New York Journal of Mathematics

New York J. Math. **29** (2023) 1060–1074.

Sublacunary sequences that are strong sweeping out

Sovanlal Mondal, Madhumita Roy and Máté Wierdl

ABSTRACT. An increasing sequence (a_n) of positive integers which satisfies $a_{n+1}/a_n \geq 1+\eta$ for some positive η is called a lacunary sequence. It has been known for over twenty years that every lacunary sequence has the strong sweeping out property which means that in every aperiodic dynamical system we can find a set E of arbitrary small measure so that

$$\limsup_N \frac{1}{N} \sum_{n \leq N} \mathbb{1}_E(T^{a_n} x) = 1$$

and

$$\liminf_N \frac{1}{N} \sum_{n \leq N} \mathbb{1}_E(T^{a_n} x) = 0$$

almost everywhere. In this paper, we improve this result by showing that if (a_n) satisfies only

$$\frac{a_{n+1}}{a_n} > 1 + \frac{1}{(\log \log n)^{1-\eta}}$$

for some positive η, then it already has the strong sweeping out property.

CONTENTS

1. Introduction and main results

Throughout this paper, we use the notation

$$[N] := \{1, 2, \dots, N\}, \text{ where } N \text{ is any positive integer.}$$

Let T be a measure preserving transformation on the probability space (X, Σ, μ). After Birkhoff's pointwise ergodic theorem was proved, naturally the question was raised whether it is possible to generalize the theorem along any sequence

Received January 18, 2021.

2020 *Mathematics Subject Classification.* 37A30 (Primary), 37A05, 37A46 (secondary).

Key words and phrases. Pointwise ergodic theorem, sublacunary sequences, strong sweeping out property, universally bad sequences.

The first author is supported by the National Science Foundation under grant number DMS-1855745.

ISSN 1076-9803/2023

(a_n) of integers instead of taking the entire sequence (n). Krengel [Kre71] was the one who first constructed a strictly increasing sequence (a_n) of positive integers so that in every aperiodic system the ergodic averages $\frac{1}{N}\sum_{n\in[N]} f(T^{a_n}x)$ diverge almost everywhere. Soon after, Bellow showed in [Bel83] that if (a_n) is a *lacunary sequence*, that is, it satisfies $\frac{a_{n+1}}{a_n} \geq 1+\eta$ for some positive η, then in every aperiodic system and for every p satisfying $1 \leq p < \infty$, there exists a function $f \in L^p$ such that the ergodic averages along (a_n) diverge a.e. Our first main result provides a growth condition for a sequence (a_n) to be pointwise bad which applies to some *sublacunary* sequences as well. Recall that a sequence (a_n) is said to be *sublacunary* if it satisfies $\lim_n \frac{a_{n+1}}{a_n} = 1$.

Theorem 1.1 (Deterministic condition)**.** *Suppose (a_n) is a sequence which satisfies*

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log\log n)^{1-\eta}} \tag{1}$$

for some $\eta > 0$. Then in every aperiodic dynamical system (X, Σ, μ, T) and every $\epsilon > 0$, there exists a set $E \in \Sigma$ with $\mu(E) < \epsilon$ such that for almost every $x \in X$, we have

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n\in[N]} \mathbb{1}_E(T^{a_n}x) = 1 \text{ and } \liminf_{N\to\infty} \frac{1}{N}\sum_{n\in[N]} \mathbb{1}_E(T^{a_n}x) = 0.$$

An example of a sequence (a_n) which satisfies the growth condition in Eq. (1) is $(a_n) = \left(\lfloor e^{\frac{n}{(\log\log n)^{1-\eta}}}\rfloor\right)_n$ for some $\eta > 0$.

Definition 1.2 (Pointwise good and bad sequence)**.** *Let $1 \leq p \leq \infty$ and (a_n) be a sequence of positive integers. We say that (a_n) is pointwise good for L^p if for every measure preserving system (X, Σ, μ, T) and every $f \in L^p(X)$ the limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n\in[N]} f(T^{a_n}x)$$

exists a.e.. Similarly, we say the sequence (a_n) is pointwise bad for L^p if for every aperiodic measure preserving system (X, Σ, μ, T) there exists a function $f \in L^p(X)$ such that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n\in[N]} f(T^{a_n}x)$$

fails to exist a.e..

Whether a sequence will be pointwise good or bad for L^p depends on many factors, such as, the speed of the sequence (a_n), the value of p, and sometimes the intrinsic arithmetic properties of the sequence (a_n). From Theorem 1.1, we can see that if a sequence grows very fast, then it will be pointwise bad even for L^∞. At the other extreme, if a sequence grows slower than any positive power of n, for example $(a_n) = (\lfloor(\log n)^c\rfloor)$ for some $c > 0$, then it is again pointwise bad for L^∞ as shown by Jones and Wierdl [JW94, Example 2.18] (see also [Loy22]). Bellow [Bel89] and Reinhold-Larsson [Rei94] proved that whether a sequence will be pointwise good for L^p or not can depend on the value of p. More precisely, they showed that for any given $1 \leq p < q \leq \infty$, there are sequences (a_n) which are pointwise good for L^q but pointwise bad for L^p. Parrish gives refinements of these results in terms of Orlicz spaces in [And11]. In general, neither the growth rate of the sequence, nor the value of p alone can determine whether the sequence is pointwise good or bad. In some cases, one has to analyze the intrinsic arithmetic properties of the sequence (a_n). One such curious example is $(n^k)_n$. A celebrated result of Bourgain [Bou88, Theorem 2] says that the sequence $(n^k)_n$ is pointwise good for L^2 when k is a positive integer. On the other hand, the sequence $(\lfloor n^k + \log n \rfloor)_n$ is known to be pointwise bad for L^2 when k is a positive integer [BKQW05, Theorem C].

After showing that polynomials are pointwise good for L^2, Bourgain showed in [Bou89] that a polynomial sequence is pointwise good for L^p for every $p > 1$, and Wierdl [Wie88] proved the same for the sequence of primes. However, the sequence of squares is pointwise bad for L^1, as was shown by Buczolich and Mauldin [BM10]. LaVictoire showed the same in [Lav11] for the sequence (n^k) of kth powers for a fixed positive integer k and for the sequence of primes. It was largely believed that there cannot be any sequence (a_n) which is pointwise good for L^1 and satisfies $(a_{n+1} - a_n) \to \infty$ as $n \to \infty$, but Buczolich disproved this conjecture in [Buc07]. Later, LaVictoire [LaV09] showed that a large class of random sequences also serve as counterexamples. It follows from the work of Urban and Zienkiewicz [UZ07] that $(\lfloor n^c \rfloor), c \in (1, 1.001)$ is pointwise good for L^1. The current best result is due to Mirek [Mir15] who showed that $\lfloor n^c \rfloor, c \in (1, \frac{30}{29})$ is pointwise good for L^1 (see also [Tro21]). It would be interesting to know if the latter result can be extended to all positive non integer c. The case of L^2 is known from [BKQW05] as well as L^p, $p > 1$. For further exposition in this area, the reader is referred to the survey article of [RW95].

Now we will give the formal definition of the strong sweeping out property.

Definition 1.3 (Strong sweeping out property)**.** *Let (a_n) be a sequence of integers. We say that a sequence (a_n) has the strong sweeping out property if in every aperiodic dynamical system (X, Σ, μ, T) and for every $\epsilon > 0$, there exists a set $E \in \Sigma$ with $\mu(E) < \epsilon$ such that for almost every $x \in X$ we have*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n\in[N]} \mathbb{1}_E(T^{a_n}x) = 1 \text{ and } \liminf_{N\to\infty} \frac{1}{N} \sum_{n\in[N]} \mathbb{1}_E(T^{a_n}x) = 0.$$

It is clear from the definition that the strong sweeping out property is a very strong type of non-convergence. In particular, if a sequence (a_n) has the strong sweeping out property then (a_n) is L^p-bad for every $p \in [1, \infty]$.

In 1996, it was shown in [ABJLRW, Corollary 1.11] that every lacunary sequence has the strong sweeping out property. Our theorem is an improvement of their result.

Theorem 1.1 can also be interpreted from a different viewpoint. It was proved by Jones and Wierdl [JW94, Corollary 2.14] that for $1 \leq p < \infty$ if a sequence (a_n) satisfies $\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log n)^{\frac{1}{p}-\eta}}$ for some $\eta > 0$, then (a_n) is pointwise bad for L^p. Theorem 1.1 can be viewed as an extension of this result to not only L^∞ but indicators as well. It is also interesting to compare our result with the results of Berkes [Ber97] on lacunary polynomials.

In the next theorem, we will give a probabilistic condition for a sequence (a_n) to have the strong sweeping out property. Before we state the result, let us explain the notion of a randomly generated sequence. Suppose (σ_n) is a sequence of positive numbers; σ_n is the probability with which n is chosen into the random sequence. More precisely, let Y_n be a sequence of $\{0, 1\}$-valued random variables on the probability space (Ω, β, P) so that $P(Y_n = 1) = \sigma_n$ and $P(Y_n = 0) = 1 - \sigma_n$. For each $\omega \in \Omega$, let A^ω be the sequence defined by the property that $n \in A^\omega$ if and only if $Y_n(\omega) = 1$. Our second main result is the following:

Theorem 1.4 (Probabilistic condition)**.** *Let $\eta > 0$ be arbitrary and*

$$\sigma_n = \frac{(\log\log\log n)^{1-\eta}}{n}.$$

Then for a.e. ω, the random sequence $A^\omega = (a_n(\omega))$ is strong sweeping out.

This result is an improvement of [JLW99, Theorem C] where the same conclusion was obtained under the stronger hypothesis $\sigma(n) = \frac{1}{n}$.

2. Proof of the main results

2.1. Notation. Let S be a finite set and let $f = (f_n)_{n\in S}$ be a sequence of numbers or functions indexed by S. We denote the arithmetic average of (f_n) by $\mathbb{A}_S f$,

$$\mathbb{A}_S f = \mathbb{A}_S f_n := \frac{1}{\#S} \sum_{n\in S} f_n \tag{2}$$

For a sequence $w = (w(n))_{n\in S}$ of numbers not identically 0, which we regard as the sequence of weights, we denote the w-weighted average of f by $\mathbb{A}_S^w f$,

$$\mathbb{A}_S^w f = \mathbb{A}_S^w f_n := \frac{1}{\sum_{n\in S} w(n)} \sum_{n\in S} w(n) f_n. \tag{3}$$

2.2. Proof of Theorem 1.1. We will consider higher dimensional torus for proving our result. Originally, such argument was used by Jones in [Jon04] to prove that the finite union of lacunary sequences has the strong sweeping out property. Later, an extension of this method was used in [Mon23] by the first author of this paper. This technique is referred to as the *grid method*.

To prove Theorem 1.1, we need the following two lemmas.

Lemma 2.1. *Let $\tilde{A} = (a_n)_{n\in[N]}$ be a finite sequence of integers which satisfies $\frac{a_{n+1}}{a_n} > 2Q$ for some $Q \le N$. Suppose that $\tilde{A} = \biguplus_{q\in[Q]} A_q$ be partition of $\tilde{A}$. Then there exists an irrational number $r \in (0,1)$ such that for all $q \in [Q]$ we have $ra_n \in I_{Q-q}$ (mod 1) whenever $a_n \in A_q$, where $I_q = \left(\frac{q-1}{Q}, \frac{q}{Q}\right)$.*

Proof. This lemma in a bit different form appeares elsewhere [JW94, Lemma 2.13], hence its proof is skipped here. □

Lemma 2.2. *Let $A = (a_n)$ be a sequence of integers which satisfies the following property: For every $C > 0$, $\epsilon > 0$ and $N_1 \in \mathbb{N}$, there exists a dynamical system (X, Σ, μ, T), a set $E \in \Sigma$ with $\mu(E) < \epsilon$, and an integer $N_2 > N_1$ such that*

$$\mu\Big\{x \in X : \max_{N_2 \le N \le N_2} \mathbb{A}_{[N]} \mathbb{1}_E(T^{a_n}x) > 1-\epsilon\Big\} \ge C\mu(E). \tag{4}$$

Then the sequence $A = (a_n)$ has the strong sweeping out property.

Proof. A version of this lemma appears elsewhere [Mon23, Theorem 3.1] already with detailed proof, so we will just outline the proof here.

First, by using Calderon's transference principle [Cal68], one can prove that if we have a maximal inequality on $\mathbb{R}$ (with respect to the transformation $\tau(n) = n+1$), then the maximal inequality transfers to any dynamical system with the same constant. Hence the hypothesis of Theorem 2.2 implies that there is a denial of a maximal inequality on $\mathbb{R}$. Now, we invoke [ABJLRW, Theorem 2.3] to finish the proof of this theorem. □

Proof of Theorem 1.1. Let $C > 0$, $N_1 \in \mathbb{N}$, and $\epsilon > 0$. By Lemma 2.2, it will be sufficient to find a set E in $\mathbb{T}^K$ with $\lambda^{(K)}(E) < \epsilon$, and an integer $N_2 > N_1$ which satisfy Eq. (4). Here $\mathbb{T}^K$ denotes the K-dimensional torus and $\lambda^{(K)}$ means the Haar-Lebesgue measure on $\mathbb{T}^K$.

Let $N > N_1$ be a very large positive integer. As

$$a_n \neq a_m \text{ for } n \neq m,$$

(a_n) can also be considered as a set. Since for $x \in (0,1)$, we have $e^x = 1 + x + O(x^2)$, we can rewrite the given condition as

$$\frac{a_{n+1}}{a_n} > e^{\frac{1}{(\log\log N)^{1-\eta}}} \text{ for } n \in [N]. \tag{5}$$

Let us choose a natural number $Q = Q(N)$ which just needs to go to ∞ as $N \to \infty$. Choose another integer $K = K(N)$ large enough so that

$$e^{\frac{K}{(\log\log N)^{1-\eta}}} > 2Q. \tag{6}$$

This implies

$$\frac{a_{n+K}}{a_n} > 2Q \text{ for } n \in [N]. \tag{7}$$

For any $k \in [K]$, define

$$A_k := \{a_n : n \equiv k \ (\text{mod } K) \text{ and } n \in [N]\}. \tag{8}$$

Observe that for each $k \in [K]$, A_k satisfies the hypothesis of Lemma 2.1. That means each A_k has the property that if it is partitioned into Q sets, e.g. $A_k = \dot{\bigcup}_{q\in[Q]} A_{k,q}$, then there is an irrational number r_k so that

$$r_k a_n \in I_{Q-q} \pmod 1 \text{ for all } a_n \in A_{k,q},\ q \in [Q], \tag{9}$$

where

$$I_q := \big(\frac{q-1}{Q}, \frac{q}{Q}\big). \tag{10}$$

The above partition $A_1, A_2, ..., A_K$ of $(a_n)_{n\in[N]}$ naturally induces a partition of the index set $[N]$ into K index sets $\mathcal{N}_k, k \in [K]$. For every $\mathsf{x} = (x_1, x_2, \dots, x_K) \in \mathbb{T}^K$, we then have

$$\mathbb{A}_J f(T^{a_n}\mathsf{x}) = \frac{1}{\#J} \sum_{k\in[K]} \sum_{n\in J\cap\mathcal{N}_k} f(T^{a_n}\mathsf{x}). \tag{11}$$

The space of action is the K dimensional torus $\mathbb{T}^K$, subdivided into little K dimensional cubes C of the form

$$C = I_{q(1)} \times I_{q(2)} \times ...... \times I_{q(k)} \text{ for some } q(k) \leq Q \text{ for } q \leq Q. \tag{12}$$

At this point it is useful to introduce the following vectorial notation to describe these cubes C. For a vector $\mathbf{q} = \big(q(1), q(2), ..., q(K)\big)$ with $q(k) \in [Q]$, define

$$I_{\mathbf{q}} := I_{q(1)} \times I_{q(2)} \times \cdots \times I_{q(K)}. \tag{13}$$

Since each component $q(k)$ can take up the values $1, 2, \dots, Q$, we divided $\mathbb{T}^K$ into Q^K cubes. We also consider the "bad" set $E \subset \mathbb{T}^K$ defined by

$$E := \cup_{k\leq K} (0,1) \times (0,1) \times \cdots \times \underbrace{(I_1 \cup I_2)}_{\text{k-th coordinate}} \times \cdots \times (0,1). \tag{14}$$

Defining the set $E_k \subset \mathbb{T}^K$ by

$$E_k := (0,1) \times (0,1) \times \cdots \times \underbrace{(I_1 \cup I_2)}_{\text{k-th coordinate}} \times \cdots \times (0,1) \tag{15}$$

we have

$$E = \cup_{k\leq K} E_k \text{ and } \lambda^{(K)}(E_k) \leq \frac{2}{Q} \text{ for every } k \leq K. \tag{16}$$

By Eq. (16), we have

$$\lambda^{(K)}(E) \leq \frac{2K}{Q}. \tag{17}$$

Since we want the measure of the "bad set" to be smaller and smaller, we must assume

$$K << Q. \tag{18}$$

Now the idea is to have averages that move each of the cubes into the support of the set E. The 2-dimensional version of the process is illustrated in Fig. 1.

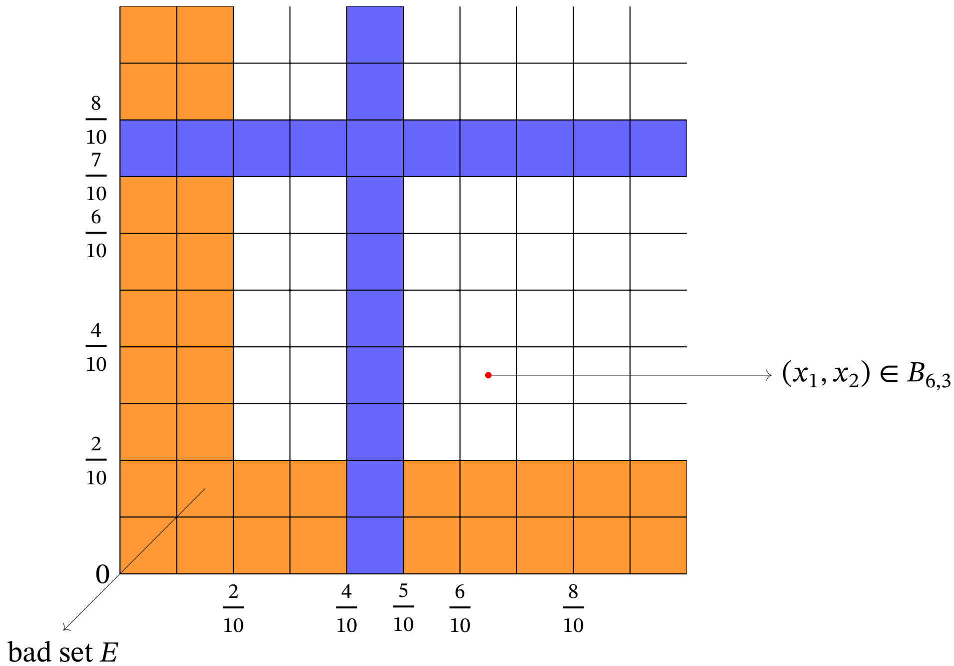

FIGURE 1. Illustration of the 2-dimensional case. Here the "bad set" E is the orange colored region. Let (x_1, x_2) be an arbitrary point (which belongs to $B_{6,3}$ in this case). We need to look at an average where $r_1 a_n \in (\frac{4}{10}, \frac{5}{10})$ for all $n \in A_1$ and $r_2 a_n \in (\frac{7}{10}, \frac{8}{10})$ for all $n \in A_2$. Then it would give us $(x_1, x_2) + (r_1 a_n, r_2 a_n) \in E$ for all $n \in A_1 \cup A_2$.

Since we have Q^K cubes, we need to have Q^K averages $\mathbb{A}_{J_i}$. This means we need to have Q^K disjoint intervals J_i of indices. The length of these intervals J_i needs to be "significant", in comparison with J_{i-1}. For our purpose, $J_i = (2^{N_1+i}, 2^{N_1+i+1})$ will be suitable. This means we need to have Q^K exponents available, which implies that

$$N \geq 2^{Q^K}. \tag{19}$$

For simplicity, we assume that

$$N = 2^{Q^K} \tag{20}$$

To make our plan work, first let us check that we can really choose such $K(N)$ which satisfies the condition (18), (20) and (6). Let us write Eq. (6) as

$$\frac{K}{(\log \log N)^{1-\eta}} > \log 2Q. \tag{21}$$

Using the assumption $N = 2^{Q^K}$, the condition in Eq. (21) becomes

$$\frac{K}{(K\log Q)^{1-\eta}} > \log 2Q. \tag{22}$$

After rearranging and ignoring the difference between $\log Q$ and $\log 2Q$, we get

$$K^{\eta} > \big(\log Q\big)^{2-\eta}, \tag{23}$$

which we can simplify a bit generously to

$$K > (\log Q)^{\frac{2}{\eta}}. \tag{24}$$

We certainly can choose K to satisfy Eq. (24) and also make sure that $\frac{K(N)}{Q(N)}$ goes to 0 as $N \to \infty$. Choose N_2 large enough so that $K(N_2)$ and $Q(N_2)$ satisfy Eq. (24) and the following:

$$\frac{2K(N_2)}{Q(N_2)} < \min\{\epsilon, \frac{1}{C}\}. \tag{25}$$

So we have Q^K cubes $C_i, i \in [Q^K]$ and Q^K intervals $J_i, i \in [Q^K]$. We match C_i with J_i. We know that C_i is of the form

$$C_i = I_{\mathrm{q}_i}, \tag{26}$$

for some K dimensional vector $\mathrm{q}_i = \big(q_i(1), q_i(2), \dots q_i(K)\big)$ with $q_i(k) \in [Q]$ for every $k \in [K]$. The interval J_i is partitioned as

$$J_i = \bigcup_{k\in[K]} (J_i \cap \mathcal{N}_k). \tag{27}$$

For a given $k \in [K]$, let us define the set of indices $\mathcal{N}_{k,q}$ for $q \le Q$, by

$$\mathcal{N}_{k,q} := \bigcup_{i\le Q^K, q_i(k)=q} (J_i \cap \mathcal{N}_k) \tag{28}$$

Since the sets $A_{k,q} := \{a_n : n \in \mathcal{N}_{k,q}\}$ form a partition of A_k, by the argument above Eq. (9), there is an irrational number r_k so that

$$r_k a_n \in I_{Q-q} \pmod 1 \text{ for } n \in \mathcal{N}_{k,q} \text{ and } q \le Q. \tag{29}$$

Define the transformation T on the K dimensional torus $\mathbb{T}^K$ by

$$T(x_1, x_2, \dots, x_K) := (x_1 + r_1, x_2 + r_2, \dots, x_K + r_K). \tag{30}$$

We claim that

$$\Big\{\mathrm{x} | \max_{i\in[Q^K]} \mathbb{A}_{J_i} \mathbb{1}_E(T^{a_n}\mathrm{x}) = 1\Big\} = \mathbb{T}^K. \tag{31}$$

Indeed, let $\mathrm{x} \in C_i$ and consider the average $\mathbb{A}_{J_i}$. Let us write

$$\mathbb{A}_{n\in J_i} \mathbb{1}_E(T^{a_n}\mathrm{x}) = \frac{1}{\#J_i} \sum_{k\le K} \sum_{J_i\cap\mathcal{N}_k} \mathbb{1}_E(T^{a_n}\mathrm{x}) \tag{32}$$

$$= \frac{1}{\#J_i} \sum_{k\le K} \sum_{n\in J_i\cap\mathcal{N}_k} \mathbb{1}_E(x_1 + r_1 a_n, x_2 + r_2 a_n, \dots, x_K + r_K a_n). \tag{33}$$

We claim that for each $k \in [K]$

$$(x_1 + r_1 a_n, x_2 + r_2 a_n, \dots x_k + r_k a_n, \dots, x_K + r_K a_n) \in E_k \text{ if } n \in J_i \cap \mathcal{N}_k \tag{34}$$

Since, $E_k \subset E$, we would have

$$\mathbb{1}_E(x_1 + r_1 a_n, x_2 + r_2 a_n, \dots x_k + r_k a_n, \dots, x_K + r_K a_n) = 1 \text{ if } n \in J_i \cap \mathcal{N}_k \tag{35}$$

which would imply that

$$\mathbb{1}_E(T^{a_n}\mathrm{x}) = 1 \text{ for all } n \in J_i. \tag{36}$$

So let us prove Eq. (34). Since $\mathrm{x} \in C_i = I_{\mathrm{q}}$, we have $x_k \in I_{q_i(k)}$ for every k. By the definition of r_k in Eq. (29) we have $r_k a_n \in I_{Q-q_i(k)}$ if $n \in J_i \cap \mathcal{N}_k$. It follows that

$$x_k + r_k a_n \in I_{q_i(k)} + I_{Q-q_i(k)} \text{ if } n \in J_i \cap \mathcal{N}_k. \tag{37}$$

Since $I_{q_i(k)} + I_{Q-q_i(k)} \subset I_1 \cup I_2$, we get

$$x_k + r_k a_n \in I_1 \cup I_2 \text{ if } n \in J_i \cap \mathcal{N}_k.$$

By the definition of E_k in (15), this implies that

$$(x_1 + r_1 a_n, x_2 + r_2 a_n, \dots x_k + r_k a_n, \dots, x_K + r_K a_n) \in E_k$$

as claimed. □

Remark 2.3. *We make two remarks here:*

(1) *Sharpness: We can prove Theorem 1.1 by replacing the assumption (1) with*

$$\frac{a_{n+1}}{a_n} > 1 + \frac{(\log\log\log n)^{2+\eta}}{\log\log n}, \quad \eta > 0.$$

(2) *Theorem 1.1 can also be proved by using* [PS10, Theorem 3.1]. *By applying this result, one can slightly weaken the hypothesis. More precisely, we can prove that any sequence (a_n) satisfying*

$$\frac{a_{n+1}}{a_n} > 1 + \frac{(\log\log\log n)^{1+\eta}}{\log\log n}, \quad \eta > 0,$$

has the strong sweeping out property.

We can generalize Theorem 1.1 for weighted ergodic averages in the following way:

Theorem 2.4. *Let $(w(n))$ be a sequence of real numbers from the interval $(0, 1]$ and denote $G(n) := \sum_{i\in[n]} w(i)$. Suppose (a_n) is a sequence of integer which satisfies*

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log\log G(n))^{1-\eta}} \tag{38}$$

for some $\eta > 0$. Then (a_n) satisfies the strong sweeping out property for the w-weighted averages $\mathbb{A}^w_{[N]} f(T^{a_n}x)$.

Proof. The proof of Theorem 1.1 will work in this case after the following obvious modifications. Instead of $\mathbb{A}_J$, we have to work with the weighted averages $\mathbb{A}_J^w$. Accordingly, the length of J_i also has to be changed. A suitable choice of J_i in this case would be following:

$$J_i = \left(G^{-1}(2^{N_1+i}), G^{-1}(2^{N_1+i+1})\right) \tag{39}$$

Now, one can reiterate the argument given in Theorem 1.1 to get the desired conclusion. □

Corollary 2.5. *If (a_n) is a sequence of integers satisfying*

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log\log\log n)^{1-\eta}}$$

for some $\eta > 0$, then (a_n) has the strong sweeping out property for the logarithmic averages

$$\mathbb{A}_{[N]}^{1/n} f(T^{a_n}x) = \frac{1}{\log N}\sum_{[N]}\frac{1}{n}f(T^{a_n}x).$$

2.3. Proof of Theorem 1.4. Now, we will prove Theorem 1.4. For any sequence A of integers, we define $A(t) := \{n \in A : n \leq t\}$. Observe that Theorem 1.4 will follow from Theorem 1.1 and the following lemma.

Lemma 2.6. *Under the hypothesis of Theorem 1.4, for a.e. ω, there exists a subsequence $B^\omega = (b_n(\omega))$ of $A^\omega = (a_n(\omega))$ such that the following holds:*

$$\lim_{t\to\infty}\frac{B^\omega(t)}{A^\omega(t)} = 1 \tag{40}$$

$$\frac{b_{n+1}(\omega)}{b_n(\omega)} > e^{\frac{1}{(\log\log n)^{1-\eta/2}}} \tag{41}$$

Proof. Let $u_n = \min\{t | \sum_{k\leq t}\sigma_k \geq n\}$. First observe that $u_n \sim e^{n(\log\log n)^{-1+\eta}}$. By the strong law of large numbers, we have for a.e. ω that

$$\lim_{n\to\infty}\frac{A^\omega(u_n)}{\sum_{u\leq u_n}\sigma_u} = 1,$$

which implies that $\lim_{n\to\infty}\frac{A^\omega(u_n)}{n} = 1$.

Clearly, $\frac{u_{n+1}}{u_n} \sim e^{\frac{1}{(\log\log n)^{1-\eta}}}$. However, this does not imply that

$$\frac{a_{n+1}(\omega)}{a_n(\omega)} > e^{\frac{1}{(\log\log n)^{1-\eta}}}.$$

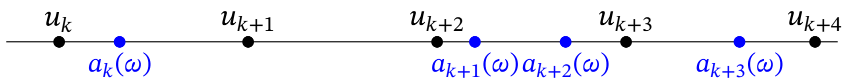

FIGURE 2. An example where A^ω may not satisfy the condition (41)

So, we need to modify our sequence. Let $v_n = e^{n(\log\log n)^{-1+\frac{\eta}{2}}}$ and $I_n = [v_n, v_{n+1}) \cap \mathbb{N}$. The properties of (v_n) that we shall use here are the following:

$$\frac{v_{n+1}}{v_n} \geq e^{\frac{1}{(\log\log n)^{1-\frac{\eta}{2}}}} \text{ and } \lim_{n\to\infty} \sum_{u\in I_n} \sigma_u = 0. \tag{42}$$

Let

$$D^\omega = (d_n(\omega)) := \{d : d \in I_n \cap A^\omega \text{ for some } n \in \mathbb{N} \text{ satisfying } I_{n+1} \cap A^\omega \neq \emptyset\}$$

and

$$E^\omega = (e_n(\omega)) := \bigcup_{n\in\mathbb{N}} \{I_n \cap A^\omega : |I_n \cap A^\omega| > 1\}.$$

In Fig. 3, $a_{k+1}(\omega) \in D^\omega$ because $I_{k+1} \cap A^\omega \neq \emptyset$. And $a_{k+3}(\omega), a_{k+4}(\omega) \in E^\omega$.

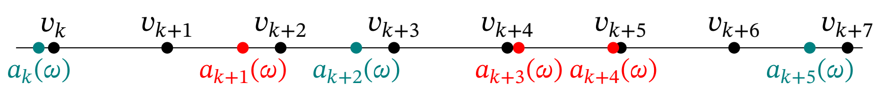

FIGURE 3. Construction of D^ω and E^ω

Define $B^\omega := A^\omega \setminus (D^\omega \cup E^\omega)$. Note that B^ω satisfies the following properties

(1) If $|B^\omega \cap I_n| = 1$ then $|B^\omega \cap I_{n+1}| = 0$.
(2) $|B^\omega \cap I_n| \leq 1$ for all n.

from which it follows that $(b_n(\omega))$ satisfies Eq. (41).

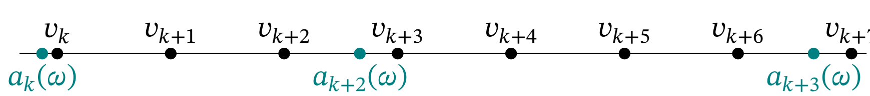

FIGURE 4. Construction of B^ω

It remains to verify Eq. (40).
It will be sufficient to show that for a.e. ω,

$$\lim_{t\to\infty} \frac{D^{\omega}(t)}{A^{\omega}(t)} = 0 \text{ and } \lim_{t\to\infty} \frac{E^{\omega}(t)}{A^{\omega}(t)} = 0.$$

By the strong law of large numbers, we need to show that

$$\lim_{t\to\infty} \frac{\mathbb{E}D^{\omega}(t)}{\sum_{n\leq t}\sigma_n} = 0 \text{ and } \lim_{t\to\infty} \frac{\mathbb{E}E^{\omega}(t)}{\sum_{n\leq t}\sigma_n} = 0.$$

Note that $D^{\omega}(t) \leq \sum\limits_{v_k\leq t} \big(\sum\limits_{n\in I_k} Y_n(\omega)\big). \sup\limits_{n\in I_{k+1}} Y_n$. Since the random variables $X_k = \sum_{n\in I_k} Y_n, k = 1, 2, 3, ...,$ are independent, it follows that

$$\begin{aligned}
\mathbb{E}D^{\omega}(t) &\leq \mathbb{E}\sum_{v_k\leq t}\Big(\sum_{n\in I_k} Y_n\Big). \sup_{n\in I_{k+1}} Y_n \\
&\leq \mathbb{E}\sum_{v_k\leq t}\Big(\sum_{n\in I_k} Y_n\Big).\Big(\sum_{n\in I_{k+1}} Y_n\Big) \\
&\leq \sum_{v_k\leq t}\big(\sum_{n\in I_k}\sigma_n\big).\big(\sum_{n\in I_{k+1}}\sigma_n\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\limsup_{t\to\infty}\frac{\mathbb{E}D^{\omega}(t)}{\sum_{n\leq t}\sigma_n} &\leq \limsup_{t\to\infty}\frac{\sum_{v_k\leq t}\big(\sum_{n\in I_k}\sigma_n\big).\big(\sum_{n\in I_{k+1}}\sigma_n\big)}{\sum_{n\leq t}\sigma_n} \\
&\leq \limsup_{k\to\infty}\Big(\sum_{n\in I_{k+1}}\sigma_n\Big)\frac{\sum_{v_k\leq t}\big(\sum_{n\in I_k}\sigma_n\big)}{\sum_{n\leq t}\sigma_n} \\
&= 0 \,\big(\text{ By Eq. (42)}\big).
\end{aligned}$$

Similarly, letting $|I_k| = l$ we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{\mathbb{E}E^{\omega}(t)}{\sum_{n\leq t}\sigma_n} \\
&= \limsup_{t\to\infty}\frac{\sum\limits_{v_k\leq t}\Big(2\sum\limits_{m\neq n\in I_k}\sigma_m\sigma_n + 3\sum\limits_{m\neq n\neq p\in I_k}\sigma_m\sigma_n\sigma_p + ... + l\sum\limits_{m_1\neq m_2\neq\cdots\neq m_l\in I_k}\sigma_{m_1}\sigma_{m_2}...\sigma_{m_l}\Big)}{\sum_{n\leq t}\sigma_n} \\
&\leq \limsup_{t\to\infty}\frac{\sum_{v_k\leq t}\Big(2\big(\sum_{m\in I_k}\sigma_m\big)^2 + 3\big(\sum_{m\in I_k}\sigma_m\big)^3 + \cdots + l\big(\sum_{m\in I_k}\sigma_m\big)^l\Big)}{\sum_{n\leq t}\sigma_n} \\
&\leq \limsup_{t\to\infty}\frac{\sum_{v_k\leq t}\big(\sum_{m\in I_k}\sigma_m\big)\cdot\Big(2\big(\sum_{m\in I_k}\sigma_m\big) + 3\big(\sum_{m\in I_k}\sigma_m\big)^2 + \cdots + l\big(\sum_{m\in I_k}\sigma_m\big)^{(l-1)}\Big)}{\sum_{n\leq t}\sigma_n} \\
&\leq \limsup_{k\to\infty}\Big(2\big(\sum_{m\in I_k}\sigma_m\big) + 3\big(\sum_{m\in I_k}\sigma_m\big)^2 + ...\, l\big(\sum_{m\in I_k}\sigma_m\big)^{(l-1)}\Big)
\end{aligned}$$

$$\leq \limsup_{k\to\infty} x \cdot \frac{(2-x)}{(1-x)^2} \Big(\text{where } x = \sum_{m\in I_k} \sigma_m \Big)$$

$$= 0 \big(\text{By Eq. (42)}\big).$$

This completes the proof. □

3. Open problems

The first problem asks if our result in Theorem 1.1 is sharp.

Problem 3.1. *Suppose the sequence (a_n) of positive integers satisfies*

$$\frac{a_{n+1}}{a_n} > 1 + \frac{1}{\log\log n} \tag{43}$$

Is (a_n) strong sweeping out?

It is known from [JLW99] that there is a pointwise good sequence (a_n) for L^2 satisfying $\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log n)^{1+\eta}}$ for every $\eta > 0$ and large enough n. We already mentioned [JW94, Corollary 2.14] that if (a_n) satisfies $\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{(\log n)^{1/2-\eta}}$ for some positive η then (a_n) is pointwise bad for L^2.

Problem 3.2. *Suppose the sequence (a_n) of positive integers satisfies*

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{\log n} \tag{44}$$

for every large enough n. Is (a_n) then pointwise bad for L^2?

References

[ABJLRW] AKCOGLU, MUSTAFA; BELLOW, ALEXANDRA; JONES, ROGER L.; LOSERT, VIKTOR, REINHOLD-LARSSON, KARIN; WIERDL, MÁTÉ. The strong sweeping out property for lacunary sequences, Riemann sums, convolution powers, and related matters. *Ergodic Theory Dynam. Systems* **16** (1996), no. 2, 207–253. MR1389623, Zbl 0851.47004, doi: 10.1017/S0143385700008798. 1062, 1064

[Bel83] BELLOW, ALEXANDRA. On "bad universal" sequences in ergodic theory. II. *Measure theory and its applications* (Sherbrooke, Que., 1982), 74–78. Lecture Notes in Math., 1033, *Springer-Verlag, Berlin*, 1983. MR0729527, Zbl 0516.00014, doi: 10.1007/BFb0099847. 1061

[Bel89] BELLOW, ALEXANDRA. Perturbation of a sequence. *Adv. Math.* **78** (1989), no. 2, 131–139. MR1029097, Zbl 0687.28010, doi: 10.1016/0001-8708(89)90030-3. 1062

[Ber97] BERKES, ISTVÁN. On the convergence of $\sum c_n f(nx)$ and the Lip 1/2 class. *Trans. Amer. Math. Soc.* **349** (1997), no. 10, 4143–4158 MR1401764, Zbl 0884.42005, doi: 10.1090/S0002-9947-97-01837-0. 1063

[BKQW05] BOSHERNITZAN, MICHAEL; KOLESNIK, GRIGORI; QUAS, ANTHONY; WIERDL, MÁTÉ. Ergodic averaging sequences. *J. Anal. Math.*. **95** (2005), 63–103. MR2145587, Zbl 1100.28010, doi: 10.1007/BF02791497. 1062

[Bou88] BOURGAIN, JEAN. On the maximal ergodic theorem for certain subsets of the integers. *Israel J. Math.* **61** (1988), no. 1, 39–72. MR0937581, Zbl 0642.28010, doi: 10.1007/BF02776301. 1062

[Bou89] BOURGAIN, JEAN. Pointwise ergodic theorems for arithmetic sets. *Inst. Hautes Études Sci. Publ. Math.* (1989), no. 69, 5–45. MR1019960, Zbl 0705.28008, doi: 10.1007/BF02698838. 1062

[Buc07] BUCZOLICH, ZOLTÁN. Universally L^1 good sequences with gaps tending to infinity. *Acta Math. Hungar.* **117** (2007), no. 1-2, 91–140. MR2356189, Zbl 1164.37001, arXiv:math/0601329, doi: 10.1007/s10474-007-6068-8. 1062

[BM10] BUCZOLICH, ZOLTÁN; MAULDIN, R. DANIEL. Divergent square averages. *Ann. Of Math. (2)* **171** (2010), no.3, 1479–1530. MR2680392, Zbl 1200.37003, arXiv:math/0504067, doi: 10.4007/annals.2010.171.1479. 1062

[Cal68] CALDERÓN, ALBERTO P. Ergodic theory and translation-invariant operators. *Proc. Nat. Acad. Sci. U.S.A.*. **59** (1968), 349–353. MR0227354, Zbl 0185.21806, doi: 10.1073/pnas.59.2.349. 1064

[Jon04] JONES, ROGER L. Strong sweeping out for lacunary sequences. *Chapel Hill Ergodic Theory Workshops*, 137-144. Contemp. Math., 356. *American Mathematical Society, Providence, RI*, 2004. ISBN:0-8218-3313-8. MR2087593, Zbl 1060.37001, doi: 10.1090/conm/356/06501. 1063

[JLW99] JONES, ROGER L.; LACEY, MICHAEL; WIERDL, MÁTÉ. Integer sequences with big gaps and the pointwise ergodic theorem. *Ergodic Theory Dynam. Systems* **19** (1999), no. 5, 1295–1308. MR1721622, Zbl 0941.28013, doi: 10.1017/S0143385799146819. 1063, 1072

[JW94] JONES, ROGER L.; WIERDL, MÁTÉ. Convergence and divergence of ergodic averages. *Ergodic Theory Dynam. Systems* **14** (1994), no. 3, 515–535. MR1293406, Zbl 0828.28008, doi: 10.1017/S0143385700008002. 1062, 1063, 1064, 1072

[Kre71] KRENGEL, ULRICH. On the individual ergodic theorem for subsequences. *Ann. Math. Statist.* **42** (1971) 1091–1095 . MR0283172, Zbl 0216.09603, doi: 10.1214/aoms/1177693338. 1061

[LaV09] LAVICTOIRE, PATRICK. An L^1 ergodic theorem for sparse random subsequences. *Math. Res. Lett.* **16** (2009), no. 5, 849–859. MR2576702, Zbl 1190.28008, arXiv:0812.3175, doi: 10.4310/MRL.2009.v16.n5.a8. 1062

[Lav11] LAVICTOIRE, PATRICK. Universally L^1-bad arithmetic sequences. *J. Anal. Math.* **113** (2011), 241–263. MR2788358, Zbl 1220.37005, arXiv:0905.3865, doi: 10.1007/s11854-011-0006-y. 1062

[Loy22] LOYD, KAITLYN. A dynamical approach to the asymptotic behavior of the sequence *Omega*(n). *Ergodic Theory And Dynamical Systems* (2022), 1-22. doi: 10.1017/etds.2022.81. 1062

[Mir15] MIREK, MARIUSZ. Weak type $(1,1)$ inequalities for discrete rough maximal functions. *J. Anal. Math.* **127** (2015), 247–281. MR3421994, Zbl 1327.42022, arXiv:1305.0575, doi: 10.1007/s11854-015-0030-4. 1062

[Mon23] MONDAL, SOVALAL. Grid method for divergence of averages. *Ergodic Theory And Dynamical Systems* (2023), 1–19. doi: 10.1017/etds.2023.29. 1063, 1064

[And11] PARRISH, ANDREW. Pointwise convergence of ergodic averages in Orlicz spaces. *Illinois J. Math.* **55** (2011), no. 1, 89–106. MR3006681, Zbl 1276.37005, arXiv:0902.4645, 1062

[PS10] PERES, YUVAL; SCHLAG, WILHELM. Two Erdős problems on lacunary sequences: chromatic number and Diophantine approximation. *Bull. Lond. Math. Soc.* **42** (2010), no. 2, 295–300. MR2601556, Zbl 1215.05074, arXiv:0706.0223, doi: 10.1112/blms/bdp126. 1068

[Rei94] REINHOLD–LARSSON, KARIN. Discrepancy of behavior of perturbed sequences in L^p spaces. *Proc. Amer. Math. Soc.* **120** (1994), no. 3, 865–874. MR1169889, Zbl 0867.28011, doi: 10.2307/2160481. 1062

[Ros91] ROSENBLATT, JOSEPH. Universally bad sequences in ergodic theory. *Almost everywhere convergence, II* (Evanston, IL, 1989) 227–245. *Academic Press, Inc., Boston, MA,* 1991. MR1131794, Zbl 0745.28011.

[RW95] ROSENBLATT, JOSEPH; WIERDL, MÁTÉ. Pointwise ergodic theorems via harmonic analysis. *Ergodic theory and its connections with harmonic analysis* (Alexandria, 1993), 3–151. London Math. Soc. Lecture Note Ser., 205. *Cambridge University Press, Cambridge,* 1995. ISBN:0-521-45999-0. MR1325697, Zbl 0848.28008, doi: 10.1017/CBO9780511574818.002. 1062

[Tro21] TROJAN, BARTOSZ. Weak type $(1, 1)$ estimates for maximal functions along 1-regular sequences of integers. *Studia Math.* **261**, (2021), no. 1, 103–108. MR4309852, Zbl 1484.37009, arXiv:2012.10416, doi: 10.4064/sm200702-21-12. 1062

[UZ07] URBAN, ROMAN; ZIENKIEWICZ, JACEK. Weak type $(1, 1)$ estimates for a class of discrete rough maximal functions. *Math. Res. Lett.* **14**, (2007) no. 2, 227–237. MR2318621, Zbl 1144.42006, doi: 10.4310/MRL.2007.v14.n2.a6. 1062

[Wie88] WIERDL, MÁTÉ. Pointwise ergodic theorem along the prime numbers. *Israel J. Math.*. **64** (1988), no. 3, 315-336. MR0995574, Zbl 0695.28007, doi: 10.1007/BF02882425. 1062

(Sovanlal Mondal) THE UNIVERSITY OF MEMPHIS, DEPARTMENT OF MATHEMATICAL SCIENCES, 373 DUNN HALL, MEMPHIS, TN 38152, USA
smondal@memphis.edu

(Madhumita Roy) THE UNIVERSITY OF MEMPHIS, DEPARTMENT OF MATHEMATICAL SCIENCES, 373 DUNN HALL, MEMPHIS, TN 38152, USA
mroy@memphis.edu

(Máté Wierdl) THE UNIVERSITY OF MEMPHIS, DEPARTMENT OF MATHEMATICAL SCIENCES, 373 DUNN HALL, MEMPHIS, TN 38152, USA
mwierdl@memphis.edu

This paper is available via http://nyjm.albany.edu/j/2023/29-42.html.