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Composition properties of hyperbolic links in handlebodies

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ABSTRACT. We consider knots and links in handlebodies that have hyperbolic complements and operations akin to composition. Cutting the complements of two such open along separating twice-punctured disks such that each of the four resulting handlebodies has positive genus, and gluing a pair of pieces together along the twice-punctured disks in their boundaries, we show the result is also hyperbolic. This should be contrasted with composition of any pair of knots in the 3-sphere, which is never hyperbolic. Similar results are obtained when both twice-punctured disks are in the same handlebody and we glue a resultant piece to itself along copies of the twice-punctured disks on its boundary. We include applications to staked links.

CONTENTS

1. Introduction

A compact orientable 3-manifold M is tg-hyperbolic if the manifold M' obtained from M by shaving off all torus boundaries and capping off all sphere boundaries with balls admits a finite volume hyperbolic metric such that all remaining boundary components are totally geodesic. For a link L in a handlebody H, we say that the pair (H, L) is *tg-hyperbolic* if the complement of an open regular neighborhood of L in H is tg-hyperbolic. By the Mostow-Prasad Rigidity Theorem, such a hyperbolic metric will only depend on the complement $H \setminus L$ up to homeomorphism, which allows us to associate a hyperbolic volume to (H, L) that is invariant under ambient isotopies of L in H.

Work of W. Thurston implies that the complement of a link in a compact orientable 3-manifold is tg-hyperbolic if and only if it contains no properly embedded essential disks, spheres, annuli or tori. A sphere is essential if it does

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not bound a ball. A disk is essential if it is not boundary parallel. A torus is essential if it is incompressible and not boundary-parallel. Annuli are essential if they are incompressible, boundary-incompressible and not boundary-parallel. In a manifold with no essential disks or spheres, an annulus that is incompressible is boundary-incompressible if and only if it is not boundary-parallel.

Examples of knots and links in handlebodies with complements that are tg-hyperbolic appear in [5], [7], [8], [9], and [12]. In [3], a large source of such examples is provided. Results from [11] can also be used to generate many more.

Let L_1 and L_2 be two links in handlebodies H_1 of genus g_1 and H_2 of genus g_2 respectively. Just as we have composition of two links in the 3-sphere, we would like to define composition of these links in handlebodies.

To that end, let $D_1 \subset H_1, D_2 \subset H_2$ be properly embedded disks twice punctured by L_1, L_2 respectively which separate balls B_1 and B_2 from H_1 and H_2 such that $B_1 \cap L_1$ and $B_2 \cap L_2$ are unknotted arcs. Discarding the balls yields two handlebodies $H_1' \subset H_1$ and $H_2' \subset H_2$. Let $L_1' = H_1' \cap L_1$ and $L_2' = H_2' \cap L_2$. Glue H_1' to H_2' along D_1 and D_2 via ϕ. Since ϕ sends the endpoints of the arc in L_1' to the endpoints of the arc in L_2', this results in a link in a handlebody, denoted $(H_1', L_1', D_1) \oplus_\phi (H_2', L_2', D_2)$ in H_3 as in Figure 1.

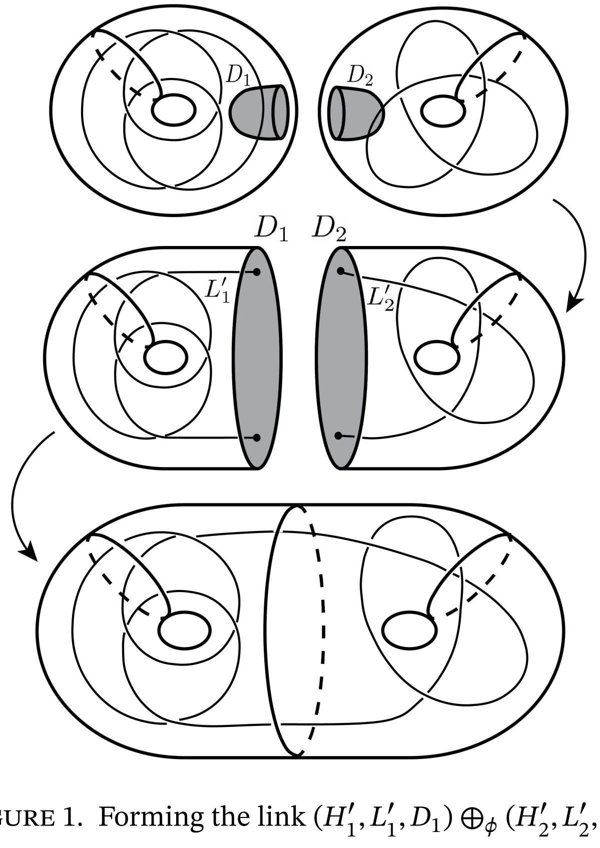

FIGURE 1. Forming the link $(H_1', L_1', D_1) \oplus_\phi (H_2', L_2', D_2)$

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In contrast to the usual composition of links, the link/handlebody pair $(H_1', L_1', D_1) \oplus_\phi (H_2', L_2', D_2)$ depends highly on D_1, D_2, and ϕ. Furthermore, while composition of links in S^3 never results in a hyperbolic link, the pair $(H_1', L_1', D_1) \oplus_\phi (H_2', L_2', D_2)$ can be tg-hyperbolic.

However, even if both $H_1 \setminus L_1$ and $H_2 \setminus L_2$ are tg-hyperbolic, it is not always true that $(H_1', L_1', D_1) \oplus_\phi (H_2', L_2', D_2)$ is tg-hyperbolic. In fact, the disks D_1 and D_2 can always be chosen so that at least one is "knotted" and there is an essential torus in the link complement associated to $(H_1', L_1', D_1) \oplus_\phi (H_2', L_2', D_2)$ as shown in Figure 2.

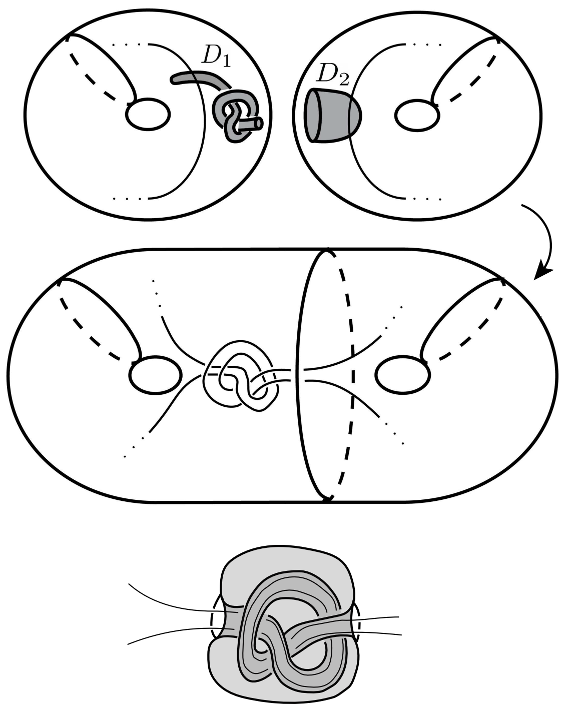

FIGURE 2. By choosing one of D_1, D_2 to be "knotted", one can create an essential torus in the complement $H_3 \setminus L_3$ which separates a knot exterior from H_3 of the form appearing in the last image.

In Section 2, we provide a method to avoid the problem with "knotted disks". In Theorem 2.1, we prove that if the two handlebody/link pairs cut along their disks appear as submanifolds of handlebody/link pairs of higher genus that are tg-hyperbolic, then the composition of the original pair is tg-hyperbolic. The presence of the rest of the higher genus tg-hyperbolic handlebodies prevents the disk from being "knotted". We also show an analogue of this result where one cuts along two separating twice punctured disks in a single handlebody and glues the resulting manifold to itself along a homeomorphism of the twice punctured disks.

In Section 3, we discuss applications. As mentioned, [3] and [11] provide many examples of tg-hyperbolic links in handlebodies, and our construction here can be applied to them to generate many more. Furthermore, these results can be applied to staked links introduced in [2], which correspond to link projections with isolated poles placed in the complementary regions, over which strands of the link cannot pass. These are equivalent to links in handlebodies.

We can also consider applications to knotoids. In [1], a definition of what it means for a planar knotoid to be hyperbolic is given in terms of a corresponding knot in a handlebody being tg-hyperbolic. So the results here can be applied to extend the known examples of hyperbolic planar knotoids.

In addition to considering knots in handlebodies, there is work that has been done on hyperbolicity of links in thickened surfaces, as in [4] and [11]. Questions about compositions have been addressed in that situation, as in [6]. Converting a method applied there to our situation can avoid the problem of knotted disks and allow composition of tg-hyperbolic links in handlebodies to be tg-hyperbolic without requiring them to be submanifolds as described above. That is, we can take a geodesic g that runs from the surface of the handlebody to the link. Then the boundary of a regular neighborhood of g, including its endpoint on the link, will be a properly embedded twice-punctured disk that cannot be knotted and therefore allows composition to yield tg-hyperbolic links in handlebodies. However, we do not include the details of the proof here.

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2. Proof of main result

For any submanifold S of a smooth manifold M, we denote by $N(S)$ a *closed* regular neighborhood of S in M and by $\mathring{N}(S)$ the interior of $N(S)$. For a space X, we denote by $|X|$ the number of connected components of X. Throughout, we use the fact that a handlebody is irreducible, which is to say that it contains no essential spheres. This is true because a handlebody can be embedded in S^3, and any sphere in S^3 cuts S^3 into two balls. So the sphere in the handlebody will bound a ball to one side.

Let H_1, H_2 be two handlebodies, each of genus at least 2, that contain links L_1 and L_2 such that $H_1 \setminus L_1$ and $H_2 \setminus L_2$ are tg-hyperbolic. Let E_1 and E_2 be properly embedded disks in H_1 and H_2, which separate H_1 and H_2 into handlebodies $H_{1,1}, H_{1,2}$ and $H_{2,1}, H_{2,2}$ of genera $g_{1,1}, g_{1,2}$ and $g_{2,1}, g_{2,2}$ respectively, where all genera are at least 1. Suppose further that E_1 and E_2 are each twice punctured by L_1 and L_2 respectively. Let $L_{i,j} = L_i \cap H_{i,j}$.

We denote by $M_{i,j} = H_{i,j} \setminus \mathring{N}(L_{i,j})$ and by $F_i = E_i \setminus \mathring{N}(L_i)$ the corresponding separating surfaces. As we will ultimately only be interested in $M_{1,1}$ and $M_{2,2}$, we will for convenience often drop the extra subscripts and write $M_{1,1}$ and $M_{2,2}$ as M_1 and M_2 respectively.

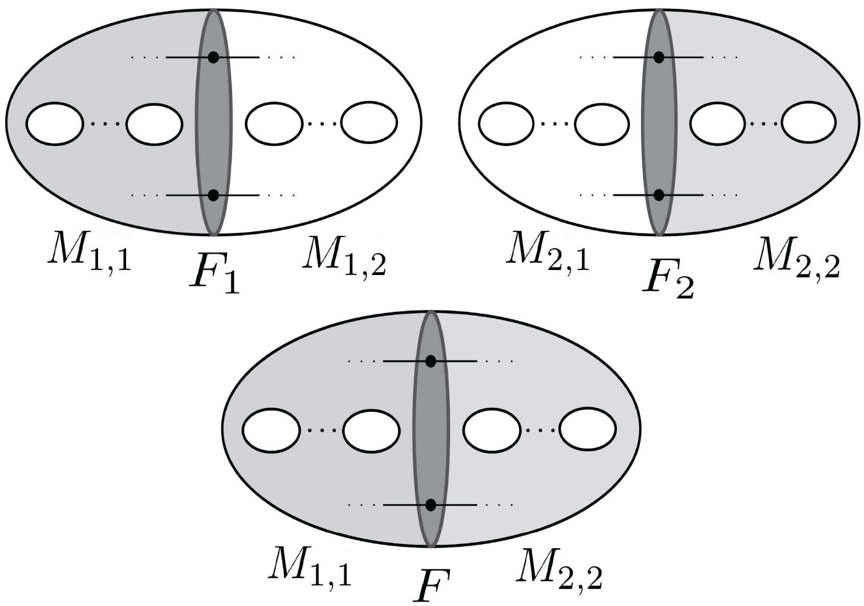

FIGURE 3. The links L_1, L_2 in H_1, H_2 respectively and the link L in the handlebody H.

Gluing $H_{1,1}$ to $H_{2,2}$ along an orientation preserving homeomorphism $\phi : F_1 \to F_2$ sending ∂E_1 to ∂E_2 and $\partial F_1 \cap \partial N(L_{1,1})$ to $\partial F_2 \cap \partial N(L_{2,2})$ yields a manifold/link pair denoted $(H_{1,1}, L_{1,1}, E_1) \oplus_\phi (H_{2,2}, L_{2,2}, E_2)$ which is a handlebody H of genus $g_{1,1} + g_{2,2}$ containing the link L formed by gluing $L_{1,1}$ and $L_{2,2}$ along their endpoints, as in Figure 3. Let H_L be the complement of $\mathring{N}(L)$ in H. We denote by F the image of F_1 and F_2 in H_L, and by E the separating disk in H corresponding to F.

There is one component of L, which we denote by K, that is cut into two arcs K_1 and K_2 by F, the arcs of which are in $H_{1,1}$ and $H_{2,2}$ respectively. We denote $\partial N(K)$ by T_K. We denote the sub-annuli of T_K corresponding to the arcs K_1 and K_2 by A_{K_1} and A_{K_2} respectively. A hyperbolic manifold is always assumed tg-hyperbolic unless otherwise stated.

Theorem 2.1. *Let L_1 and L_2 be links in H_1 and H_2 such that $H_1 \setminus L_1$ and $H_2 \setminus L_2$ are tg-hyperbolic, with $E_1 \subset H_1$ and $E_2 \subset H_2$ twice-punctured disks separating each of H_1 and H_2 into handlebodies, all of positive genus. If $\phi : E_1 \to E_2$ is a homeomorphism sending ∂E_1 to ∂E_2 and sending punctures to punctures, then $(H, L) = (H_{1,1}, L_{1,1}, E_1) \oplus_\phi (H_{2,2}, L_{2,2}, E_2)$ is tg-hyperbolic.*

To prove Theorem 2.1, it is enough to show that since $H_1 \setminus \mathring{N}(L_1)$ and $H_2 \setminus \mathring{N}(L_2)$ contain no essential disks, spheres, annuli and tori, the same holds for $H \setminus \mathring{N}(L)$. In the remainder of this section, we rule out these four kinds of essential surfaces with a sequence of lemmas.

Lemma 2.2. *The surfaces F, F_1, andF_2 are incompressible and boundary incompressible in H_L.*

Proof. We show that F is incompressible and boundary incompressible. The same reasoning immediately applies to F_1 and F_2, as we only use that M_1 and M_2 are submanifolds of the hyperbolic manifolds $H_1 \setminus \mathring{N}(L_1)$ and $H_2 \setminus \mathring{N}(L_2)$ respectively.

Suppose that F is compressible. Then there is some nontrivial circle $C \subset F$ which bounds a disk D' in M_1 or M_2. Suppose $D' \subset M_1$ and let D be the disk in E bounded by C. Suppose that D is punctured once by L. Then the sphere $D \cup D'$ is punctured once by K_1, a contradiction. Suppose next that D is punctured twice by L. Then K_1 is contained in the 3-ball bounded by $D \cup D'$ in H_1, so K_1 can be pushed into a neighborhood of E by an isotopy fixing the endpoints of K_1. Hence M_1 contains a properly embedded disk that is essential since the boundary of the disk, which is isotopic to ∂E_1, splits ∂H_1 into two surfaces of positive genus. This contradicts the fact that that $H_1 \setminus \mathring{N}(L_1)$ is hyperbolic. We reach the analogous contradictions if $D' \subset M_2$, since $H_2 \setminus \mathring{N}(L_2)$ is hyperbolic.

Suppose next that F is boundary compressible. Then there is a nontrivial arc $\alpha \subset F$ which together with an arc $\beta \subset \partial H_L$ bounds a disk D in M_1 or M_2 such that $D \cap F = \alpha$. Suppose $D \subset M_1$. There are two cases.

Case 1: The arc β is in A_{K_1}. If β is trivial in A_{K_1}, then we can isotope D so that $\partial D \subset F$, which yields a compression disk for F since α was a nontrivial arc in F, a contradiction. If β is nontrivial in A_{K_1}, then it is a spanning arc of A_{K_1}. Thus, K_1 together with an arc in E bounds a disk in M_1. Thus we can push K_1 onto F in H_L through an isotopy fixing the endpoints of K_1. Once we have moved K_1 out of the way, we can construct an essential disk in M with boundary isotopic in ∂H_L to ∂E, which contradicts that $H_1 \setminus \mathring{N}(L_1)$ is tg-hyperbolic.

Case 2: The arc β is in ∂H. Suppose D is separating in $H_{1,1}$. Since D is disjoint from A_{K_1}, D separates M_1 into two regions, each of which contains an endpoint of K_1. Since K_1 is connected, this is a contradiction.

Suppose D is not separating in $H_{1,1}$. The arc α separates an annulus A from F such that $A^* = A \cup D$ is a properly embedded annulus in H_L with one boundary component a meridian on T_K and another boundary component on ∂H. Since D is not separating in $H_{1,1}$, $\partial A^* \cap \partial H$ is nontrivial in ∂H, thus A^* is an essential annulus in M_1, which contradicts that $H_1 \setminus \mathring{N}(L_1)$ is hyperbolic.
Since $H_2 \setminus \mathring{N}(L_2)$ is hyperbolic, we reach the analogous contradictions if $D \subset M_2$, and thus F is boundary incompressible. □

Lemma 2.3. *The manifold H_L is irreducible.*

Proof. Suppose H_L contains an essential sphere S. Suppose first that $S \cap F = \emptyset$. Then $S \subset M_1$ or $S \subset M_2$, which implies that one of $H_1 \setminus \mathring{N}(L_1)$ or $H_2 \setminus \mathring{N}(L_2)$ contains an essential sphere, a contradiction.

Suppose next that $S \cap F \neq \emptyset$. We assume that $|S \cap F|$ is minimal among all essential spheres in H_L. An innermost circle C of $S \cap F$ in S bounds a disk D in S such that $D \cap F = C$. Since F is incompressible, C bounds a disk D' in F. Then we can view $D \cup D'$ as a sphere in $H_{1,1}$ or $H_{2,2}$, which from the last case must bound a ball in H_L. Thus, we can push D to D' and slightly beyond, pushing any other intersections of S with D' out of the way as well, to reduce $|S \cap F|$, contradicting minimality. □

Lemma 2.4. *The manifold H_L is boundary irreducible.*

Proof. Suppose ∂H_L has a compressing disk D'. Suppose first that $\partial D' \subset \partial N(L)$. Then the sphere given by $\partial N(D' \cup K')$, where K' is the corresponding component of L, does not bound a ball to either side, contradicting the fact we have already eliminated essential spheres in H_L.

Suppose now that $\partial D' \subset \partial H$. If $D' \cap F = \emptyset$, then one of $\partial H_{1,1}$ or $\partial H_{2,2}$ has a compression disk in M_1 or M_2 respectively, which contradicts that $H_1 \setminus \mathring{N}(L_1)$ and $H_2 \setminus \mathring{N}(L_2)$ are hyperbolic. Thus we can assume that $D' \cap F \neq \emptyset$, and we further assume that $|D' \cap F|$ is minimal among all compression disks of ∂H_L. Then by incompressibility of F, the elements of $D' \cap F$ are all arcs. By minimality of $|D' \cap F|$, an outermost arc of $D' \cap F$ in D' is then nontrivial in F, as otherwise by doing a surgery we could find a compression disk D'' of ∂H_L with $|D'' \cap F| < |D' \cap F|$. This outermost arc cuts a disk from D' that gives a boundary compression for F, a contradiction. □

Lemma 2.5. *The manifold H_L does not contain an essential annulus A with $A \cap F = \emptyset$.*

Proof. Suppose H_L contains such an annulus, and assume without loss of generality that $A \subset M_1$. We can view A as a properly embedded annulus $\overline{A}$ in $H_1 \setminus \mathring{N}(L_1)$ which we will show is essential $H_1 \setminus \mathring{N}(L_1)$, a contradiction to its being tg-hyperbolic.

Suppose $\overline{A}$ is compressible in $H_1 \setminus \mathring{N}(L_1)$. Then a nontrivial simple closed curve $\gamma \subset \overline{A}$ bounds a disk D in $H_1 \setminus \mathring{N}(L_1)$. We assume that $|D \cap F_1|$ is minimal among all compression disks of $\overline{A}$ in $H_1 \setminus \mathring{N}(L_1)$. Note that the components of $D \cap F_1$ are circles. If $D \cap F_1 = \emptyset$, then $D \subset M_1$, which implies that A is compressible in H_L, a contradiction. If $D \cap F_1 \neq \emptyset$, by incompressibility of F_1, an innermost circle of $D \cap F_1$ in D is trivial in F_1, hence by irreducibility of H_L, we can reduce $|D \cap F_1|$ by an isotopy, contradicting minimality.

Thus, $\overline{A}$ is boundary compressible in $H_1 \setminus \mathring{N}(L_1)$. (Note that if $\overline{A}$ is boundary parallel, then it is boundary compressible.) Therefore, both boundary components of $\overline{A}$ must be on the same component of ∂H_L. We consider two cases.

Case 1: The annulus A has both boundary components on ∂H. Suppose $\overline{A}$ is boundary compressible in $H_1 \setminus \mathring{N}(L_1)$. Then a nontrivial arc in $\overline{A}$ together with an arc in ∂H_1 bounds a disk D in $H_1 \setminus \mathring{N}(L_1)$. We assume $|D \cap F_1|$ is minimal among all boundary compressing disks of $\overline{A}$. If $D \cap F_1 = \emptyset$, then $D \subset M_1$, which implies that A is boundary compressible in H_L, a contradiction. If $|D \cap F_1| \neq \emptyset$, by incompressibility of F_1 and minimality, the components of $D \cap F_1$ are arcs. An outermost arc in D must be nontrivial in F_1, as otherwise, we could find a boundary compression disk D' of $\overline{A}$ in $H_1 \setminus \mathring{N}(L_1)$ with $|D' \cap F_1| < |D \cap F_1|$, a contradiction. But then we have a boundary compression disk for F_1 in H_L, a contradiction to Lemma 2.2.

Case 2: The annulus A has both boundary components on $\partial N(L)$. Suppose first that the components ∂A are on a single torus component of $\partial N(L)$ in M_1, and that $\overline{A}$ is boundary compressible in $H_1 \setminus \mathring{N}(L_1)$. A nontrivial arc in $\overline{A}$ together with an arc in $\partial N(L_1)$ bounds a disk D in $H_1 \setminus \mathring{N}(L_1)$. Note that the components of $D \cap F_1$ are circles, thus repeating the minimality argument from Case 1 it follows that $\overline{A}$ is boundary compressible in H_L, a contradiction.

Suppose next that the components of ∂A are both in T_K. Since $A \cap F = \emptyset$, both components of ∂A are $(1,0)$ curves in T_K. Suppose $\overline{A}$ is boundary compressible in $H_1 \setminus \mathring{N}(L_1)$, then a nontrivial arc α in $\overline{A}$ together with an arc $\beta \subset T_K$ bounds a disk D in $H_1 \setminus \mathring{N}(L_1)$. Again, choose D such that $|D \cap F_1|$ is minimal.

If $\beta \cap F = \emptyset$, the components of $|D \cap F_1|$ are circles, and thus we reach a contradiction by repeating the minimality argument from Case 1 and obtaining a boundary compression for A in H_L. If $\beta \cap F \neq \emptyset$, then β intersects $\partial N(L_1) \cap M_{1,2}$ in at least one arc. Thus, D must intersect F in at least one arc. Choosing an outermost arc on D, we obtain a disk in $D \cap M_{1,2}$ with a boundary consisting of two arcs, one a nontrivial arc in F and one in $\partial N(L_1) \cap M_{1,2}$. This contradicts boundary incompressibility of F_1. □

Lemma 2.6. *The manifold H_L contains no essential annuli.*

Proof. Suppose H_L contains an essential annulus A. We assume that $|A \cap F|$ is minimal among all essential annuli in H_L. From Lemma 2.5, we can assume that $A \cap F \neq \emptyset$. There are three cases.

Case 1: The annulus A has boundary components $\partial_1 A, \partial_2 A$ in ∂H. By minimality and incompressibility and boundary incompressibility of F, the components of $A \cap F$ are all either nontrivial circles in A and F or all nontrivial arcs in A and F.

(1a) The components of $A \cap F$ are all nontrivial circles in A and F. Then up to isotopy, the boundary components $\partial_1 A, \partial_2 A$ do not intersect F. Suppose some component of ∂A, say $\partial_1 A$, is in M_1. Then a circle C in $A \cap F$ together with $\partial_1 A$ bounds an annulus $A^* \subset A$ in M_1 such that $A^* \cap F = C$.

Let D denote the disk in E bounded by C. Suppose D is punctured once by L. Then H_L contains a properly embedded once-punctured disk $D \cup A^*$ which can be pushed off E to yield an essential annulus in M_1, contradicting Lemma 2.5.

Suppose D is punctured twice by L. Then we can slide C along E out to ∂H. Hence we obtain an annulus A^{**} that is entirely contained in M_1.

So, A^{**} is a properly embedded annulus in M_1, which is incompressible since $\partial_1 A, C$ are nontrivial in A. Hence by Lemma 2.5, it is boundary compressible in H_L and both boundary curves are on ∂H_1.

Doing the boundary compression on A^{**} yields a disk with boundary on ∂H_1. If the boundary of the disk is trivial on H_1, as happens when the two boundaries of A^{**} are parallel on ∂H_1, then we can form a sphere from the disk and another disk on ∂H_1. Irreducibility of H_L implies we can then isotope A to lower the number of intersections with F, a contradiction.

If the boundary of the disk is nontrivial on H_1, we contradict boundary irreducibility of H_L.

(1b) The components of $A \cap F$ are nontrivial arcs in both A and F. Then A is cut by F into disks in M_1 and M_2 with boundaries that consist of two opposite sides in F and two opposite sides in ∂H. Let $D_1 \subset M_1$ be one such disk. Let $R \subset F$ be a rectangle such that two opposite sides of R are the components of $D_1 \cap F$, and the other two sides are disjoint curves in ∂E. Then $D_1 \cup R$ is either a properly embedded Möbius band Q or a properly embedded annulus $A_1 \subset M_1$ in H_L.

We begin with the case it is an annulus, which we claim is essential in H_L. By minimality of $|A \cap F|$, A_1 is incompressible, as otherwise we could push D_1 through F.

Suppose A_1 is boundary compressible in H_L. Then a nontrivial arc $\alpha \subset A_1$ bounds a disk D in H_L with an arc $\beta \subset \partial H$. We suppose $|D \cap F|$ is minimal among all boundary compression disks of A_1 in H_L. By minimality and incompressibility of F, the components of $D \cap F$ are arcs. Up to isotopy we can assume that $\alpha \subset D_1$ or $\alpha \subset R$. In the former case D provides a boundary compression of A, a contradiction. Suppose now that $\alpha \subset F$. If D does not intersect F in an arc distinct from α, then D provides a boundary compression of F, a contradiction. If $D \cap F \neq \emptyset$, then an outermost arc in D of $D \cap F$ is nontrivial in F, as otherwise by doing a surgery we could find a boundary compression disk D' of A_1 along α with $|D' \cap F| < |D \cap F|$. This yields a boundary compression of F, a contradiction. If A_1 were boundary parallel in H_L, it would be boundary compressible, hence A_1 is an essential annulus in H_L contained in M_1, which contradicts Lemma 2.5.

Suppose now that $D_1 \cup R$ is a Möbius band Q. Then the boundary of a regular neighborhood of Q is an annulus A_2. It cannot compress in the regular neighborhood of Q since that is a solid torus, and the boundaries of A_2 are isotopic to twice the core curve of the solid torus. It cannot compress to the outside of the regular neighborhood of Q because either component of the boundary of the annulus links the core curve of the annulus, due to the twisting of the Möbius band. If the core curve bounded a disk, that disk would not intersect the boundary curves of the annulus, which would contradict the linking. And it is boundary incompressible for the same reasons that A_1 is, also contradicting Lemma 2.5.

Case 2: The annulus A has boundary components $\partial_1 A$ and $\partial_2 A$ on $\partial N(L)$. There are two subcases.

(2a) Both $\partial_1 A$ and $\partial_2 A$ lie on the torus components $T_{K_{1,i}}$ and $T_{K_{2,j}}$ where $T_{K_{1,i}}$ is a torus component of $\partial N(L)$ contained completely in M_1, and $T_{K_{2,j}}$ is a torus component of $\partial N(L)$ contained completely in M_2. By minimality of $|A \cap F|$ and incompressibility of F, the components of $A \cap F$ are circles which are nontrivial in both A and F. A circle C in $A \cap F$ bounds a subannulus A^* of A with $\partial_1 A$ such that $A^* \cap F = C$ which is incompressible since C and $\partial_1 A$ are nontrivial in A.

Suppose $A^* \subset M_1$. Let D denote the disk in E bounded by C. If D is punctured once, we can take the union of it with A^*, and then H_L contains an essential annulus in M_1 with one boundary component on $T_{K_{1,i}}$ and another boundary component on T_K. If D is punctured twice, we can glue the annulus $F \setminus D$ to A^* to obtain an annulus essential in H_L and contained in M_1 with one boundary component on $T_{K_{1,i}}$ and the other boundary component on ∂H. Both cases contradict Lemma 2.5. We reach the analogous contradictions if $A^* \subset M_2$.

(2b) The annulus A has at least one boundary component $\partial_1 A$ on K. Suppose first that $\partial_1 A$ is a $(1,0)$ curve in T_K. Then $\partial_2 A$ is either a $(1,0)$ curve in T_K or lies in some $T_{K_{1,i}}$ or $T_{K_{2,j}}$. By minimality of $|A \cap F|$ and incompressibility of F, the components of $A \cap F$ are circles which are nontrivial in A and F. A circle C in $A \cap F$ bounds a subannulus A^* of A with $\partial_1 A$ such that $A^* \cap F = C$. Note A^* is incompressible since C and $\partial_1 A$ are nontrivial in A.

Suppose, without loss of generality, that $A^* \subset M_1$. Let D denote the disk in E bounded by C. Suppose first that D is punctured once. Then we obtain a new annulus A'^* by gluing D onto A^*, with both boundaries now meridians on

T_K. We can view A'^* as a properly embedded annulus in M_1 which is boundary compressible in H_L by Lemma 2.5.

By irreducibility of H_L, the annulus must be boundary parallel. If it is boundary parallel to the M_1 side of H_L, then we can use that to isotope A along T_K and reduce its number of intersection curves with F, a contradiction to minimality. It cannot be boundary parallel to the other side as the boundary of the handlebody is to that side.

If D is punctured twice, then H_L contains an essential annulus in M_1 with one boundary component on T_K and the other boundary component on ∂H. this contradicts Lemma 2.5.

Suppose next that $\partial_1 A$ is a (p, q)-curve in T_K with $|q| > 0$. If $\partial_2 A \subset T_K$, then all components of $A \cap F$ are nontrivial arcs in A. If there is an innermost arc of $A \cap F$ in F that is trivial in F, then A is boundary compressible, contradicting its essentiality.

So all arcs in $A \cap F$ are nontrivial and parallel on F. Each component of $A \cap M_1$ is a disk with boundary consisting of four arcs, two in $\partial N(K)$ and two in F. Let D be one of them. The two arcs on its boundary in F cut a disk D' from F that has two arcs on its boundary also in $\partial N(K)$. Then $D \cup D'$ is either a properly embedded Möbius band Q or an annulus A'. We consider the annulus possibility first.

If A' is compressible, then we can use the compression disk together with half of A' to obtain a disk with boundary consisting of two arcs, one in F and one in $\partial N(K)$. But this contradicts the boundary-incompressibility of F.

If A' is boundary compressible by a disk D'', we can take the arc in $D'' \cap A'$ to be in $D' \subset F$, therefore obtaining a boundary compression of F. So A' is a essential annulus that does not intersect F. Therefore the existence of A' contradicts Lemma 2.5.

If $D \cup D'$ is a Möbius band Q, then the boundary of Q must be a meridian on T_K as it is entirely contained in M_1 and cannot be trivial as then we would have a projective plane embedded in M_1 which we could embed in S^3, a contradiction.

The boundary of a regular neighborhood of Q is an annulus A''. It is incompressible to the inside of the regular neighborhod of Q as that is a solid torus, with the core curve of the annulus going around the core curve of the solid torus twice. It is incompressible to the outside as the boundaries are meridian curves on T_K. It is boundary incompressible as any boundary compression would yield a boundary compression for F, a contradiction. So again, the existence of an essential annulus A'' that misses F contradicts Lemma 2.5.

Suppose $\partial_2 A$ is in some $T_{K_{1,i}}$. Then there must be an intersection arc in $A \cap F$ that cuts a disk from A with one boundary in F and the other boundary in $\partial N(K_2)$. We can use it to push K_2 onto E by an isotopy in H_L fixing the endpoints of K_2. This implies that H_L contains a compressing disk in M_2 with boundary isotopic in ∂H to ∂E. We reach the analogous contradiction if $\partial_2 A$ is in some $T_{K_{2,j}}$.

Case 3: The annulus A has a boundary component $\partial_1 A$ on $\partial N(L)$ and a boundary component $\partial_2 A$ on ∂H.

Let J be the component of L with regular neighborhood boundary that A intersects. Then the boundary of a regular neighborhood of $A \cup \partial N(J)$ is an annulus A' with both of its boundaries in ∂H. The boundaries of A' are two parallel nontrivial curves on the boundary of H that are also parallel to the one boundary of A on ∂H. Thus A' must be incompressible.

If A' is boundary compressible, then do the boundary compression on the annulus A' to obtain a disk D'' with boundary in ∂H. By boundary-irreducibility of H_L, D'' would have to have trivial boundary in ∂H. The boundary compression has the impact on $\partial A'$ of surgering the two curves along an arc running from one to the other. Surgering two nontrivial parallel curves on a surface of genus at least two along an arc that is not in the annulus between the curves yields a nontrivial curve. So the boundary compression cannot be to that side. Thus the boundary compression must be to the side of the annulus in ∂H shared by the two curves. But this side is a solid torus missing its core curve J, preventing a boundary compression to that side. So A' is an essential annulus in H_L with both boundaries on ∂H, contradicting Case 1. □

Lemma 2.7. *The manifold H_L contains no essential torus.*

Proof. Suppose H_L contains an essential torus $\mathcal{T}$. We assume that $|\mathcal{T} \cap F|$ is minimal among all essential tori in H_L.

Suppose first that $\mathcal{T} \cap F = \emptyset$. Then $\mathcal{T} \subset M_1$ or $\mathcal{T} \subset M_2$. For convenience, we assume $\mathcal{T} \subset M_1$. Then we can view $\mathcal{T}$ as a torus $\overline{\mathcal{T}}$ in $H_1 \setminus \mathring{N}(L_1)$ which we show is essential.

Suppose $\overline{\mathcal{T}}$ is boundary parallel in $H_1 \setminus \mathring{N}(L_1)$. Since ∂H_1 has genus at least 2, $\overline{\mathcal{T}}$ must be parallel to a component of $\partial N(L_1)$. If it is boundary parallel to a component J, then $\overline{\mathcal{T}}$ must separate a solid torus from H_1 that has J as its core curve. Since F is to the side of $\overline{\mathcal{T}}$ that H is, the solid torus cannot intersect F_1 either. So both the solid torus and J are in M_1, and $\overline{T}$ is boundary parallel in H_L, contrary to our assumption.

Suppose $\overline{\mathcal{T}}$ is compressible in $H_1 \setminus \mathring{N}(L_1)$. Then a nontrivial curve $\gamma \subset \overline{\mathcal{T}}$ bounds a disk D in $H_1 \setminus \mathring{N}(L_1)$. We assume that $|D \cap F_1|$ is minimal among all compression disks of $\overline{\mathcal{T}}$ in $H_1 \setminus \mathring{N}(L_1)$. Note that the components of $D \cap F$ are circles.

If $D \cap F_1 = \emptyset$, then $D \subset M_1$, which implies that $\mathcal{T}$ is compressible in H_L, a contradiction. If $D \cap F_1 \neq \emptyset$, by incompressibility of F, an innermost circle of $D \cap F_1$ in D is trivial in F_1, hence by irreducibility of H_L, we can reduce $|D \cap F_1|$ by an isotopy, contradicting minimality. It follows that $\overline{\mathcal{T}}$ is essential in $H_1 \setminus \mathring{N}(L_1)$, which contradicts that $H_1 \setminus \mathring{N}(L_1)$ is hyperbolic. Since $H_2 \setminus \mathring{N}(L_2)$ is hyperbolic, we reach the analogous contradictions if $\mathcal{T} \subset M_2$.

Suppose next that $\mathcal{T} \cap F \neq \emptyset$. By minimality of $|\mathcal{T} \cap F|$ and incompressibility of F, the components of $\mathcal{T} \cap F$ are circles which are nontrivial in $\mathcal{T}$ and F.

Let A_C be an annulus which is a connected component of $M_1 \cap \mathcal{T}$ with boundary two circles in $F \cap \mathcal{T}$. We claim the boundaries of A_C are two disjoint circles C_1 and C_2 which bound disjoint disks in E punctured once by L. Suppose otherwise. Then two circles $C_1, C_2 \subset A_C \cap F$ bound disks $D_1, D_2 \subset E$ such that $D_2 \subset D_1$. If D_2 is punctured once and D_1 is punctured twice by L, then we can glue D_2 and a slightly moved D_1 to A_C to obtain a sphere in H that is punctured three times by L. Thus D_1, D_2 are both punctured once or twice by L.

Suppose D_1 and D_2 are both punctured twice. Then by adding the annuli in $F \setminus D_i$ to A_C, we obtain an annulus A'_C with boundary in ∂H. By the same reasoning as in the proof of Case 1 in the proof of Lemma 2.6, A'_C is boundary compressible in M_1 and we can push A_C through F to reduce $|A \cap F|$, contradicting minimality.

Suppose D_1 and D_2 are both punctured once. The circles C_1 and C_2 bound an annulus $A_{C,F}$ in F which is not punctured by L.

By gluing the punctured disks D_1 and D_2 onto A_C, and sliding the D_1 portion just off F, we obtain a new annulus $\overline{A}_C$ with boundaries on A_{K_1}. This annulus $\overline{A}_C \subset M_1$ is properly embedded in H_L with $\partial \overline{A}_C \subset T_K$. The boundaries of $\overline{A}_C$ are meridians on T_K that bound an annulus $A'_{C,F} \subset A_{K_1}$ which is obtained from $A_{C,F}$ by an isotopy in M_1. Note $\overline{A}_C$ is incompressible in H_L as A_C is incompressible, and hence by Lemma 2.5 it is boundary compressible in H_L. Thus a nontrivial arc α in $\overline{A}_C$ bounds a disk D_β in H_L with an arc $\beta \subset T_K$.

If β is not a nontrivial arc in $A'_{C,F}$, it intersects A_{K_2} in a nontrivial arc. In that case D_β becomes a compressing disk for the torus $\overline{A}_C \cup (T_K \setminus A'_{C,F}$. Doin the compression yields a sphere in H_L that separates K from ∂H, a contradiction to irreducibility of H_L.

If β is not a nontrivial arc in $A'_{C,F}$, the disk D_β lies in the region contained in M_1 that $\overline{A}_C$ separates from H_L. We can thus push D_β by an isotopy to obtain a boundary compression disk for A_C in M_1, hence A_C is boundary compressible in M_1 and boundary parallel (since the boundary compressing arc in M_1 is a nontrivial arc in $A_{C,F}$) and we can push it through F to reduce $|A \cap F|$, a contradiction.

We reach the analogous contradictions if $A_C \subset M_2$. Thus, we can assume the boundaries of A_C are two disjoint circles which bound disjoint disks in E punctured once by L.

If there were more than one such annulus in M_1 and one such in M_2, then following along the annuli, one after the other as we travel along a longitude of $\mathcal{T}$, we would have to have them cycle one inside the next as they pass through F, and the torus could never close up. So there is only one to each side of F and

$\mathcal{T}$ is cut into two incompressible (since the elements of $\mathcal{T} \cap F$ are nontrivial in $\mathcal{T}$) annuli $\mathcal{A}_1 \subset M_1, \mathcal{A}_2 \subset M_2$.

If we glue the punctured disks D_1 and D_2 to $\mathcal{A}_1$ we obtain an incompressible annulus, which must then be boundary parallel to $\partial N(K)$ by Lemma 2.6. The same holds for $\mathcal{A}_2$, implying the torus $\mathcal{T}$ is boundary parallel, a contradiction to its being essential. □

A situation where Theorem 2.1 is easily applicable is when $H_1 = H_2$, and $L_1 = L_2$. See Figure 4.

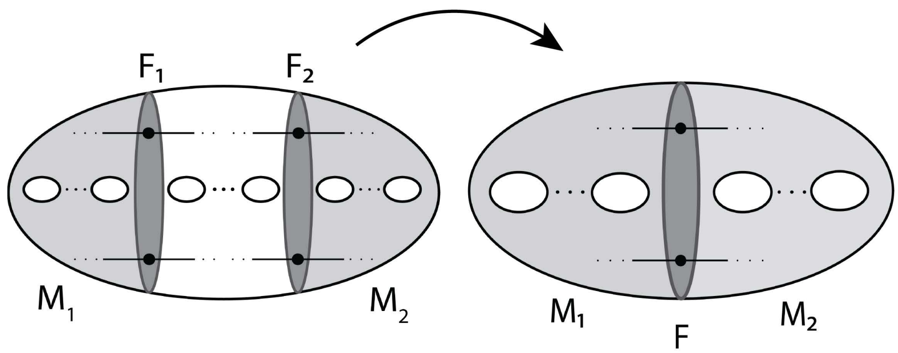

FIGURE 4. Applying Theorem 2.1 to two pieces in a single handlebody.

Corollary 2.8. *Let (H, L) be a handlebody/link pair that is tg-hyperbolic. Let E_1 and E_2 be two disjoint twice-punctured separating disks in H. Then cutting along the two disks, the piece with both disks on the boundary can be discarded and the two pieces with one disk along the boundary, assuming they are positive genus, can be glued together along those disks, and the resulting handlebody/link pair will be tg-hyperbolic.*

Note that the intermediate piece that is being removed need not have positive genus. So, we can remove appropriate tangles from a tg-hyperbolic link in a handlebody and still preserve tg-hyperbolicity. Thus, in order to determine tg-hyperbolicity of a link in a handlebody, all such tangles could be removed and if the resulting simplified link is not tg-hyperbolic because of the presence of an essential sphere, disk, annulus or torus, neither could the original link have been.

The ideas in the proof of Theorem 2.1 extend to a different setting, where we cut a handlebody into three pieces along disks E_1 and E_2 and glue one piece to itself along the copies of E_1 and E_2.

Suppose L_1 is a link in a handlebody H_1 and (H_1, L_1) is tg-hyperbolic. Suppose E_1 and E_2 are two nontrivial separating disks in H_1 each punctured twice by L_1, which together separate a handlebody $H_{1,2}$ of genus $g_{1,2}$ from two disjoint handlebodies $H_{1,1}, H_{1,3}$ of genus $g_{1,1}, g_{1,3}$ respectively, with all these genera positive. Let $M_{1,i} = H_{1,i} \setminus \mathring{N}(L_1), F_i = E_i \setminus \mathring{N}(L_1)$. Let $L_{1,2} = L_1 \cap H_{1,2}$.

Gluing the subsets F_1, F_2 of $\partial M_{1,2}$ together by an orientation preserving homeomorphism $\phi : F_1 \to F_2$ sending ∂E_1 to ∂E_2 and $\partial F_1 \cap \partial N(L_1)$ to $\partial F_2 \cap \partial N(L_2)$ yields a link complement $H_L = H \setminus \mathring{N}(L)$ in the handlebody H of genus $g_{1,2} + 1$ as in Figure 5. We denote by F the image of F_1 and F_2 in H_L.

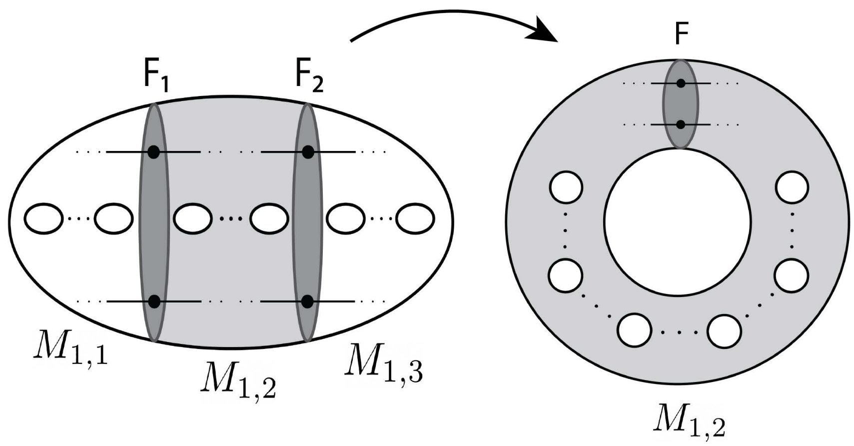

FIGURE 5. Gluing M to itself by a homeomorphism $F_1 \to F_2$.

Theorem 2.9. *Suppose $H_1 \setminus L_1$ is tg-hyperbolic, and $E_1 \cap L_1 = E_1 \cap K, E_2 \cap L_1 = E_2 \cap K'$, where K and K' are two distinct components of L_1, then H_L is tg-hyperbolic.*

Theorem 2.9 follows from the same arguments as Theorem 2.1. Namely, the surfaces F, F_1, and F_2 are incompressible and boundary incompressible, and we can use this to reach the analogous contradictions from Lemmas 2.2-2.7. The requirement that the punctures of E_1 and E_2 correspond to two distinct components K and K' of L must be introduced to force an annulus with boundary in ∂H that intersects F in nontrivial arcs to be cut into disks with two opposite sides in F. Without this condition the result does not hold in general, as shown in Figure 6.

3. Applications

3.1. Staked links. Links in handlebodies are directly related to the theory of staked links defined in [2]. (These links are also called tunnel links as in [10] or starred links as in as-of-yet unpublished work of N. Gügümcü and L. Kauffman.) In this section we will only work with staked links in S^2. A *staked link* is a pair $(L_D, \{p_i\}_{1\leq i\leq n})$ of a link diagram $L_D \subset S^2$ together with a finite collection $\{p_i\}_{1\leq i\leq n}$ of *isolated poles*, which are distinct points $p_1, \dots, p_n \in S^2$ such that

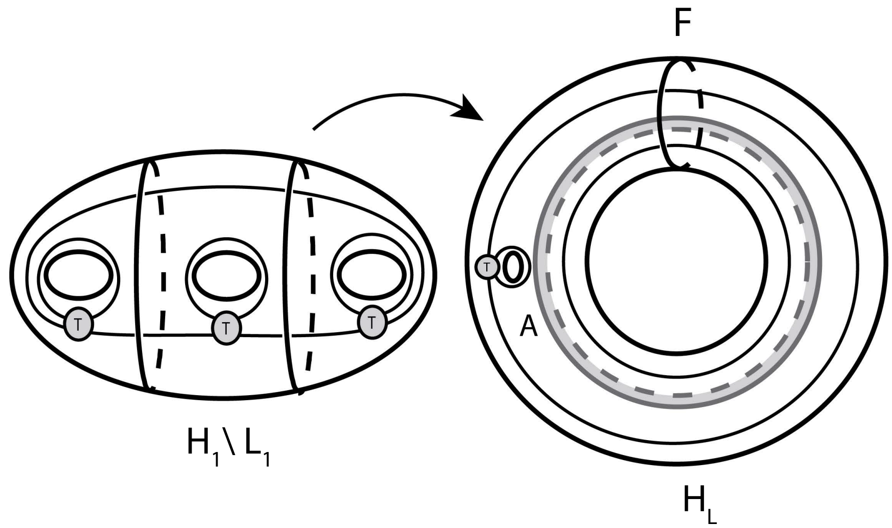

FIGURE 6. A counterexample to Theorem 2.9 when the condition on the punctures of E_1, E_2 is removed. Here T is an alternating tangle which can be chosen to satisfy the conditions of Theorem 1.6 of [3] (appearing in the next section) so that $H_1 \setminus L_1$ tg-hyperbolic. After cutting and gluing, H_L contains an essential annulus A with boundary in ∂H as shown (perpendicular to the page), which intersects F in a single nontrivial arc and which separates one component of the link.

each p_i lies in a connected component of $S^2 \setminus L_D$. Staked links are considered up to Reidemeister moves that do not pass strands over elements of $\{p_i\}_{1 \leq i \leq n}$. A staked link determines a link in a handlebody of genus $n-1$ as follows. Choose open disks $D_1, \dots, D_n \subset S^2 \setminus L_D$ containing $p_1, \dots, p_n$ respectively, such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Then $D_L := S^2 \setminus (\cup_{i=1}^n D_i)$ is the closure of a $n-1$ punctured disk and L_D determines a link $\overline{L}_D$ in the handlebody $D_L \times [0,1]$ as shown in Figure 7. A staked link $(L_D, \{p_i\}_{1 \leq i \leq n})$ is tg-hyperbolic if $(D_L \times [0,1], \overline{L}_D)$ is hyperbolic as in Section 1.

Given a staked link $(L_D, \{p_i\}_{1 \leq i \leq n})$, any simple closed loop $\gamma : [0,1] \to S^2$ with $\gamma(0) = \gamma(1) = p_i$ determines a proper non self-intersecting arc $a_\gamma \subset S^2 \setminus (\cup_{i=1}^n D_i)$ with $\partial a_\gamma \subset \partial D_i$, and hence a proper separating disk $a_\gamma \times [0,1]$ in $D_L \times [0,1]$, as in Figure 8. If γ intersects L_D twice, this disk could come from a gluing operation satisfying the conditions of Theorem 2.1, hence Theorem 2.1 gives a way to check if a complicated staked link is hyperbolic by checking if it is cut by γ into pieces which come from hyperbolic staked links.

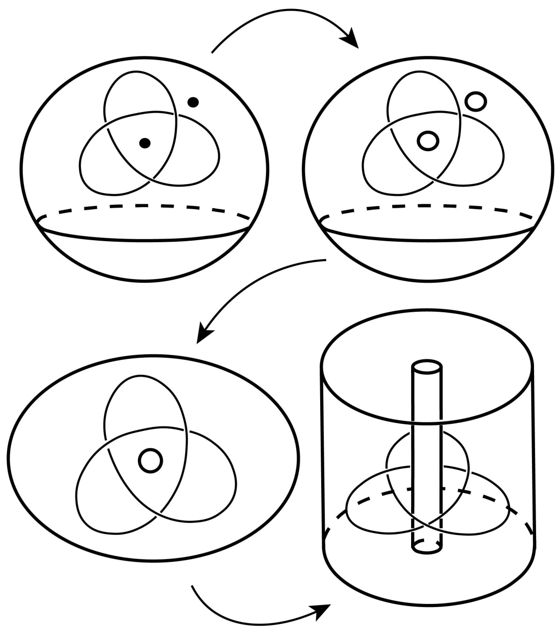

FIGURE 7. A staked link $L_D \subset S^2$ with n stakes determines a link $\overline{L}_D$ in a handlebody of genus $n-1$.

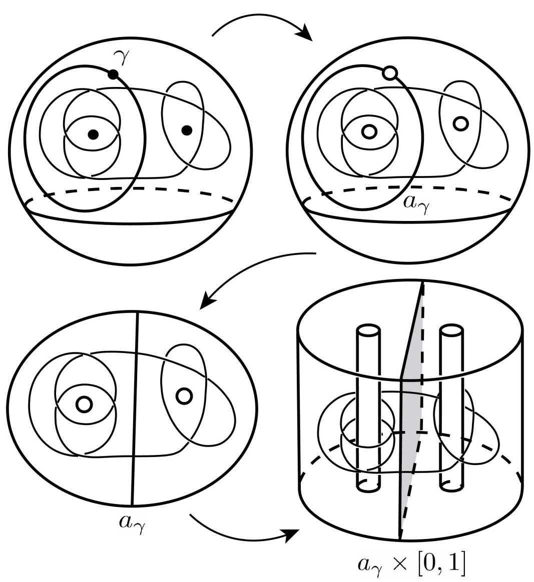

FIGURE 8. A simple closed loop γ based at a pole of a staked knot determines a separating disk in the corresponding handlebody.

3.2. Alternating links. To show a link in a handlebody (H, L) is tg-hyperbolic, it is sufficient to show that H can be given a product structure $H \cong F \times [0,1]$, where F is the closure of a disk punctured some nonzero number of times, such that the projection of L to the surface $F \times \{1/2\}$ is alternating and satisfies conditions as follows.

Theorem 3.1 (Theorem 1.6 in [3]). *Let F be a projection surface with nonempty boundary which is not a disk, and let $L \subset F \times I$ be a link with a connected, reduced, alternating projection diagram $\pi(L) \subset F \times \{1/2\}$ with at least one crossing. Let $M = (F \times I) \setminus N(L)$. Then M is tg-hyperbolic if and only if the following four conditions are satisfied:*

- *(i) $\pi(L)$ is weakly prime on $F \times \{1/2\}$;*
- *(ii) the interior of every complementary region of $(F \times \{1/2\}) \setminus \pi(L)$ is either an open disk or an open annulus;*
- *(iii) if regions R_1 and R_2 of $(F \times \{1/2\}) \setminus \pi(L)$ share an edge, then at least one is a disk;*
- *(iv) there is no simple closed curve α in F that intersects $\pi(L)$ exactly in a nonempty collection of crossings, such that for each such crossing, α bisects the crossing and the two opposite complementary regions meeting at that crossing that do not intersect α near that crossing are annuli.*

By weakly prime we mean that there is no simple closed curve on the projection surface that crosses the link twice and that bounds a disk that contains crossings. Note that each of these conditions is easily checked for the projection.

In the notations of Section 2, this gives a simple way to show that (H_1, L_1) and (H_2, L_2) are tg-hyperbolic. Note that Theorem 2.1 gives the expected behavior when both L_1, L_2 are alternating and K_1, K_2 glue together so that K is alternating. In particular, Theorem 2.1 can apply in the general situation of gluing an alternating piece to a non-alternating piece.

As an example, for any weakly prime alternating tangle T as in Figure 9 other than 0 or 1 crossing or a horizontal sequence of bigons, (which do not satisfy the conditions of the theorem), we can form the piece M_T. Then if we take any other hyperbolic knot in a handlebody of positive genus, and split it into two pieces of positive genus by a twice-punctured disk, we can glue either resulting piece to the piece M_T and still generate a tg-hyperbolic handlebody/link pair.

3.3. Planar knotoids. Knotoids are a variation on knots given by projections of line segments defined up to Reidemeister moves and disallowing strands to pass over or under the endpoints of the segment. When the projection surface is a plane, we say the knotoid is a planar knotoid. In [1], two definitions of hyperbolicity of planar knotoids were given. The first, which is called the planar reflected doubling map, associates to the knotoid a link in a genus three handlebody. If the complement of the link is tg-hyperbolic, the knotoid is said to be hyperbolic under the reflected doubling map. The second, which is called

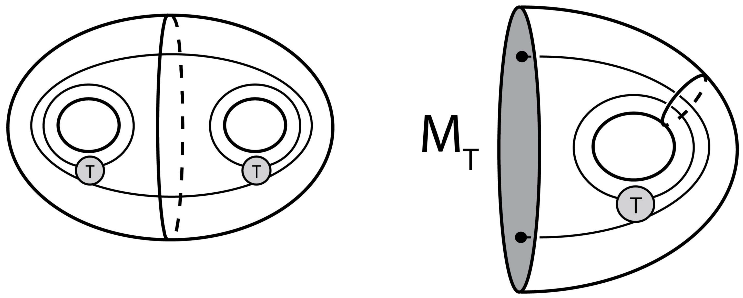

FIGURE 9. If T is an alternating tangle satisfying simple restrictions, the genus 2 handlebody/link pair depicted is tg-hyperbolic, so we can glue M_T to any other piece from a hyperbolic handlebody/link pair to obtain another tg-hyperbolic handlebody/link pair.

the planar gluing map, associates to the knotoid a link in a genus two handlebody. Again, if the complement of the link is tg-hyperbolic, the knotoid is said to be hyperbolic under the gluing map. Proposition 2.5 in [1] proves that hyperbolicity of a planar knotoid under the reflected doubling map implies hyperbolicity under the gluing map but not vice versa. Further, the volume under the reflected doubling map is always at least as large as the volume under the gluing map. Theorem 2.1 together with the results from [3] can provide many examples of planar knotoids that are hyperbolic under either of the two constructions.

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