New York Journal of Mathematics
New York J. Math. **30** (2024) 24–37.

Characteristic polynomials and finite dimensional representations of simple Lie algebras

Amin Geng, Shoumin Liu and Xumin Wang

ABSTRACT. In this paper, we prove the correspondence between finite dimensional representations of a simple Lie algebra and their associated characteristic polynomials. We will also define a monoid structure on these characteristic polynomials related to the tensor products of the representations. Furthermore, the factorization of characteristic polynomials sheds new light on the structure of simple Lie algebras and their Borel subalgebras.

CONTENTS

1. Introduction

For several matrices $A_1, \dots, A_n$ of equal size, their characteristic polynomial is defined as

$$f_A(z) = det(z_0 I + z_1 A_1 + \cdots + z_n A_n), \quad z = (z_0, \dots, z_n) \in \mathbb{C}^{n+1},$$

which has been investigated since the late 19th and early 20th century in problems related to group determinant and determinantal representations. We refer to [4, 5, 8, 6, 7, 9] for some illustrations of this topic. However, studying characteristic polynomial for several general matrices is a new frontier in linear algebra. In [18], the notion of projective spectrum of operators is defined by R. Yang

Received August 22, 2023.
2020 *Mathematics Subject Classification.* 17B05,17B10.
Key words and phrases. Lie algebra, characteristic polynomial, finite dimensional representation, monoid.
Liu is funded by the NSFC (Grant No. 11971181, Grant No.12271298).

ISSN 1076-9803/2024

through the multiparameter pencil, and multivariable homogeneous characteristic polynomials have been studied. Fruitful results have been obtained in [2, 10, 11, 13, 1]. It is natural to consider similar topics for finite dimensional Lie algebras. For a Lie algebra $\mathfrak{g}$ with a basis $\{x_1, \dots, x_n\}$, the characteristic polynomial of its adjoint representation

$$f_{\mathfrak{g}} = \det(z_0 I + z_1 \mathrm{ad}x_1 + \cdots + z_n \mathrm{ad}x_n)$$

is investigated in [1]. It is shown that $f_{\mathfrak{g}}$ is invariant under the automorphism group $Aut(\mathfrak{g})$. Let $\phi : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{gl}(V)$ be an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$, which $\mathfrak{sl}(2, \mathbb{C})$ is Lie algebra of all 2-by-2 matrices with zero trace over $\mathbb{C}$ and V is a $(m + 1)$-dimensional complex vector space. The characteristic polynomial

$$f_\phi(z_0, z_1, z_2, z_3) = \begin{cases} z_0 \prod_{l=1}^{m/2} \left(z_0^2 - 4l^2(z_1^2 + z_2 z_3)\right) & 2 \mid m \\ \prod_{l=0}^{(m-1)/2} \left(z_0^2 - (2l+1)^2(z_1^2 + z_2 z_3)\right) & 2 \nmid m \end{cases} \tag{1.1}$$

is obtained in [3, 14, 12]. When the homomorphism ϕ is an arbitrary finite dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$, we let $d_{n,\phi}$ denote the dimension of the eigenspace of $\phi(h)$ for the eigenvalue n, $n \in \mathbb{Z}$. The characteristic polynomial

$$f_\phi(z_0, z_1, z_2, z_3) = z_0^{d_{0,\phi}} \prod_{n \geq 1} \left(z_0^2 - n^2(z_1^2 + z_2 z_3)\right)^{d_{n,\phi}} \tag{1.2}$$

is obtained in [16], where the authors proved that there is one to one correspondence between finite dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$ and their characteristic polynomials. One wonders whether similar results hold for simple Lie algebra.

The paper is sketched as the following. Section 2 presents the definition of the characteristic polynomials with respect to finite dimensional representations of a simple Lie algebra. It is shown that the latter can be reconstructed through its characteristic polynomials. In Section 3, similar to [16, Section 5], we show that the characteristic polynomials of a simple Lie algebra can be endowed with a commutative monoid structure compatible with the tensor product of representations. In Section 4, some results about the characteristic polynomials of $\mathfrak{sl}(2, \mathbb{C})$ acting on the classical simple Lie algebras are obtained. In Section 5.3, we calculate the rank of the spectral matrices for the Borel subalgebras of simple Lie algebras.

2. Decomposition of the representations of simple Lie algebra through characteristic polynomials

We first recall some basics for Lie algebras which can be found in [15]. Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, Φ be the root system of $\mathfrak{g}$, and Π=$\{\alpha_1, \dots, \alpha_n\}$ be simple roots of Φ. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\{h_{\alpha_1}, \dots, h_{\alpha_n}\}$ be a basis of $\mathfrak{h}$ corresponding to Π, and E_α be the root vector for each $\alpha \in \Phi$. It is well known that the set $\mathcal{A}$=$\{h_{\alpha_1}, \dots, h_{\alpha_n}, E_\alpha(\alpha \in \Phi)\}$ is a canonical basis of $\mathfrak{g}$, and we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Phi} \mathbb{C}E_\alpha.$$

Suppose $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional linear representation of $\mathfrak{g}$, with V being a $\mathfrak{g}$-module.

Definition 2.1. The polynomial

$$f_\phi(z_0, z_1, \dots, z_n, z_\alpha) = \det\left(z_0 I + \sum_{i=1}^{n} \phi(h_{\alpha_i}) z_i + \sum_{\alpha\in\Phi} \phi(E_\alpha) z_\alpha\right)$$

is called the characteristic polynomial of $\mathfrak{g}$ with respect to the basis $\mathcal{A}$ and the representation ϕ, where $z_0, z_1, \dots, z_n, z_\alpha$ are indeterminants.

Since the representations of a simple Lie algebra $\mathfrak{g}$ are closely related to its Cartan subalgebra $\mathfrak{h}$, we restrict the representation ϕ of $\mathfrak{g}$ to $\mathfrak{h}$ and obtain a new polynomial as follows.

Definition 2.2. Let $\mathfrak{g}$ and ϕ be as above and define

$$\tilde{f}_\phi(\tilde{z}) = \det\left(z_0 I + \sum_{i=1}^{n} \phi(h_{\alpha_i}) z_i\right) = f_\phi(z_0, z_1, \dots, z_n, 0, \dots, 0)$$

with $\tilde{z} = (z_0, z_1, \dots, z_n)$. We call $\tilde{f}_\phi(\tilde{z})$ the linearization of $f_\phi(z)$.

For convenience, the polynomials $f_\phi(z)$ and $\tilde{f}_\phi(\tilde{z})$ are denoted as f_ϕ and $\tilde{f}_\phi$, respectively. We gather these two kinds of polynomials in the two sets,

$$\mathbf{CP}_\mathfrak{g} = \{f_\phi(z)\}_\phi, \qquad \widetilde{\mathbf{CP}_\mathfrak{g}} = \{\tilde{f}_\phi(\tilde{z})\}_\phi,$$

where ϕ runs over all finite dimensional representations of $\mathfrak{g}$. For these two sets, analogous to the proof of [15, Section 22.5, Proposition A], the following proposition can be obtained.

Proposition 2.3. *The map*

$$\rho : \mathbf{CP}_\mathfrak{g} \to \widetilde{\mathbf{CP}_\mathfrak{g}},$$
$$\rho(f_\phi) = \tilde{f}_\phi,$$

is a bijective map.

Proof. For the representation module V for ϕ, we have the complete decomposition of $V = \oplus_{i=1}^{t} V_i$ by Weyl's theorem [15, Theorem 6.3], where each V_i is an irreducible module of $\mathfrak{g}$ with highest weight β_i. Write $\phi = \oplus_{i=1}^{t} \phi_i$, where ϕ_i is the representation corresponding to V_i. Then we have

$$f_\phi = \prod_{i=1}^{t} f_{\phi_i}. \tag{2.1}$$

By [15, Section 20], each V_i can be written as $V_i = \bigoplus_r \mathbb{C}\mathcal{V}_r$ with $r \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$ being a weight of V_i, such that $h(\mathcal{V}_r) = r(h)\mathcal{V}_r$ for any $h \in \mathfrak{h}$. This implies that $\mathfrak{h}$ is diagonalizable on V_i. We denote the set of those eigenvectors $\mathcal{V}_r$ by Γ_i. It follows that

$$\tilde{f}_\phi = \prod_{i=1}^{t} \prod_{\mathcal{V}_r \in \Gamma_i} \left(z_0 + r(h_{\alpha_1})z_1 + \cdots + r(h_{\alpha_n})z_n\right) = \prod_{i=1}^{t} \tilde{f}_{\phi_i}. \tag{2.2}$$

For the module V, let $\Gamma = \bigcup_{i=1}^{t} \Gamma_i$. Once the set Γ is known, we can find one dominant weight eigenvector $\mathcal{V}_{\beta_0}$ in Γ for which β_0 is one of the highest weights in Γ. Therefore the $\mathcal{V}_{\beta_0}$ will generate a unique irreducible module of $\mathfrak{g}$, which we denote by W. Then W is an irreducible component of V. Without loss of generality, suppose the module W is isomorphic to the module V_1. Thus we can consider $\Gamma' = \Gamma \backslash \Gamma_1$. We can repeat the above operation to determine all the irreducible components of V. Therefore, the polynomial f_ϕ can be obtained by formula (2.1) through the algorithm on Γ.

Once the polynomial $\tilde{f}_\phi$ is fixed, we can uniquely write it as products of linear polynomials with the coefficient of z_0 being 1 as in (2.2), and each linear factor can determine a linear functional in $\mathfrak{h}^*$, because $\{h_{\alpha_i}\}_{i=1}^{n}$ is a basis of $\mathfrak{h}$. Then we can determine all the weights for the Γ in this way. Therefore, by the algorithm on the set Γ, the map ρ is an one to one correspondence between $\mathbf{CP}_\mathfrak{g}$ and $\widetilde{\mathbf{CP}_\mathfrak{g}}$. □

By the algorithm in the proof of the Proposition 2.3, the following theorem holds.

Theorem 2.4. *Let ϕ and ψ be two finite dimensional representations of a finite dimensional complex simpe Lie algebra $\mathfrak{g}$. Then ϕ and ψ are isomorphic if and only if $f_\phi = f_\psi$.*

Remark 2.5. In fact, the characteristic polynomials might differ if we choose different bases for the Lie algebra. For the adjoint representation of a finite dimensional Lie algebra, it is proved that the characteristic polynomial is invariant under the automorphism of the Lie algebra in [1, Theorem 2.3]. But it does not hold for arbitrary finite dimensional representations, especially for the outer automorphisms, which can produce non-trivial actions on their representations.

3. Monoid structures on $\mathbf{CP}_\mathfrak{g}$ and $\widetilde{\mathbf{CP}_\mathfrak{g}}$

In this section, we mainly study the algebraic structure of $\mathbf{CP}_\mathfrak{g}$ and $\widetilde{\mathbf{CP}_\mathfrak{g}}$ defined through tensor products of the representations of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a simple Lie algebra with simple roots $\Pi = \{\alpha_i\}_{i=1}^{n}$ and Cartan subalgebra $\mathfrak{h}$, having a basis $\{h_{\alpha_i}\}_{i=1}^{n}$. Let $\mathrm{rep}(\mathfrak{g})$ be the monoidal category of finite dimensional representations of $\mathfrak{g}$. Let ϕ be an object in $\mathrm{rep}(\mathfrak{g})$, Γ_ϕ be all weights of ϕ with $|\Gamma_\phi|$ being its size. For each $\lambda \in \Gamma_\phi$, we let d_λ denote the multiplicity of λ in ϕ, which is the dimension of the eigenspace of ϕ for λ.

The set $\widetilde{\mathbf{CP}_{\mathfrak{g}}}$ can be endowed with the monoid structure by a multiplication defined below.

Definition 3.1. Let ϕ, φ be two objects in rep($\mathfrak{g}$), and f_ϕ and f_φ be their characteristic polynomials with $\tilde{f}_\phi = \rho(f_\phi)$ and $\tilde{f}_\varphi = \rho(f_\varphi)$ being their linearizations, respectively. Suppose that $\Gamma_\phi = \{\lambda_j\}$, $\Gamma_\varphi = \{\mu_k\}$ are all weights of ϕ, φ, respectively, and d_{λ_j}, d_{μ_k} are their multiplicities in ϕ, φ, respectively. Let

$$\lambda_{ji} = \lambda_j(h_{\alpha_i}), \qquad \mu_{ki} = \mu_k(h_{\alpha_i})$$

for $j = 1, \dots, |\Gamma_\phi|$, $k = 1, \dots, |\Gamma_\varphi|$, $i = 1, \dots, n$. It can be seen that

$$\tilde{f}_\phi = \prod_{\lambda_j \in \Gamma_\phi} \left(z_0 + \sum_{\alpha_i \in \Pi} \lambda_j(h_{\alpha_i}) z_i \right)^{d_{\lambda_j}} = \prod_{\lambda_j \in \Gamma_\phi} \left(z_0 + \sum_{i=1}^{n} \lambda_{ji} z_i \right)^{d_{\lambda_j}},$$

$$\tilde{f}_\varphi = \prod_{\mu_k \in \Gamma_\varphi} \left(z_0 + \sum_{\alpha_i \in \Pi} \mu_k(h_{\alpha_i}) z_i \right)^{d_{\mu_k}} = \prod_{\mu_k \in \Gamma_\varphi} \left(z_0 + \sum_{i=1}^{n} \mu_{ki} z_i \right)^{d_{\mu_k}}.$$

Define $\tilde{f}_\phi * \tilde{f}_\varphi \in \mathbb{C}[z_0, z_1, \dots, z_n]$ for $\tilde{f}_\phi$ and $\tilde{f}_\varphi$ by the formula

$$\tilde{f}_\phi * \tilde{f}_\varphi = \prod_{\lambda_j \in \Gamma_\phi, \mu_k \in \Gamma_\varphi} \left(z_0 + \sum_{i=1}^{n} (\lambda_{ji} + \mu_{ki}) z_i \right)^{d_{\lambda_j} d_{\mu_k}}. \tag{3.1}$$

Here, we call the polynomial $\tilde{f}_\phi * \tilde{f}_\varphi$ the resolution product of $\tilde{f}_\phi$ and $\tilde{f}_\varphi$.

Proposition 3.2. *Let $\tilde{f}_\phi$, $\tilde{f}_\varphi$ be two polynomials in $\widetilde{\mathbf{CP}_{\mathfrak{g}}}$. Then*

$$\tilde{f}_\phi * \tilde{f}_\varphi = \tilde{f}_{\phi \otimes \varphi}.$$

Proof. Suppose that $\{v_{\lambda_j}^t\}_{\lambda_j \in \Gamma_\phi}$ with $t = 1, \dots, d_{\lambda_j}$, $\{w_{\mu_k}^s\}_{\mu_k \in \Gamma_\varphi}$ with $s = 1, \dots, d_{\mu_k}$ are bases of representation ϕ and φ, respectively, such that each $h_{\alpha_i} \in \mathfrak{h}$ is diagonalizable under these two bases for ϕ and φ, respectively. This implies that

$$\phi(h_{\alpha_i})(v_{\lambda_j}^t) = \lambda_{ji} v_{\lambda_j}^t, \quad \varphi(h_{\alpha_i})(w_{\mu_k}^s) = \mu_{ki} w_{\mu_k}^s.$$

It is known that $\{v_{\lambda_j}^t \otimes w_{\mu_k}^s\}_{\lambda_j \in \Gamma_\phi, \mu_k \in \Gamma_\varphi}$ is a basis of $\phi \otimes \varphi$ for $t = 1, \dots, d_{\lambda_j}$, $s = 1, \dots, d_{\mu_k}$. By the definition of tensor products of the representations of Lie algebra, we have

$$\begin{aligned}
& \phi \otimes \varphi(h_{\alpha_i})(v_{\lambda_j}^t \otimes w_{\mu_k}^s) \\
= \ & (\phi(h_{\alpha_i}) \otimes I + I \otimes \varphi(h_{\alpha_i}))(v_{\lambda_j}^t \otimes w_{\mu_k}^s) \\
= \ & \phi(h_{\alpha_i})(v_{\lambda_j}^t) \otimes w_{\mu_k}^s + v_{\lambda_j}^t \otimes \varphi(h_{\alpha_i})(w_{\mu_k}^s) = (\lambda_{ji} + \mu_{ki})(v_{\lambda_j}^t \otimes w_{\mu_k}^s).
\end{aligned}$$

Let

$$\phi \otimes \varphi(h_{\alpha_i})_{v^t_{\lambda_j} \otimes w^s_{\mu_k}} = \lambda_{ji} + \mu_{ki} \tag{3.2}$$

for $1 \leq j \leq |\Gamma_\phi|$, $1 \leq k \leq |\Gamma_\varphi|$, $1 \leq i \leq n$, and $d_{\lambda_j \tilde{+} \mu_k}$ denote the dimension of eigenvector space for $\phi \otimes \varphi(h_{\alpha_i})$ of the eigenvalue $\lambda_{ji} + \mu_{ki}$ spanned by the vectors $v^t_{\lambda_j} \otimes w^s_{\mu_k}$, for $t = 1, \dots, d_{\lambda_j}$ and $s = 1, \dots, d_{\mu_k}$. Therefore, it follows that

$$d_{\lambda_j \tilde{+} \mu_k} = d_{\lambda_j} d_{\mu_k}. \tag{3.3}$$

By equations (3.2) and (3.3), one can rewrite the polynomials $\tilde{f}_\phi$, $\tilde{f}_\varphi$ and $\tilde{f}_{\phi\otimes\varphi}$ as the follows,

$$\begin{aligned}
\tilde{f}_\phi &= \prod_{\lambda_j \in \Gamma_\phi,} \left(z_0 + \sum_{i=1}^{n} \lambda_{ji} z_i \right)^{d_{\lambda_j}}, \\
\tilde{f}_\varphi &= \prod_{\mu_k \in \Gamma_\varphi,} \left(z_0 + \sum_{i=1}^{n} \mu_{ki} z_i \right)^{d_{\mu_k}}, \\
\tilde{f}_{\phi\otimes\varphi} &= \prod_{\lambda_j \in \Gamma_\phi, \mu_k \in \Gamma_\varphi} \left(z_0 + \sum_{i=1}^{n} \phi \otimes \varphi(h_{\alpha_i})_{v^t_{\lambda_j} \otimes w^s_{\mu_k}} z_i \right)^{d_{\lambda_j \tilde{+} \mu_k}} \\
&= \prod_{\lambda_j \in \Gamma_\phi, \mu_k \in \Gamma_\varphi} \left(z_0 + \sum_{i=1}^{n} (\lambda_{ji} + \mu_{ki}) z_i \right)^{d_{\lambda_j} d_{\mu_k}} = \tilde{f}_\phi * \tilde{f}_\varphi.
\end{aligned}$$

□

Theorem 3.3. *The set $\widetilde{\mathbf{CP}_{\mathfrak{g}}}$ is a commutative monoid under the resolution product with the unit element z_0.*

Proof. By Proposition 3.2, the set $\widetilde{\mathbf{CP}_{\mathfrak{g}}}$ is closed under the resolution product. For three representations ϕ, ψ, φ of $\mathfrak{g}$, it is known that

$$\phi \otimes (\psi \otimes \varphi) \simeq (\phi \otimes \psi) \otimes \varphi.$$

By Proposition 3.2, it follows that

$$\tilde{f}_\phi * (\tilde{f}_\psi * \tilde{f}_\varphi) = \tilde{f}_{\phi\otimes(\psi\otimes\varphi)} = \tilde{f}_{(\phi\otimes\psi)\otimes\varphi} = (\tilde{f}_\phi * \tilde{f}_\psi) * \tilde{f}_\varphi.$$

Let φ_0 denote the trivial representation of dimension 1 of $\mathfrak{g}$, then $\tilde{f}_{\varphi_0} = z_0$. For each representation ϕ of $\mathfrak{g}$, we have $\phi \simeq \phi \otimes \varphi_0 \simeq \varphi_0 \otimes \phi$, so it follows that $\tilde{f}_\phi = \tilde{f}_\phi * z_0 = z_0 * \tilde{f}_\phi$. At the end, the relation $\tilde{f}_\phi * \tilde{f}_\psi = \tilde{f}_\psi * \tilde{f}_\phi$ holds for $\phi \otimes \psi \simeq \psi \otimes \phi$. □

Let ϕ and φ be two representations of $\mathfrak{g}$ of finite dimension. By Proposition 2.3, there is one to one correspondence between $\mathbf{CP}_{\mathfrak{g}}$ and $\widetilde{\mathbf{CP}_{\mathfrak{g}}}$. Define

$$f_\phi * f_\varphi = \rho^{-1}(\tilde{f}_\phi * \tilde{f}_\varphi) \tag{3.4}$$

which is called the resolution product of f_ϕ and f_φ. By Proposition 3.2, we have

$$f_\phi * f_\varphi = \rho^{-1}(\tilde{f}_\phi * \tilde{f}_\varphi) = \rho^{-1}(\tilde{f}_{\phi\otimes\varphi}) = f_{\phi\otimes\varphi}.$$

Combining with Theorem 3.3, the theorem below holds.

Theorem 3.4. *The set $\mathbf{CP}_{\mathfrak{g}}$ is a commutative monoid under the resolution product with the unit element z_0. Furthermore, the monoids $\mathbf{CP}_{\mathfrak{g}}$ and $\widetilde{\mathbf{CP}_{\mathfrak{g}}}$ are isomorphic under their resolution product structures through the linearization map.*

Remark 3.5. Here we recall the definition of the formal characters $\mathbb{Z}[\Lambda]$ of a simple Lie algebra $\mathfrak{g}$ from [15, Section 22.5]. Let $\Lambda \subseteq \mathfrak{h}^*$ be the set of integral weight lattice, namely all $\lambda \in \mathfrak{h}^*$ which $\langle \lambda, \alpha \rangle \in \mathbb{Z}(\alpha \in \Phi)$. Let $\mathbb{Z}[\Lambda]$ be the free abelian group with $\{e(\mu)\}_{\mu\in\Lambda}$ being its basis. From [15, Section 22.5], the abelian group $\mathbb{Z}[\Lambda]$ has a commutative ring structure through decreeing that $e(\lambda)e(\mu) = e(\lambda+\mu)$. For $\lambda \in \Lambda^+$, we suppose that $V(\lambda)$ is an irreducible finite dimensional module of $\mathfrak{g}$ with the highest weight λ and $\Pi(\lambda)$ denote the set of its weights. We define the formal character for $V(\lambda)$ as

$$ch_\lambda = \sum_{\mu\in\Pi(\lambda)} m_\lambda(\mu)e(\mu),$$

where $m_\lambda(\mu)$ is the multiplicity of μ in $V(\lambda)$, namely the dimension of the eigenspace corresponding to the weight μ. If $V = V(\lambda_1) \oplus \cdots V(\lambda_t)$, we define its formal character as

$$ch_V = \sum_{i=1}^{t} ch_{\lambda_i}.$$

Define a map τ from $\mathbf{CP}_{\mathfrak{g}}$ to $\mathbb{Z}[\Lambda]$ by

$$\tau(f_\phi) = ch_V,\ f_\phi \in \mathbf{CP}_{\mathfrak{g}},$$

where ϕ is the representation of $\mathfrak{g}$ acting on the module V. By the Theorem 3.4 and [15, Section 22.5, Proposition B], it follows that

$$\tau(f_\phi)\tau(f_\varphi) = ch_V + ch_W, \quad \tau(f_\phi) * \tau(f_\varphi) = ch_V ch_W,$$

where ϕ and φ are representations of $\mathfrak{g}$ acting on V and W, respectively.

4. The adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$ on simple Lie algebras

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, $\{h, e_1, e_2\}$ be a canonical basis of $\mathfrak{sl}(2,\mathbb{C})$, and $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ be a Lie algebra embedding. Suppose that $\mathrm{ad}\circ\phi$ is the composition of ϕ and the adjoint representation ad of $\mathfrak{g}$. In this section, our main concern is the characteristic polynomial $f_{\mathrm{ad}\circ\phi}(z_0, z_1, z_2, z_3)$ of $\mathrm{ad}\circ\phi$. In [16], the authors have calculated the case for simple Lie algebra type A_{n-1}. In this section, we're going to calculate this for other types.

In the following, a general formula in Theorem 4.1 will be presented. We first apply the formula to the complex simple Lie algebra of type C_n. Analogous results for other types are summarized in Table 1.

Here we recall the Chevalley basis for $\mathfrak{g}$ from [15, Section 25.2]. Any basis

$$\{E_\alpha,\ \alpha \in \Phi,\ h_i,\ i = 1, \dots, n\}$$

is called a Chevalley basis of $\mathfrak{g}$ if it satisfies:

(i) $[E_\alpha, E_{-\alpha}] = h_\alpha$.
(ii) If $\alpha, \beta, \alpha+\beta \in \Phi$, $[E_\alpha, E_{-\beta}] = c_{\alpha,\beta} E_{\alpha+\beta}$, then $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$.
(iii) The coefficient $c_{\alpha,\beta}$ in (ii) satisfies

$$c^2_{\alpha,\beta} = q(r+1)\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)},$$

where $\beta - r\alpha, \ldots, \beta + q\alpha$ is the α-string through β.

In [15, Scetion 25.2, Propostion], the existence of a Chevalley basis of $\mathfrak{g}$ is presented. From [15, Scetion 25.2, Theorem] we know that for any $\alpha \in \Phi$,

$$[h_i, E_\alpha] = \langle \alpha, \alpha_i \rangle E_\alpha,$$

where $h_i = h_{\alpha_i}$, $\{\alpha_i\}_{i=1}^n$ is a base for Φ.

Theorem 4.1. *Let $\mathfrak{g}$ be as above, Φ be its root system, and $\lambda \in \Phi$. Let $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ be the Lie algebra embedding defined by $\phi(h) = H_\lambda$, $\phi(e_1) = E_\lambda$, $\phi(e_2) = E_{-\lambda}$. Then*

$$f_{\mathrm{ad}\circ\phi}(z_0, z_1, z_2, z_3) = z_0^{k_0} \prod_{i=1}^{3} \left(z_0^2 - i^2(z_1^2 + z_2 z_3)\right)^{k_i}, \tag{4.1}$$

where k_i represents the multiplicity of the eigenvalue i of $\mathrm{ad}H_\lambda$*. Furthermore, we have $dim\mathfrak{g} = k_0 + 2(k_1 + k_2 + k_3)$.*

Proof. In view of formula (1.2), we need to compute the eigenvalues of $\mathrm{ad}H_\lambda$ and their multiplicities. If we choose the canonical basis to be a Chevalley basis, it is known that for $\beta \in \Phi$,

$$[H_\lambda, E_\beta] =< \beta, \lambda > E_\beta, \ with < \beta, \lambda >= \frac{2(\beta,\lambda)}{(\lambda,\lambda)},$$

where (β, λ) and (λ, λ) are the canonical inner products. For $\mathfrak{g}$ is a simple Lie algebra, the possible values for $< \beta, \lambda >$ are $0, \pm 1, \pm 2, \pm 3$. Since

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Phi} \mathbb{C} E_\alpha, \quad [H_\lambda, \mathfrak{h}] = 0,$$

for $\mathrm{ad}H_\lambda$, we have all its possible eigenvalues being $0, \pm 1, \pm 2, \pm 3$. Therefore, formula (4.1) follows from formula (1.2). ☐

Remark 4.2. Let $\mathfrak{g}, \Phi$ be as above. For each root $\alpha, \beta \in \Phi$, if α, β have the same length, there is an element g of Weyl group with respect to $\mathfrak{g}$, such that $g\alpha = \beta$, which can be extended to an automorphism of $\mathfrak{g}$. By Corollary 2.5, we see that the characteristic polynomial is invariant under g. Thus the tuple (k_0, k_1, k_2, k_3) of powers in (4.1) has at most two possible sets of values depending on whether λ being long or short.

Remark 4.3. Let $\{\epsilon_i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. From [15], the root system Φ of type C_n can be realized in $\mathbb{R}^n$, and it consists of $2n$ long roots $\pm 2\epsilon_i$ $(1 \le i \le n)$ and $2n^2 - 2n$ short roots $\pm\epsilon_i \pm \epsilon_j$ $(1 \le i < j \le n)$, with simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\} = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\}$.

Theorem 4.4. *Let $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ be the Lie algebra embedding defined by $\phi(h) = H_\lambda$, $\phi(e_1) = E_\lambda$, $\phi(e_2) = E_{-\lambda}$ with $\lambda \in \Phi$, where $\mathfrak{g}$ is the complex simple Lie algebra of type* C_n. *Suppose that* $\mathrm{ad}\circ\phi$ *be the composition of ϕ and the adjoint representation* ad *of $\mathfrak{L}$. Then*

$$f_{\mathrm{ad}\circ\phi} = z_0^{2n^2-3n+2}\left(z_0^2-(z_1^2+z_2z_3)\right)^{2n-2}\left(z_0^2-4(z_1^2+z_2z_3)\right) \tag{4.2}$$

with λ being a long root, and

$$f_{\mathrm{ad}\circ\phi} = z_0^{2n^2-7n+10}\left(z_0^2-(z_1^2+z_2z_3)\right)^{4n-8}\left(z_0^2-4(z_1^2+z_2z_3)\right)^3 \tag{4.3}$$

with λ being a short root.

Proof. By Remark 4.2, without loss of generality, suppose that $\lambda{=}2\epsilon_n$ for the long root case. By Theorem 4.1, it is natural to calculate the multiplicity of each eigenvalue, whose possible values are $0, \pm1, \pm2$. Through $\frac{2(\beta,2\epsilon_n)}{(2\epsilon_n,2\epsilon_n)}$, we can find the number of E_β for the eigenvalue $<\beta,\alpha>$ by realizing Φ as in the Remark 4.3. Hence, it follows that $k_0 = 2n^2-3n+2$, $k_1 = 2n-2$, and $k_2 = 1$.

In a similar way, we assume $\lambda{=}\epsilon_1-\epsilon_2$ for the short root case. By calculating $\frac{2(\beta,\epsilon_1-\epsilon_2)}{(\epsilon_1-\epsilon_2,\epsilon_1-\epsilon_2)}$, it can be seen that $k_0 = 2n^2-8n+10$, $k_1 = 4n-8$, and $k_2 = 3$. So the formulas follow from Theorem 4.1. □

We list the crucial values k_0(including the dimension of the Cartan subalgebra), k_1, k_2, k_3 in Table 1 for all types of simple Lie algebras. We call the quadruple (k_0,k_1,k_2,k_3) the power index for the embedding ϕ from $\mathfrak{sl}(2,\mathbb{C})$ into $\mathfrak{g}$. In the table, the letters α and γ stand for a long root and a short root, respectively.

From the table, it can be seen that once we know the the dimensions of the Cartan subalgebras of the simple Lie algebras, then the table gives us enough information to classify them into different types. Hence the corollary below holds.

Corollary 4.5. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two finite dimensional complex simple Lie algebras, and $f_{\mathfrak{g}_1}$ and $f_{\mathfrak{g}_2}$ be the corresponding characteristic polynomials associated to their adjoint representations, respectively. Then $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2$ if and only if $f_{\mathfrak{g}_1} = f_{\mathfrak{g}_2}$ up to a change of basis of $\mathfrak{g}_2$.*

Proof. By the Theorem 4.4, it is sufficient to prove that finite dimensional complex simple Lie algebras can be distinguished by the power indices listed in the Table 1.

We first start from type E_8, whose power index is $(134, 56, 1, 0)$, by the values of k_1, k_2, k_3, the only possible types are A_{29} and D_{16}. But the values k_0 of type A_{29} and D_{16} are different. And by verifying type G_2, F_4, E_6, E_7 in the same way, we see that their power indices are distinct.

For the type A_n and D_m, we obtain two equations,

$$2n-2 = 4m-8, \quad n^2-2n+2 = 2m^2-9m+14.$$

TABLE 1. *The values of* $k_0,\ k_1,\ k_2,\ k_3$

		k_0	k_1	k_2	k_3
$Type$	$Root$	$<\beta,\alpha>=0$	$<\beta,\alpha>=1$	$<\beta,\alpha>=2$	$<\beta,\alpha>=3$
A_n	α	n^2-2n+2	$2n-2$	1	0
B_n	α	$2n^2-7n+10$	$4n-6$	1	0
	γ	$2n^2-3n+2$	0	$2n-1$	0
C_n	α	$2n^2-3n+2$	$2n-2$	1	0
	γ	$2n^2-7n+10$	$4n-8$	3	0
D_n	α	$2n^2-9n+14$	$4n-8$	1	0
G_2	α	4	4	1	0
	γ	4	2	1	2
F_4	α	22	14	1	0
	γ	22	8	7	0
E_6	α	36	20	1	0
E_7	α	67	32	1	0
E_8	α	134	56	1	0

And the solution is $n = m = 3$. For the type B_n and C_m, in the similar way, we have $n = m = 2$. Therefore the results coincide with the fact that $A_3 \cong D_3$, $B_2 \cong C_2$. $\square$

5. Borel subalgebras, parabolic subalgebras and spectral matrix

In this section, we focus on the rank of spectral matrices and parabolic subalgebras. Let $\mathfrak{L}$ be a subalgebra of $\mathfrak{g}$ with a basis $\mathcal{B} = \{v_1, \dots, v_n\}$. Hu and Zhang obtained the following theorem in [14].

Theorem 5.1. *$\mathfrak{L}$ is solvable if and only if the characteristic polynomial of $\mathfrak{L}$ is completely reducible with respect to any finite dimensional representation and any basis.*

Definition 5.2. Let $\mathfrak{L}$ be a solvable Lie algebra with a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ and $f_{\mathfrak{L}}(z)$ be its characteristic polynomial for the adjoint representation. By Theorem 5.1, one may write

$$f_{\mathfrak{L}}(z) = \prod_{j=1}^{n}\left(z_0 + \sum_{i=1}^{n}\lambda_{ij}z_i\right).$$

The spectral matrix for $\mathfrak{L}$ with respect to the basis $\mathcal{B}$ is defined as $\lambda_{\mathfrak{L}}=(\lambda_{ij})_{n\times n}$, namely

$$\lambda_{\mathfrak{L}} = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1n} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nn} \end{pmatrix}.$$

Definition 5.3. Let $\mathfrak{g}$ be a simple Lie Algebra, with $\mathfrak{h}$ being a Cartan subalgebra of $\mathfrak{g}$, and Φ^+ being the positive root system of $\mathfrak{g}$. A Borel subalgebra of $\mathfrak{g}$ is a subalgebra

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in\Phi^+} \mathbb{C}E_\alpha,$$

which is the maximal solvable subalgebra of $\mathfrak{g}$. Any subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ containing a Borel subalgebra is called a parabolic subalgebra of $\mathfrak{g}$.

By computing the characteristic polynomial of the Borel subalgebra of a simple Lie algebra, the theorem below follows.

Theorem 5.4. *Suppose* $\mathfrak{b}$ *is a Borel subalgebra of* $\mathfrak{g}$. *Then* $\mathrm{rank}\lambda_{\mathfrak{b}} = \dim\mathfrak{h}$.

Proof. Take a basis of $\{h_1, \dots, h_n\}$ for $\mathfrak{h}$, and suppose that $\mathrm{ht}(\alpha)$ is the height of $\alpha \in \Phi^+$ relative to some set of simple roots. Arrange the positive roots in Φ^+ by their heights such that $\Phi^+ = \{\alpha_1, \dots, \alpha_s\}$ with $s \geq j \geq i \geq 1$ indicates $\mathrm{ht}(\alpha_j) \geq \mathrm{ht}(\alpha_i)$. Let $E_{\alpha_1}, \dots, E_{\alpha_s}$ be the corresponding roots. Since

$$\begin{aligned}
&[\mathfrak{h}, \mathfrak{h}] = 0,\\
&[h_i, E_{\alpha_j}] = \alpha_j(h_i)E_{\alpha_j}, \forall 1 \leq i \leq n, 1 \leq j \leq s,\\
&[E_{\alpha_i}, E_{\alpha_j}] = N_{ij}E_{\alpha_i+\alpha_j}, 1 \leq i, j \leq s, \alpha_i + \alpha_j \in \Phi^+,\\
&[E_{\alpha_i}, E_{\alpha_j}] = 0, 1 \leq i, j \leq s, \alpha_i + \alpha_j \notin \Phi^+,
\end{aligned}$$

we see $\mathrm{ht}(\alpha_i + \alpha_j) > \mathrm{ht}(\alpha_i)$ and $\mathrm{ht}(\alpha_i + \alpha_j) > \mathrm{ht}(\alpha_j)$ if $\alpha_i + \alpha_j \in \Phi^+$. Under the ordered basis $\mathcal{B} = \{h_1, \dots, h_n, E_{\alpha_1}, \dots, E_{\alpha_s}\}$, we have that

$$\begin{aligned}
&\mathrm{ad}h_i = \begin{pmatrix} 0 & 0 \\ 0 & A_i \end{pmatrix} \quad with \quad A_i = \begin{pmatrix} \alpha_1(h_i) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_s(h_i) \end{pmatrix},\\
&\mathrm{ad}E_{\alpha_j} = \begin{pmatrix} 0I_{n\times n} & & \\ \vdots & \ddots & \\ * & \cdots & 0 \end{pmatrix},
\end{aligned}$$

and adE_{α_j} is a strict lower triangular matrix. Then the characteristic polynomial for the adjoint representation with the ordered basis $\{h_1, \dots, h_n, E_{\alpha_1},$ $\dots, E_{\alpha_s}\}$ of $\mathfrak{b}$ is

$$f_{\mathfrak{b}}(z) = \begin{vmatrix} z_0 I_{n\times n} & 0 \\ * & C \end{vmatrix}$$

with

$$C = \begin{pmatrix} z_0 + \sum_{i=1}^n \alpha_1(h_i)z_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_0 + \sum_{i=1}^n \alpha_s(h_i)z_i \end{pmatrix},$$

which implies

$$f_{\mathfrak{b}}(z) = z_0^n \prod_{j=1}^{s} \left(z_0 + \sum_{i=1}^{n} \alpha_j(h_i)z_i \right).$$

Therefore, we have

$$\operatorname{rank}\lambda_{\mathfrak{b}} = \operatorname{rank}\begin{pmatrix} 0I_{n\times n} & & & \\ & \alpha_1(h_1) & \cdots & \alpha_1(h_n) \\ & \vdots & \ddots & \vdots \\ & \alpha_s(h_1) & \cdots & \alpha_s(h_n) \end{pmatrix} = \dim\mathfrak{h}.$$

□

Remark 5.5. It also can be obtained through the formula $Rank\lambda_{\mathfrak{b}}$=dim $\mathfrak{b}/Nil(\mathfrak{b})$ ([1, proposition 4.5]) by proving $Nil(\mathfrak{b})$=$\sum_{\alpha\in\Phi^+}\mathbb{C}E_\alpha$, which can be done by analyzing its structure. For more information about the nilpotent radical $Nil(\mathfrak{b})$, we refer the reader to [17, Chapter 5].

Theorem 5.6. *Let $\mathfrak{p}$ be a parabolic subalgebra of a complex simple Lie algebra $\mathfrak{g}$. Then $\mathfrak{p}$ is a Borel subalgebra if and only if its characteristic polynomial of any finite dimensional representation is a product of linear factors.*

Proof. If $\mathfrak{p}$ is a Borel subalgebra of $\mathfrak{g}$, then $\mathfrak{p}$ is solvable. By Lie's Theorem, there exists a basis of the complex linear space V such that the matrix of $\mathfrak{p}$ is upper triangular relative to the basis. Therefore the necessity holds. On the other hand, by Theorem 5.1, if the characteristic polynomial of a linear representation of $\mathfrak{p}$ is a product of linear factors, then $\mathfrak{p}$ is solvable. □

Remark 5.7. From this paper, we see that the characteristic polynomials of the representations of a complex simple Lie algebra have a profound meaning for the representation monoidal category of the Lie algebra. There are many interesting topics about these polynomials, such as how to present these polynomials precisely, finding the link between the coefficients and the representations, how to factorize the tensor products through the resolution products of their characteristic polynomials and so on. Therefore, we need more efforts to work on these topics.

References

[1] Azari Key, Fatemeh; Yang, Rongwei. Spectral invariants for finite dimensional Lie algebras. *Linear Alg. Appl.* **611** (2021), 148–170. MR4190618, Zbl 1484.17005, arXiv:2004.00551, doi: 10.1016/j.laa.2020.10.024. 25, 27, 35

[2] Chagouel, Isaak; Stessin, Michael; Zhu, Kehe. Geometric spectral theory for compact operators. *Trans. Amer. Math. Soc.* **368** (2016), no. 3, 1559–1582. MR3449218, Zbl 1337.47007, arXiv:1309.4375, doi: 10.1090/tran/6588. 25

[3] CHEN, ZHIQI; CHEN, XUEQING; DING, MING. On the characteristic polynomial of $\mathfrak{sl}(2, \mathbb{F})$. *Linear Algebra Appl.* **579** (2019), 237–243. MR3959733, Zbl 1462.17011, doi: 10.1016/j.laa.2019.05.036 25

[4] CURTIS, CHARLES W. Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer. History of Mathematics, 15. *American Mathematical Society, Providence, RILondon Mathematical Society, London*, 1999. xvi+287 pp. ISBN:0-8218-9002-6. MR1715145, Zbl 0939.01007, doi: 10.1090/hmath/015. 24

[5] DEDEKIND, RICHARD. Gesammelte mathematische Werke. Bände I-III. Herausgegeben von Robert Fricke, Emmy Noether und Öystein Ore. *Chelsea Publishing Co., New York,* 1968. MR0237282. 24

[6] DICKSON, LEONARD E. Determination of all general homogeneous polynomials expressible as determinants with linear elements. *Trans. Amer. Math. Soc.* **22** (1921), no. 2, 167–179. MR1501168, JFM 48.0099.02, doi: 10.1090/S0002-9947-1921-1501168-0. 24

[7] DICKSON, LEONARD E. An elementary exposition of Frobenius theory of group-characters and group-determinants. *Ann. of Math. (2)* **4** (1902), no. 1, 25–49. MR1502293, JFM 33.0149.03, doi: 10.2307/1967149. 24

[8] DIXON, ALFRED C. Note on the reduction of a ternary quantic to a symmetrical determinant. *Proc. Cambridge Philos. Soc.* **11** (1902), 350–351. JFM 33.0140.04. 24

[9] FROBENIUS, GEORG. Ueber vertauschbare Matrizen. *Sitzungsberichte der Königlich Preussischen,Akademie der Wissenschaften zu Berlin* (1896), 601–614. JFM 27.0109.04. 24

[10] GRIGORCHUK, ROSTISLAV I.; YANG, RONGWEI. Joint spectrum and the infinite dihedral group. *Proc. Steklov Inst. Math.* **297** (2017), no. 1, 145–178; translation from: *Tr. Mat. Inst. Steklova* **297**, *Poryadok i Khaos v Dinamicheskikh Sistemakh* (2017), 165–200. ISBN:5-7846-0142-3 ISBN:978-5-7846-0142-1. MR3695412, Zbl 1462.47003, arXiv:1605.01547, doi: 10.1134/S0081543817040095. 25

[11] HE, WEI; YANG, RONGWEI. Projective spectrum and kernel bundle. *Sci. China Math.* **58** (2015), no. 11, 2362-2372 MR3426136, Zbl 1334.47008, doi: 10.1007/s11425-015-5043-z. 25

[12] HU, ZHIGUANG. Eigenvalues and eigenvectors of a class of irreducible tridiagonal matrices. *Linear Algebra Appl.* **619** (2021), 328–337. MR4231560, Zbl 1495.15013, doi: 10.1016/j.laa.2021.03.014. 25

[13] HU, ZHIGUANG; YANG, RONGWEI. On the charactersitic polynomials of multiparameter pencils. *Linear Algebra Appl.* **558** (2018), 250–263. MR3854197, Zbl 1398.15006, doi: 10.1016/j.laa.2018.08.036. 25

[14] HU, ZHIGUANG; ZHANG, PHILIP B. Determinant and charactreristic polynomials of Lie algebras. *Linear Algebra Appl.* **563** (2019), 426–439. MR3880055, Zbl 1405.17018, doi: 10.1016/j.laa.2018.11.015. 25, 33

[15] HUMPHREYS, JAMES E. Introduction to Lie algebras and representation theory. Graduate Texts in Mathematics, 9. *Springer-Verlag, New York-Berlin*, 1972. xii+169 pp. MR0323842, Zbl 0254.17004, doi: 10.1007/978-1-4612-6398-2. 25, 26, 27, 30, 31

[16] JIANG, TIANYI; LIU, SHOUMIN. Characteristic polynomials and finitely dimensional representations of $\mathfrak{sl}(2, \mathbb{C})$. *Linear Algebra Appl.* **647** (2022), 78–88. MR4413328, Zbl 1498.17012, arXiv:2112.07933, doi: 10.1016/j.laa.2022.04.008. 25, 30

[17] KNAPP, ANTHONY W. Lie groups beyond an introduction. Second edition. Progress in Mathematics, 140. *Birkhäuser Boston, Inc., Boston, MA*, 2002. xviii+812 pp. ISBN: 0-8176-4259-5 MR1920389, Zbl 1075.22501, doi: 10.1007/978-1-4757-2453-0. 35

[18] YANG, RONGWEI. Projective spectrum in Banach algebras. *J. Topo. Anal.* **1** (2009), no. 3, 289–306. MR2574027, Zbl 1197.47015, arXiv:0804.0387, doi: 10.1142/S1793525309000126. 24

(Amin Geng) School of Mathematics, Shandong University, Shanda Nanlu 27, Jinan 250100, Shandong Province, China
gengamin@163.com

(Shoumin Liu) School of Mathematics, Shandong University, Shanda Nanlu 27, Jinan 250100, Shandong Province, China
s.liu@sdu.edu.cn

(Xumin Wang) School of Mathematics, Shandong University, Shanda Nanlu 27, Jinan 250100, Shandong Province, China
202320303@mail.sdu.edu.cn

This paper is available via http://nyjm.albany.edu/j/2024/30-2.html.