) &= \{f \in L^1(\mathbb{R}^n) : f_\pm \in H^1_{L_{N_\pm}}(\mathbb{R}^n_\pm)\} \\ &= \{f \in L^1(\mathbb{R}^n) : f_{\pm,e} \in H^1(\mathbb{R}^n)\}, \end{aligned}$$

where $H^1(\mathbb{R}^n)$ denotes the classical Hardy space defined by the maximal function.

Proof. For each $x \in \mathbb{R}^n_\pm$, note that

$$\begin{aligned} \sup_{t>0} |\exp(-t\sqrt{L_N})f(x)| &= \sup_{t>0} |\exp(-t\sqrt{L_{N_\pm}})f_\pm(x)| \\ &= \sup_{t>0} |\exp(-t\sqrt{L})f_{\pm,e}(x)|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t\sqrt{L_N})f(x)|dx \\ &\quad= \int_{\mathbb{R}^n_+} \sup_{t>0} |\exp(-t\sqrt{L_{N_+}})f_+(x)|dx + \int_{\mathbb{R}^n_-} \sup_{t>0} |\exp(-t\sqrt{L_{N_-}})f_-(x)|dx \\ &\quad= \int_{\mathbb{R}^n_+} \sup_{t>0} |\exp(-t\sqrt{L})f_{+,e}(x)|dx + \int_{\mathbb{R}^n_-} \sup_{t>0} |\exp(-t\sqrt{L})f_{-,e}(x)|dx \\ &\quad= \frac{1}{2}\int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t\sqrt{L})f_{+,e}(x)|dx + \frac{1}{2}\int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t\sqrt{L})f_{-,e}(x)|dx, \end{aligned}$$

which completes the proof. □

3.3. BMO space. The space of functions of bounded mean oscillation (BMO) naturally arises as the class of functions whose deviation from their means over cubes is bounded. In [8, 9], Duong and Yan first introduce the BMO space associated with operator. We say that a locally integrable function f is in the class $M(\Omega)$, if there exists $\beta > 0$ such that

$$\int_\Omega \frac{|f(x)|}{(1+|x|)^{n+\beta}}dx < \infty.$$

Definition 3.5. For $-1/2 < \alpha < \theta/n$, [2] a function $f \in M(\Omega)$ is said to be in $\mathrm{BMO}^\alpha_{\mathcal{L}}(\Omega)$, the Morrey-Campanato space associated with $\mathcal{L}$, if

$$\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L}}(\Omega)} = \sup_{Q\subset\Omega} \frac{1}{|Q|^\alpha}\left(\frac{1}{|Q|}\int_Q |f(x) - \exp(-r_Q\sqrt{\mathcal{L}})f(x)|^2dx\right)^{1/2} < \infty,$$

where $\Omega = \mathbb{R}^n$ for $\mathcal{L} = L/L_N$, and $\Omega = \mathbb{R}^n_\pm$ for $\mathcal{L} = L_{N_\pm}$.

[2]Here θ denotes the Hölder index of the Poisson kernel $p_L(t,x,y)$.

Note that $(\mathrm{BMO}^{\alpha}_{\mathcal{L}}(\Omega), \|\cdot\|_{\mathrm{BMO}^{\alpha}_{\mathcal{L}}(\mathbb{R}^n)})$ is a semi-normed vector space, with the semi-norm vanishing on the kernel space $\mathcal{K}_{\mathcal{L}}$ defined by

$$\mathcal{K}_{\mathcal{L}} = \{f \in M(\Omega) : \exp(-t\sqrt{\mathcal{L}})f = f, \forall t > 0\}.$$

In this paper, the Morrey-Campanato space $\mathrm{BMO}^{\alpha}_{\mathcal{L}}(\Omega)$ is understood to be modulo kernel space $\mathcal{K}_{\mathcal{L}}$; see [5, 4, 9, 20, 29, 33] for more detials about this subject.

In order to analyze these Morrey-Campanato spaces, let us introduce the following classical Campanato space. For $-1/2 < \alpha < \theta/n$, the classical Campanato space $\mathrm{BMO}^{\alpha}(\mathbb{R}^n)$ is defined as the class of all locally square integrable functions f endowed with the norm

$$\|f\|_{\mathrm{BMO}^{\alpha}(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^{\alpha}} \left(\frac{1}{|Q|}\int_Q |f(x) - f_Q|^2 dx\right)^{1/2} < \infty,$$

where f_Q stands for the mean (or average) of f over Q. In fact, when $\alpha = 0$, the Campanato space BMO^0 reduces to the BMO space $\mathrm{BMO}(\mathbb{R}^n)$ of John and Nirenberg (see [22]), and when $0 < \alpha < \theta/n$, it coincides with the Lipschitz space $\Lambda_{\alpha}(\mathbb{R}^n)$ (see [14]). Moreover, when $-1/2 < \alpha < 0$, the norm of a Campanato function is equivalent to the following Morrey norm

$$\|f\|_{L^{2,\alpha}(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^{\alpha}} \left(\frac{1}{|Q|}\int_Q |f(x)|^2 dx\right)^{1/2};$$

see [32].

Therefore, the Campanato norm $\|\cdot\|_{\mathrm{BMO}^{\alpha}(\mathbb{R}^n)}$ unifies the Morrey norm $\|\cdot\|_{L^{2,\alpha}(\mathbb{R}^n)}$, the BMO norm $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^n)}$ and the Lipschitz norm $\|\cdot\|_{\Lambda_{\alpha}(\mathbb{R}^n)}$ by assigning different values to α.

The following proposition reveals the connection between the Neumann Morrey-Campanato space and the classical one.

Proposition 3.6. *Let $-1/2 < \alpha < \theta/n$. The Neumann Morrey-Campanato space $\mathrm{BMO}^{\alpha}_{L_N}(\mathbb{R}^n)$ can be described in the following way*

$$\begin{aligned}\mathrm{BMO}^{\alpha}_{L_N}(\mathbb{R}^n) &= \{f \in M(\mathbb{R}^n) : f_{\pm} \in \mathrm{BMO}^{\alpha}_{L_{N_{\pm}}}(\mathbb{R}^n_{\pm})\}\\ &= \{f \in M(\mathbb{R}^n) : f_{\pm,e} \in \mathrm{BMO}^{\alpha}_{L}(\mathbb{R}^n)\}\\ &= \{f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : f_{\pm,e} \in \mathrm{BMO}^{\alpha}(\mathbb{R}^n)\}.\end{aligned}$$

Proof. We divide our proof in following four steps

$$\begin{array}{ccc}\mathrm{BMO}^{\alpha}_{L_N}(\mathbb{R}^n) & \longrightarrow & \mathrm{BMO}^{\alpha}_{L_{N_{\pm}}}(\mathbb{R}^n_{\pm})\\ \nwarrow & & \swarrow\\ \mathrm{BMO}^{\alpha}_{L}(\mathbb{R}^n) & \longleftrightarrow & \mathrm{BMO}^{\alpha}(\mathbb{R}^n)\end{array}$$

Step 1. From $\mathrm{BMO}^{\alpha}_{L_N}(\mathbb{R}^n)$ to $\mathrm{BMO}^{\alpha}_{L_{N_{\pm}}}(\mathbb{R}^n_{\pm})$.

Pick a function $f \in \mathrm{BMO}^{\alpha}_{L_N}(\mathbb{R}^n)$, then we have

$$\|f_{\pm}\|_{\mathrm{BMO}^{\alpha}_{L_{N_{\pm}}}(\mathbb{R}^n_{\pm})}$$

$$
\begin{aligned}
&= \sup_{Q\subset\mathbb{R}^n_\pm} \frac{1}{|Q|^\alpha}\left(\frac{1}{|Q|}\int_Q |f_\pm(x)-(\exp(-r_Q\sqrt{L_N})f)_\pm(x)|^2dx\right)^{1/2}\\
&= \sup_{Q\subset\mathbb{R}^n_\pm} \frac{1}{|Q|^\alpha}\left(\frac{1}{|Q|}\int_Q |f(x)-\exp(-r_Q\sqrt{L_N})f(x)|^2dx\right)^{1/2}\\
&\le \|f\|_{\mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)},
\end{aligned}
$$

which yields $f_\pm \in \mathrm{BMO}^\alpha_{L_{N_\pm}}(\mathbb{R}^n_\pm)$.

Step 2. From $\mathrm{BMO}^\alpha_{L_{N_\pm}}(\mathbb{R}^n_\pm)$ to $\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$.

Assume $f_+ \in \mathrm{BMO}^\alpha_{L_{N_+}}(\mathbb{R}^n_+)$. For every $Q\subset\mathbb{R}^n$, it is sufficient to show that

$$
\int_Q |f_{+,e}(x)-\exp(-r_Q\sqrt{L})f_{+,e}(x)|^2dx \le 2|Q|^{1+2\alpha}\|f_+\|^2_{\mathrm{BMO}^\alpha_{L_{N_+}}(\mathbb{R}^n_+)}.
$$

If $Q_+\neq\emptyset$ and $Q_-\neq\emptyset$, then there holds by (2.9) that

$$
\begin{aligned}
&\int_Q |f_{+,e}(x)-\exp(-r_Q\sqrt{L})f_{+,e}(x)|^2dx\\
&\quad= \int_{Q_+} |f_+(x)-\exp(-r_Q\sqrt{L_{N_+}})f_+(x)|^2dx\\
&\qquad+ \int_{Q_-} |f_+(\tilde{x})-\exp(-r_Q\sqrt{L_{N_+}})f_+(\tilde{x})|^2dx\\
&\quad\le 2\int_{\widehat{Q}_+} |f_+(x)-\exp(-r_Q\sqrt{L_{N_+}})f_+(x)|^2dx\\
&\quad\le 2|\widehat{Q}_+|^{1+2\alpha}\|f_+\|^2_{\mathrm{BMO}^\alpha_{L_{N_+}}(\mathbb{R}^n_+)}\\
&\quad= 2|Q|^{1+2\alpha}\|f_+\|^2_{\mathrm{BMO}^\alpha_{L_{N_+}}(\mathbb{R}^n_+)}.
\end{aligned}
$$

If $Q_+=\emptyset$ or $Q_-=\emptyset$, then we can deduce from a similar argument above that

$$
\int_Q |f_{+,e}(x)-\exp(-r_Q\sqrt{L})f_{+,e}(x)|^2dx \le |Q|^{1+2\alpha}\|f_+\|^2_{\mathrm{BMO}^\alpha_{L_{N_+}}(\mathbb{R}^n_+)}.
$$

Therefore, one has $f_{\pm,e}\in\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ with desired norm control.

Step 3. From $\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ to $\mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)$.

Assuming $f_{\pm,e}\in\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$, for each $Q\subset\mathbb{R}^n$, we arrive at

$$
\begin{aligned}
&\int_Q |f(x)-\exp(-r_Q\sqrt{L_N})f(x)|^2dx\\
&\quad= \int_{Q_+} |f_+(x)-\exp(-r_Q\sqrt{L_{N_+}})f_+(x)|^2dx\\
&\qquad+ \int_{Q_-} |f_-(x)-\exp(-r_Q\sqrt{L_{N_-}})f_-(x)|^2dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{Q_+} |f_{+,e}(x) - \exp(-r_Q\sqrt{L})f_{+,e}(x)|^2 dx \\
&\quad + \int_{Q_-} |f_{-,e}(x) - \exp(-r_Q\sqrt{L})f_{-,e}(x)|^2 dx \\
&\leq |Q|^{1+2\alpha}\|f_{+,e}\|^2_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} + |Q|^{1+2\alpha}\|f_{-,e}\|^2_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)},
\end{aligned}
$$

which implies $f \in \mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)$.

Step 4. $\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ vs $\mathrm{BMO}^\alpha(\mathbb{R}^n)$.

When $-1/2 < \alpha < 0$, $\alpha = 0$ or $0 < \alpha < \theta/n$, the space $\mathrm{BMO}^\alpha(\mathbb{R}^n)$ reduces to the Morrey space, the BMO space and the Lipschitz space, respectively. These spaces can be characterized by the Morrey-Campanato space $\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ associated with L; see [5, 7, 8, 9] for more details.

Based on the above arguments, we obtain the desired conclusion. □

3.4. HMO space. To introduce the HMO space related to the Neumann problem, we shall recall the classical HMO space; see [11, 17, 18, 33, 34] for examples. For $-1/2 < \alpha < \theta/n$, a harmonic function $u(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ is said to be in $\mathrm{HMO}^\alpha_L(\mathbb{R}^n \times \mathbb{R}_+)$, the harmonic mean oscillation space associated with L, if

$$
\|u\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^\alpha}\left(\int_0^{r_Q}\frac{1}{|Q|}\int_Q |t\nabla_{x,t}u(x,t)|^2 dx\frac{dt}{t}\right)^{1/2} < \infty.
$$

Definition 3.7. For $-1/2 < \alpha < \theta/n$, a Neumann harmonic function $u(x,t)$ defined on $\Omega \times \mathbb{R}_+$ is said to be in $\mathrm{HMO}^\alpha_{\mathcal{L}}(\Omega \times \mathbb{R}_+)$, the Neumann harmonic mean oscillation space associated with $\mathcal{L}$, if

$$
\|u\|_{\mathrm{HMO}^\alpha_{\mathcal{L}}(\Omega\times\mathbb{R}_+)} = \sup_{Q\subset\Omega} \frac{1}{|Q|^\alpha}\left(\int_0^{r_Q}\frac{1}{|Q|}\int_Q |t\nabla_{x,t}u(x,t)|^2 dx\frac{dt}{t}\right)^{1/2} < \infty,
$$

where $\Omega = \mathbb{R}^n$ for $\mathcal{L} = L_N$, and $\Omega = \mathbb{R}^n_\pm$ for $\mathcal{L} = L_{N_\pm}$.

Proposition 3.8. *For $-1/2 < \alpha < \theta/n$, the Neumann HMO space $\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n \times \mathbb{R}_+)$ can be described in the following way*

$$
\begin{aligned}
\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+) &= \{u \in W^{1,2}(\mathbb{R}^n\times\mathbb{R}_+) : u_\pm \in \mathrm{HMO}^\alpha_{L_{N_\pm}}(\mathbb{R}^n_\pm\times\mathbb{R}_+)\} \\
&= \{u \in W^{1,2}(\mathbb{R}^n\times\mathbb{R}_+) : u_{\pm,e} \in \mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)\}.
\end{aligned}
$$

Proof. We divide our proof in following three steps

$$
\begin{array}{ccc}
\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+) & \longrightarrow & \mathrm{HMO}^\alpha_{L_{N_\pm}}(\mathbb{R}^n_\pm\times\mathbb{R}_+) \\
\nwarrow & & \swarrow \\
& \mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+) &
\end{array}
$$

Step 1. From $\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+)$ to $\mathrm{HMO}^\alpha_{L_{N_\pm}}(\mathbb{R}^n_\pm\times\mathbb{R}_+)$.

Pick a function $u \in \mathrm{HMO}^{\alpha}_{L_N}(\mathbb{R}^n)$. Evidently, its restriction $u_\pm$ is also a Neumann harmonic function on $\mathbb{R}^n_\pm \times \mathbb{R}_+$. Moreover, it holds that

$$\begin{aligned}
\|u_\pm\|_{\mathrm{HMO}^{\alpha}_{L_{N_\pm}}(\mathbb{R}^n_\pm\times\mathbb{R}_+)} &= \sup_{Q\subset\mathbb{R}^n_\pm} \frac{1}{|Q|^\alpha}\left(\int_0^{r_Q}\frac{1}{|Q|}\int_Q |t\nabla_{x,t}u(x,t)|^2 dx\frac{dt}{t}\right)^{1/2}\\
&\le \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^\alpha}\left(\int_0^{r_Q}\frac{1}{|Q|}\int_Q |t\nabla_{x,t}u(x,t)|^2 dx\frac{dt}{t}\right)^{1/2}\\
&= \|u\|_{\mathrm{HMO}^{\alpha}_{L_N}(\mathbb{R}^n\times\mathbb{R}_+)},
\end{aligned}$$

which yields $u_\pm \in \mathrm{HMO}^{\alpha}_{L_{N_\pm}}(\mathbb{R}^n_\pm \times \mathbb{R}_+)$.

Step 2. From $\mathrm{HMO}^{\alpha}_{L_{N_\pm}}(\mathbb{R}^n_\pm \times \mathbb{R}_+)$ to $\mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n \times \mathbb{R}_+)$.

Assume $u_\pm \in \mathrm{HMO}^{\alpha}_{L_{N_\pm}}(\mathbb{R}^n_\pm \times \mathbb{R}_+)$. Since $u_\pm$ is Neumann harmonic in $\mathbb{R}^n_\pm \times \mathbb{R}_+$, the even extension of $u_\pm$ is harmonic in $\mathbb{R}^n \times \mathbb{R}_+$. For each $Q \subset \mathbb{R}^n$, there holds

$$\begin{aligned}
&\int_0^{r_Q}\int_Q |t\nabla_{x,t}u_{+,e}(x,t)|^2 dx\frac{dt}{t}\\
&\quad= \int_0^{r_Q}\int_{Q_+} |t\nabla_{x,t}u_+(x,t)|^2 dx\frac{dt}{t} + \int_0^{r_Q}\int_{Q_-} |t\nabla_{x,t}u_+(\tilde{x},t)|^2 dx\frac{dt}{t}\\
&\quad\le 2|\widehat{Q}_+|^{1+2\alpha}\|u_+\|^2_{\mathrm{HMO}^{\alpha}_{L_{N_+}}(\mathbb{R}^n_+\times\mathbb{R}_+)}\\
&\quad= 2|Q|^{1+2\alpha}\|u_+\|^2_{\mathrm{HMO}^{\alpha}_{L_{N_+}}(\mathbb{R}^n_+\times\mathbb{R}_+)},
\end{aligned}$$

which implies $u_{+,e} \in \mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n \times \mathbb{R}_+)$ and $u_{-,e} \in \mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n \times \mathbb{R}_+)$ similarly.

Step 3. From $\mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n \times \mathbb{R}_+)$ to $\mathrm{HMO}^{\alpha}_{L_N}(\mathbb{R}^n \times \mathbb{R}_+)$.

Assume $u_{\pm,e} \in \mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n \times \mathbb{R}_+)$. Noticing that $u_{\pm,e}$ is an even function with respect to x_n, we obtain that its derivative is an odd function, and hence $u = u_+ + u_-$ satisfies the Neumann boundary condition automatically. For every $Q \subset \mathbb{R}^n$, it follows

$$\begin{aligned}
&\int_0^{r_Q}\int_Q |t\nabla_{x,t}u(x,t)|^2 dx\frac{dt}{t}\\
&\quad\le \int_0^{r_Q}\int_{\widehat{Q}_+} |t\nabla_{x,t}u_{+,e}(x,t)|^2 dx\frac{dt}{t} + \int_0^{r_Q}\int_{\widehat{Q}_-} |t\nabla_{x,t}u_{-,e}(x,t)|^2 dx\frac{dt}{t}\\
&\quad\le |\widehat{Q}_+|^{1+2\alpha}\|u_{+,e}\|^2_{\mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n\times\mathbb{R}_+)} + |\widehat{Q}_-|^{1+2\alpha}\|u_{-,e}\|^2_{\mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n\times\mathbb{R}_+)}\\
&\quad= |Q|^{1+2\alpha}\|u_{+,e}\|^2_{\mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n\times\mathbb{R}_+)} + |Q|^{1+2\alpha}\|u_{-,e}\|^2_{\mathrm{HMO}^{\alpha}_{L}(\mathbb{R}^n\times\mathbb{R}_+)},
\end{aligned}$$

which yields $u \in \mathrm{HMO}^{\alpha}_{L_N}(\mathbb{R}^n \times \mathbb{R}_+)$.

Based on the above arguments, we obtain the desired conclusion. $\square$

4. Neumann harmonic function with Lebesgue trace

This section is devoted to solving the elliptic equation of Neumann type with the Lebesgue initial value.

Theorem 4.1. *Let $1 < p < \infty$. A function $u(x,t)$ is Neumann harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L_N}) f(x)$ for some $f \in L^p(\mathbb{R}^n)$.

Proof. Step 1. Necessity: from solution to trace.

If $u(\cdot,t) \in L^p(\mathbb{R}^n)$, then $u_{\pm,e}(\cdot,t) \in L^p(\mathbb{R}^n)$ by Proposition 3.2. Noticing $u_{+,e}(x,t)$ is harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ and

$$\sup_{t>0} \|u_{+,e}(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C,$$

we deduce from the characterization of the Poisson integral with Lebesgue trace (see Appendix A) that

$$u_{+,e}(x,t) = \exp(-t\sqrt{L})g(x)$$

for some $g \in L^p(\mathbb{R}^n)$. Moreover, it holds

$$\begin{aligned} g(x',-x_n) &= \lim_{t\to 0^+} \exp(-t\sqrt{L})g(x',-x_n) = \lim_{t\to 0^+} u_{+,e}(x',-x_n,t) \\ &= \lim_{t\to 0^+} u_{+,e}(x',x_n,t) = \lim_{t\to 0^+} \exp(-t\sqrt{L})g(x',x_n) = g(x',x_n), \end{aligned}$$

which yields g is an even function with respect to x_n. Therefore, for all $x \in \mathbb{R}^n_+$, we have

$$\begin{aligned} u_+(x,t) &= u_{+,e}(x,t) = \exp(-t\sqrt{L})g(x) \\ &= \exp(-t\sqrt{L})g_{+,e}(x) = \exp(-t\sqrt{L_{N_+}})g_+(x). \end{aligned}$$

Similarly, for $u_{-,e}(\cdot,t) \in L^p(\mathbb{R}^n)$, there exists an even function $h \in L^p(\mathbb{R}^n)$ such that

$$u_{-,e}(x,t) = \exp(-t\sqrt{L})h(x)$$

and, for each $x \in \mathbb{R}^n_-$, it holds

$$u_-(x,t) = \exp(-t\sqrt{L_{N_-}})h_-(x).$$

Finally, by letting $f = g_+ + h_-$, we see that $f_{+,e} = g_{+,e} = g \in L^p(\mathbb{R}^n)$ and $f_{-,e} = h_{-,e} = h \in L^p(\mathbb{R}^n)$, which together with Proposition 3.2 implies $f \in L^p(\mathbb{R}^n)$. It can been easily seen that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} u(x,t) &= \exp(-t\sqrt{L_{N_+}})g_+(x) + \exp(-t\sqrt{L_{N_-}})h_-(x) \\ &= \exp(-t\sqrt{L_{N_+}})f_+(x) + \exp(-t\sqrt{L_{N_-}})f_-(x) \\ &= (\exp(-t\sqrt{L_N})f)_+(x) + (\exp(-t\sqrt{L_N})f)_-(x) \end{aligned}$$

$$= \exp(-t\sqrt{L_N})f(x).$$

Therefore, the necessity of Theorem 4.1 follows.

Step 2. Sufficiency: from trace to solution.

If $f \in L^p(\mathbb{R}^n)$, then $f_{\pm,e} \in L^p(\mathbb{R}^n)$ by Proposition 3.2. Note that

$$\begin{aligned} u_\pm(x,t) &= (\exp(-t\sqrt{L_N})f)_\pm(x) \\ &= \exp(-t\sqrt{L_{N_\pm}})f_\pm(x) = \exp(-t\sqrt{L})f_{\pm,e}(x)\chi_{\mathbb{R}^n_\pm}(x), \end{aligned}$$

which tells us that u is Neumann harmonic in $\mathbb{R}^n \times \mathbb{R}_+$. By the Poisson upper bound on $p_L(t,x,y)$, and Minkowski's inequality, we arrive at

$$\begin{aligned} \sup_{t>0} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} &= \sup_{t>0} \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} p_{L_N}(t,x,y)f(y)dy \right|^p dx \right)^{1/p} \\ &\leq C \sup_{t>0} \left[\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \frac{t}{(t+|y|)^{n+1}} |f(x-y)|dy \right)^p dx \right]^{1/p} \\ &\leq C \sup_{t>0} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|^p dx \right)^{1/p} \frac{t}{(t+|y|)^{n+1}} dy \\ &\leq C\|f\|_{L^p(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of the sufficiency and hence Theorem 4.1. □

5. Neumann harmonic function with Hardy trace

In this section, we consider the Neumann harmonic function with Hardy trace. For a harmonic function u in $\mathbb{R}^n \times \mathbb{R}_+$, let u^* denote its maximal function defined by

$$u^*(x) = \sup_{t>0} |u(x,t)|.$$

Theorem 5.1. *A function $u(x,t)$ is Neumann harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u^*\|_{L^1(\mathbb{R}^n)} \leq C$$

if and only if $u(x,t) = \exp(-t\sqrt{L_N})f(x)$ for some $f \in H^1_{L_N}(\mathbb{R}^n)$.

Remark 5.2. When A = identity matrix, Stein [35, page 119, Proposition 1] characterized the harmonic function with H^p trace for all $0 < p < \infty$. He proved that, for $0 < p < \infty$, a function $u(x,t)$ is harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with $\|u^*\|_{L^p(\mathbb{R}^n)} \leq C$ if and only if $u(x,t) = \exp(-t\sqrt{\Delta})f(x)$ for some $f \in H^p(\mathbb{R}^n)$, where Δ is the negative Laplacian on $\mathbb{R}^n$. When $0 < p \leq 1$, the Hardy space $H^p(\mathbb{R}^n)$ is a proper subset of $L^p(\mathbb{R}^n)$, and when $p > 1$, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. In order to emphasize the difference between Hardy space and Lebesgue space, we only consider the left endpoint case $p = 1$. In fact, our Theorem 5.1 also holds for $p > 1$. It is worth pointing out that $H^p_{L_N}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ provided $p > 1$, and hence

$$\left(\int_{\mathbb{R}^n} \sup_{t>0} |u(x,t)|^p dx\right)^{1/p} \le C \Leftrightarrow \sup_{t>0}\left(\int_{\mathbb{R}^n} |u(x,t)|^p dx\right)^{1/p} \le C$$

for arbitrary Neumann harmonic function u.

Proof of Theorem 5.1. Step 1. Necessity: from solution to trace.

If $u^* \in L^1(\mathbb{R}^n)$, then $(u_{\pm,e})^* = (u^*)_{\pm,e} \in L^1(\mathbb{R}^n)$ by Proposition 3.2. Noticing $u_{+,e}(x,t)$ is harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ and

$$\|(u_{+,e})^*\|_{L^1(\mathbb{R}^n)} \le C,$$

we deduce from the characterization of the Poisson integral with Hardy trace (see Appendix A) that

$$u_{+,e}(x,t) = \exp(-t\sqrt{L})g(x)$$

for some $g \in H^1_L(\mathbb{R}^n)$. Moreover, it holds

$$\begin{aligned} g(x', -x_n) &= \lim_{t\to 0^+} \exp(-t\sqrt{L})g(x', -x_n) = \lim_{t\to 0^+} u_{+,e}(x', -x_n, t) \\ &= \lim_{t\to 0^+} u_{+,e}(x', x_n, t) = \lim_{t\to 0^+} \exp(-t\sqrt{L})g(x', x_n) = g(x', x_n), \end{aligned}$$

which yields g is an even function with respect to x_n. Therefore, for all $x \in \mathbb{R}^n_+$, we have

$$\begin{aligned} u_+(x,t) &= u_{+,e}(x,t) = \exp(-t\sqrt{L})g(x) \\ &= \exp(-t\sqrt{L})g_{+,e}(x) = \exp(-t\sqrt{L_{N_+}})g_+(x). \end{aligned}$$

Similarly, for $(u_{-,e})^* \in L^1(\mathbb{R}^n)$, there exists an even function $h \in H^1_L(\mathbb{R}^n)$ such that

$$u_{-,e}(x,t) = \exp(-t\sqrt{L})h(x)$$

and, for each $x \in \mathbb{R}^n_-$, it holds

$$u_-(x,t) = \exp(-t\sqrt{L_{N_-}})h_-(x).$$

Finally, by letting $f = g_+ + h_-$, we see that $f_{+,e} = g_{+,e} = g \in H^1_L(\mathbb{R}^n)$ and $f_{-,e} = h_{-,e} = h \in H^1_L(\mathbb{R}^n)$, which together with Proposition 3.4 implies $f \in H^1_{L_N}(\mathbb{R}^n)$. It can been easily seen that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} u(x,t) &= \exp(-t\sqrt{L_{N_+}})g_+(x) + \exp(-t\sqrt{L_{N_-}})h_-(x) \\ &= \exp(-t\sqrt{L_{N_+}})f_+(x) + \exp(-t\sqrt{L_{N_-}})f_-(x) \\ &= (\exp(-t\sqrt{L_N})f)_+(x) + (\exp(-t\sqrt{L_N})f)_-(x) \\ &= \exp(-t\sqrt{L_N})f(x). \end{aligned}$$

Therefore, the necessity of Theorem 4.1 follows.

Step 2. Sufficiency: from trace to solution.

If $f \in H^1_{L_N}(\mathbb{R}^n)$, then $f_{\pm,e} \in H^1_L(\mathbb{R}^n)$ by Proposition 3.4. Note that

$$\begin{aligned} u_\pm(x,t) &= (\exp(-t\sqrt{L_N})f)_\pm(x) \\ &= \exp(-t\sqrt{L_{N_\pm}})f_\pm(x) = \exp(-t\sqrt{L})f_{\pm,e}(x)\chi_{\mathbb{R}^n_\pm}(x), \end{aligned}$$

which tells us that u is Neumann harmonic in $\mathbb{R}^n \times \mathbb{R}_+$.

Moreover, it holds that

$$\|u^*\|_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t\sqrt{L_N})f(x)|dx = \|f\|_{H^1_{L_N}(\mathbb{R}^n)},$$

which completes the proof. □

6. Neumann harmonic function with BMO trace

The following theorem is the main result in this section.

Theorem 6.1. *Let $-1/2 < \alpha < \theta/n$. A function $u(x,t)$ is Neumann harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u\|_{\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L_N})f(x)$ for some $f \in \mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)$.

Remark 6.2. In Theorem 6.1, when the Neumann boundary condition is removed, the Laplace version of this result can reduce to the classical one of Fabes-Johnson-Neri [11, Theorem 1.0] and Jiang-Xiao-Yang [18, Theorem 1.1]; see [23] for the fractional case. They proved that, for $-1/2 < \alpha < 1/n$, a function $u \in \mathrm{HMO}^\alpha_\Delta(\mathbb{R}^n \times \mathbb{R}_+)$ if and only if $u(x,t) = \exp(-t\sqrt{\Delta})f(x)$ for some $f \in \mathrm{BMO}^\alpha_\Delta(\mathbb{R}^n)$. Moreover there exists a constant $C > 0$ such that

$$C^{-1}\|f\|_{\mathrm{BMO}^\alpha_\Delta(\mathbb{R}^n)} \le \|u\|_{\mathrm{HMO}^\alpha_\Delta(\mathbb{R}^n\times\mathbb{R}_+)} \le C\|f\|_{\mathrm{BMO}^\alpha_\Delta(\mathbb{R}^n)}.$$

Proof of Theorem 6.1. Step 1. Necessity: from solution to trace.

If $u \in \mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n \times \mathbb{R}_+)$, then $u_{\pm,e} \in \mathrm{HMO}^\alpha_L(\mathbb{R}^n \times \mathbb{R}_+)$ by Proposition 3.8.

From Appendix A, we know that $u_{+,e}(x,t) = \exp(-t\sqrt{L})g(x)$ for some $g \in \mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ with

$$\|g\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} \le C\|u_{+,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)}.$$

It can been easily seen

$$\begin{aligned} g(x',-x_n) &= \lim_{t\to0^+} \exp(-t\sqrt{L})g(x',-x_n) = \lim_{t\to0^+} u_{+,e}(x',-x_n,t) \\ &= \lim_{t\to0^+} u_{+,e}(x',x_n,t) = \lim_{t\to0^+} \exp(-t\sqrt{L})g(x',x_n) = g(x',x_n), \end{aligned}$$

which yields g is an even function with respect to x_n, and hence

$$\|g_{+,e}\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} = \|g\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} \le C\|u_{+,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)}.$$

Moreover, for every $x \in \mathbb{R}^n_+$, one has

$$\begin{aligned} u_+(x,t) = u_{+,e}(x,t) &= \exp(-t\sqrt{L})g(x) \\ &= \exp(-t\sqrt{L})g_{+,e}(x) = \exp(-t\sqrt{L_{N_+}})g_+(x). \end{aligned}$$

Similarly, for $u_{-,e} \in \mathrm{HMO}^\alpha_L(\mathbb{R}^n \times \mathbb{R}_+)$, there exists an even function $h \in \mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ such that $u_{-,e}(x,t) = \exp(-t\sqrt{L})h(x)$ with

$$\|h_{-,e}\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} \le C\|u_{-,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)}$$

and, for every $x \in \mathbb{R}^n_-$, it holds

$$u_-(x,t) = \exp(-t\sqrt{L_{N_-}})h_-(x).$$

Finally, by letting $f = g_+ + h_-$, we see that $f_{+,e} = g_{+,e} = g \in \mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ and $f_{-,e} = h_{-,e} = h \in \mathrm{BMO}^\alpha_L(\mathbb{R}^n)$, which together with Propositions 3.6 and 3.8 yields $f \in \mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)$ with

$$\begin{aligned} \|f\|_{\mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)} &\le \|f_{+,e}\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} + \|f_{-,e}\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} \\ &= \|g_{+,e}\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} + \|h_{-,e}\|_{\mathrm{BMO}^\alpha_L(\mathbb{R}^n)} \\ &\le C\|u_{+,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)} + C\|u_{-,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)} \\ &\le C\|u\|_{\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+)}. \end{aligned}$$

Moreover, it can been easily seen that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} u(x,t) &= \exp(-t\sqrt{L_{N_+}})g_+(x) + \exp(-t\sqrt{L_{N_-}})h_-(x) \\ &= \exp(-t\sqrt{L_{N_+}})f_+(x) + \exp(-t\sqrt{L_{N_-}})f_-(x) \\ &= (\exp(-t\sqrt{L_N})f)_+(x) + (\exp(-t\sqrt{L_N})f)_-(x) \\ &= \exp(-t\sqrt{L_N})f(x). \end{aligned}$$

Therefore, the necessity of Theorem 6.1 follows.

Step 2. Sufficiency: from trace to solution.

If $f \in \mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)$, then $f_{\pm,e} \in \mathrm{BMO}^\alpha_L(\mathbb{R}^n)$ by Proposition 3.6. Note that

$$\begin{aligned} u(x,t) &= (\exp(-t\sqrt{L_N})f)_+(x) + (\exp(-t\sqrt{L_N})f)_-(x) \\ &= \exp(-t\sqrt{L_{N_+}})f_+(x) + \exp(-t\sqrt{L_{N_-}})f_-(x) \\ &= \exp(-t\sqrt{L})f_{+,e}(x)\chi_{\mathbb{R}^n_+}(x) + \exp(-t\sqrt{L})f_{-,e}(x)\chi_{\mathbb{R}^n_-}(x), \end{aligned}$$

and hence

$$u_{\pm,e}(x,t) = \exp(-t\sqrt{L})f_{\pm,e}(x).$$

This implies u is Neumann harmonic in $\mathbb{R}^n \times \mathbb{R}_+$. From Appendix A, Propositions 3.6 and 3.8, we deduce

$$\|u\|_{\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+)} \le \|u_{+,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)} + \|u_{-,e}\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)}$$

$$\begin{aligned} &\le C\|f_{+,e}\|_{\mathrm{BMO}_L^\alpha(\mathbb{R}^n)} + C\|f_{-,e}\|_{\mathrm{BMO}_L^\alpha(\mathbb{R}^n)} \\ &\le C\|f\|_{\mathrm{BMO}_{L_N}^\alpha(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of the sufficiency and hence Theorem 6.1. □

7. Final remarks

This paper will end by pointing out some further results. Since it is intended solely as a brief review and not as a rigorous development, the pertinent results are stated without proof.

7.1. The elliptic equation of Dirichlet type. Let us first substitute the Dirichlet boundary condition for the Neumann one, and then consider the following problem

$$\begin{cases} -\partial_t^2 u(x,t) + Lu(x,t) = 0, & x \in \mathbb{R}^n,\ t > 0, \\ u(x',0,t) = 0, & x' \in \mathbb{R}^{n-1},\ t > 0, \\ u(x) = f(x), & x \in \mathbb{R}^n. \end{cases}$$

In this case, the reflection method is to use the odd extension rather than the even one.

Denote by L_D the corresponding Dirichlet elliptic operator on $\mathbb{R}^n$, and let $\{\exp(-t\sqrt{L_D})\}_{t>0}$ be the the Poisson semigroups associated with Δ_D. Note that the conservative property

$$\exp(-t\sqrt{\mathcal{L}})1 = 1$$

does not hold for $\mathcal{L} = L_D$ or $L_{D_\pm}$.

Similar to the Neumann case, we can define some function classes related to the Dirichlet problem; for example $H^1_{L_D}(\mathbb{R}^n)$, $\mathrm{BMO}^\alpha_{L_D}(\mathbb{R}^n)$ and $\mathrm{HMO}^\alpha_{L_D}(\mathbb{R}^n \times \mathbb{R}_+)$. Therefore, we can derive the following result by means of these function classes.

Theorem 7.1. *For different Lebesgue, Hardy and BMO traces, the following statements are valid.*

(i) *Let $1 < p < \infty$. A function $u(x,t)$ is Dirichlet harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L_D})f(x)$ for some $f \in L^p(\mathbb{R}^n)$.

(ii) *A function $u(x,t)$ is Dirichlet harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u^*\|_{L^1(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L_D})f(x)$ for some $f \in H^1_{L_D}(\mathbb{R}^n)$.

(iii) *Let $-1/2 < \alpha < \theta/n$. A function $u(x,t)$ is Dirichlet harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u\|_{\mathrm{HMO}^\alpha_{L_D}(\mathbb{R}^n \times \mathbb{R}_+)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L_D})f(x)$ for some $f \in \mathrm{BMO}^\alpha_{L_D}(\mathbb{R}^n)$.

7.2. The parabolic equation of Neumann/Dirichlet type. In paper [40], Zhang and Yang consider the following boundary value problem for the heat equation

$$\begin{cases}\partial_t u(x,t)+\Delta u(x,t)=0, & x\in\mathbb{R}^n,\ t>0,\\ \partial_{x_n}u(x',0,t)=0, & x'\in\mathbb{R}^{n-1},\ t>0,\\ u(x,0)=f(x), & x\in\mathbb{R}^n,\end{cases}$$

where Δ is the negative Laplacian on $\mathbb{R}^n$. They proved that a caloric function (the solution to the heat equation) defined on $\mathbb{R}^n\times\mathbb{R}_+$ with the Neumann boundary condition satisfies the parabolic Carleson measure condition if and only if it can be represented as the Gaussian integral of a BMO function associated with Δ_N. They define the solution space $\mathrm{TMO}_{\Delta_N}(\mathbb{R}^n\times\mathbb{R}_+)$ as follows.

Definition 7.2. A Neumann caloric function $u(x,t)$ defined on $\mathbb{R}^n\times\mathbb{R}_+$ is said to be in $\mathrm{TMO}_{\Delta_N}(\mathbb{R}^n\times\mathbb{R}_+)$, the temperature mean oscillation space associated with Δ_N, if

$$\begin{aligned}\|u\|_{\mathrm{TMO}_{\Delta_N}(\mathbb{R}^n\times\mathbb{R}_+)}&=\max\|u_\pm\|_{\mathrm{TMO}_{\Delta_{N_\pm}}(\mathbb{R}^n_\pm\times\mathbb{R}_+)}\\ &=\max\sup_{Q\subset\mathbb{R}^n_\pm}\left(\int_0^{r_Q^2}\frac{1}{|Q|}\int_Q|\nabla_x u_\pm(x,t)|^2dxdt\right)^{1/2}<\infty.\end{aligned}$$

However, in the classical case, for a caloric function $u(x,t)$ without the Neumann boundary condition, its TMO norm is related to the parabolic Carleson measure on the whole space $\mathbb{R}^n\times\mathbb{R}_+$, rather than its restriction $u_\pm(x,t)$ on the half-space $\mathbb{R}^n_\pm\times\mathbb{R}_+$. In fact, by repeating the arguments in the proof of Proposition 3.8, the Neumann TMO space can be described in the following way

$$\begin{aligned}\mathrm{TMO}_{\Delta_N}(\mathbb{R}^n)&=\{u\in W^{1,2}(\mathbb{R}^n):\ u_{\pm,e}\text{ is a caloric function}\\ &\qquad\text{and }|\nabla_x u_{\pm,e}|^2dxdt\text{ is a parabolic Carleson measure}\}\\ &=\{u\in W^{1,2}(\mathbb{R}^n):\ u\text{ is a Neumann caloric function}\\ &\qquad\text{and }|\nabla_x u|^2dxdt\text{ is a parabolic Carleson measure}\}.\end{aligned}$$

Analogously, we can consider the following parabolic equation of Neumann or Dirichlet type

$$\begin{cases}\partial_t u(x,t)-\mathrm{div}A\nabla u(x,t)=0, & x\in\mathbb{R}^n,\ t>0,\\ \partial_{x_n}u(x',0,t)/u(x',0,t)=0, & x'\in\mathbb{R}^{n-1},\ t>0,\\ u(x,0)=f(x), & x\in\mathbb{R}^n,\end{cases}$$

where the matrix A is even with respect to the n-th variable and satisfies the admissible caloric condition, namely, for each caloric function $w(x,t)$ on $\mathbb{R}^n\times\mathbb{R}_+$, it holds

$$\int_{\mathbb{R}^n\times\mathbb{R}_+}\beta(\nabla w\partial_{x_n}\varphi+\nabla\varphi\partial_{x_n}w)dxdt=2\int_{\mathbb{R}^n\times\mathbb{R}_+}a^{nn}\partial_{x_n}w\partial_{x_n}\varphi dxdt,$$

for any $\phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$.

To state the main result, let us introduce some function classes related to the Neumann/Dirichlet heat equation.

Definition 7.3. A function $f \in L^1(\Omega)$ is said to be in $H^1_{\mathcal{L}}(\Omega)$, the Hardy space associated with $\mathcal{L}$, if

$$\|f\|_{H^1_{\mathcal{L}}(\Omega)} = \int_{\mathbb{R}^n} \sup_{t>0} |\exp(-t\mathcal{L})f(x)|dx < \infty,$$

where $\mathcal{L} = L_N/L_D$ for the Neumann/Dirichlet problem.

Definition 7.4. For $-1/2 < \alpha < \theta/n$,[3] a function $f \in M(\mathbb{R}^n)$ is said to be in $\mathrm{BMO}^\alpha_{\mathcal{L}}(\mathbb{R}^n)$, the Morrey-Campanato space associated with $\mathcal{L}$, if

$$\|f\|_{\mathrm{BMO}^\alpha_{\mathcal{L}}(\mathbb{R}^n)} = \sup_{Q\subset\mathbb{R}^n} \frac{1}{|Q|^\alpha}\left(\frac{1}{|Q|}\int_Q |f(x) - \exp(-r_Q^2\mathcal{L})f(x)|^2 dx\right)^{1/2} < \infty,$$

where $\mathcal{L} = L_N/L_D$ for the Neumann/Dirichlet problem.

Theorem 7.5. *For different Lebesgue, Hardy and BMO traces, the following statements are valid.*

(i) *Let $1 < p < \infty$. A function $u(x,t)$ is Neumann/Dirichlet caloric in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-tL_N)f(x)/\exp(-tL_D)f(x)$ for some $f \in L^p(\mathbb{R}^n)$.

(ii) *A function $u(x,t)$ is Neumann/Dirichlet caloric in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u^*\|_{L^1(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-tL_N)f(x)/\exp(-tL_D)f(x)$ for some $f \in H^1_{L_N}(\mathbb{R}^n)/H^1_{L_D}(\mathbb{R}^n)$.

(iii) *Let $-1/2 < \alpha < \theta/n$. A function $u(x,t)$ is Neumann/Dirichlet caloric in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u\|_{\mathrm{HMO}^\alpha_{L_N}(\mathbb{R}^n\times\mathbb{R}_+)/\mathrm{HMO}^\alpha_{L_D}(\mathbb{R}^n\times\mathbb{R}_+)} \le C$$

if and only if $u(x,t) = \exp(-tL_N)f(x)/\exp(-tL_D)f(x)$ for some $f \in \mathrm{BMO}^\alpha_{L_N}(\mathbb{R}^n)/\mathrm{BMO}^\alpha_{L_D}(\mathbb{R}^n)$.

Appendix A: Some classical conclusions

In this appendix, we provide some classical results about the boundary value problem for the elliptic/parabolic equation.

[3] Here θ denotes the Hölder index of the heat kernel $h_L(t,x,y)$.

Theorem A.1 *Let $1 < p < \infty$. A function $u(x,t)$ is harmonic/caloric in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\sup_{t>0} \|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L})f(x)/\exp(-tL)f(x)$ for some $f \in L^p(\mathbb{R}^n)$.

Proof. For this conclusion, see [25, 27] for more details. □

Theorem A.2 *A function $u(x,t)$ is harmonic/caloric in $\mathbb{R}^n \times \mathbb{R}_+$ with*

$$\|u^*\|_{L^1(\mathbb{R}^n)} \le C$$

if and only if $u(x,t) = \exp(-t\sqrt{L})f(x)/\exp(-tL)f(x)$ for some $f \in H^1_L(\mathbb{R}^n)$.

Proof. We only consider the elliptic case since the proof of the parabolic case is similar.[4]

Step 1. Necessity: from solution to trace.

Suppose that $u(x,t)$ is harmonic in $\mathbb{R}^n \times \mathbb{R}_+$ with $\|u^*\|_{L^1(\mathbb{R}^n)} \le C$. We first claim that

$$u(x,t+s) = \exp(-t\sqrt{L})(u(\cdot,s))(x). \tag{7.1}$$

To this end, define

$$H(x,t) = u(x,t+s) - \exp(-t\sqrt{L})(u(\cdot,s))(x).$$

It follows from the mean value property of the harmonic function that

$$\begin{aligned} |u(x,t)| &\le C \fint_{t/2}^{3t/2} \fint_{B(x,t/2)} |u(y,s)|dyds \\ &\le C \fint_{t/2}^{3t/2} \fint_{B(x,t/2)} |u^*(y)|dyds \le \frac{C}{t^n}\|u^*\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

and from the conservative property of $\exp(-t\sqrt{L})$ that

$$|\exp(-t\sqrt{L})(u(\cdot,s))(x)| \le \int_{\mathbb{R}^n} p_L(t,x,y)|u(y,s)|dy \le \frac{C}{s^n}\|u^*\|_{L^1(\mathbb{R}^n)}.$$

By these estimates, we see that

$$|H(x,t)| \le \frac{C}{(t+s)^n}\|u^*\|_{L^1(\mathbb{R}^n)} + \frac{C}{s^n}\|u^*\|_{L^1(\mathbb{R}^n)} \le \frac{C}{s^n}\|u^*\|_{L^1(\mathbb{R}^n)}$$

with

$$H(x,0) = \lim_{t\to0} H(x,t) = \lim_{t\to0} u(x,t+s) - \lim_{t\to0} \exp(-t\sqrt{L})(u(\cdot,s))(x) = 0.$$

[4]For the proof of the parabolic case, we refer the readers to the book [15, Chapter 8] for example.

This means $H(x,t)$ is a bounded harmonic function on $\mathbb{R}^n\times\mathbb{R}_+$ with zero trace. Therefore, one can employ the reflection method to define a bounded harmonic function on the whole space $\mathbb{R}^n\times\mathbb{R}$ as follows

$$\mathcal{H}(x,t)=\begin{cases}H(x,t), & t\geq 0,\\ -H(x,-t), & t<0.\end{cases}$$

However, the Liouville theorem tells us that there is no bounded harmonic function on $\mathbb{R}^n\times\mathbb{R}$ other than constant, which indicates

$$u(x,t+s)-\exp(-t\sqrt{L})(u(\cdot,s))(x)=H(x,t)\equiv 0$$

by noting $\mathcal{H}(x,0)=0$. The claims follows.

With (7.1) in hand, the remainder of the arguments is analogous to that in [28, Theorem 1.2] and is left to the reader.

Step 2. Sufficiency: from trace to solution.

If $f\in H^1_L(\mathbb{R}^n)$, then $u(x,t)=\exp(-t\sqrt{L})f(x)$ is well-defined harmonic function on $\mathbb{R}^n\times\mathbb{R}_+$ with

$$\|u^*\|_{L^1(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\sup_{t>0}|\exp(-t\sqrt{L})f(x)|dx=\|f\|_{H^1_L(\mathbb{R}^n)},$$

which completes the proof. □

Theorem A.3 *Let $-1/2<\alpha<\theta/n$. A function $u(x,t)$ is harmonic/caloric in $\mathbb{R}^n\times\mathbb{R}_+$ with*

$$\|u\|_{\mathrm{HMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)/\mathrm{TMO}^\alpha_L(\mathbb{R}^n\times\mathbb{R}_+)}\leq C$$

if and only if $u(x,t)=\exp(-t\sqrt{L})f(x)/\exp(-tL)f(x)$ for some $f\in\mathrm{BMO}^\alpha_L(\mathbb{R}^n)$.

Proof. See [21, 24] for the elliptic case and [26] for the parabolic case. □

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