New York Journal of Mathematics
New York J. Math. **29** (2023) 874–910.

On a twisted Jacquet module of $\mathrm{GL}(6, F)$ over a finite field

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Abstract. Let F be a finite field and $G = \mathrm{GL}(6, F)$. In this paper, we give an explicit description of a certain twisted Jacquet module of an irreducible cuspidal representation of G.

Contents

1. Introduction

Let F be a finite field and $G = \mathrm{GL}(n, F)$. Let P be a parabolic subgroup of G with Levi decomposition $P = MN$. Let π be any irreducible finite dimensional complex representation of G and ψ be an irreducible representation of N. Let $\pi_{N,\psi}$ be the sum of all irreducible representations of N inside π, on which π acts via ψ. It is easy to see that $\pi_{N,\psi}$ is a representation of the subgroup M_ψ of M, consisting of those elements in M which leave the isomorphism class of ψ invariant under the inner conjugation action of M on N. The space $\pi_{N,\psi}$ is called the *twisted Jacquet module* of the representation π. It is an interesting question to understand for which irreducible representations π, the twisted Jacquet module $\pi_{N,\psi}$ is non-zero and to understand its structure as a module for M_ψ.

In an earlier work of ours [1], inspired by the work of Prasad in [4], we studied the structure of a certain twisted Jacquet module of a cuspidal representation of $\mathrm{GL}(4, F)$. Recently, we have realised that the structure of the module for $\mathrm{GL}(2n, F)$ can also be studied using the more sophisticated theory of

Received January 21, 2023.
2020 *Mathematics Subject Classification.* 20G40 (Primary); 20G05 (Secondary).
Key words and phrases. Cuspidal representations, Twisted Jacquet module.
Research of the corresponding author Kumar Balasubramanian is supported by the SERB grant: MTR/2019/000358.

ISSN 1076-9803/2023

derivatives of the p-adic group GL(n) studied in the classical paper of Bernstein-Zelevinskiĭ (see [2]). In this paper, we continue our study of the twisted Jacquet module for a cuspidal representation of GL(6, F) and use elementary methods from linear algebra to calculate its structure. The calculations in the GL(6, F) case are much more involved than in the case of GL(4, F) and we hope that some of these ideas could be used to give an alternative proof in the GL($2n$, F) case as well, using simple techniques from linear algebra. We refer the reader to Section 1 in [1] for a more elaborate introduction to the problem.

Before we state our result, we set up some notation. Let $G = \mathrm{GL}(6, F)$ and P be the maximal parabolic subgroup of G with Levi decomposition $P = MN$, where $M \simeq \mathrm{GL}(3, F) \times \mathrm{GL}(3, F)$ and $N \simeq \mathrm{M}(3, F)$. We write F_6 for the unique field extension of F of degree 6. Let ψ_0 be a fixed non-trivial additive character of F. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $\psi_A : N \to \mathbb{C}^\times$ be the character of N given by

$$\psi_A\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) = \psi_0(\mathrm{Tr}(AX)). \tag{1.1}$$

Let $H_A = M_1 \times M_2$ where M_1 is the Mirabolic subgroup of GL(3, F) and $M_2 = w_0 M_1^\top w_0^{-1}$ where $w_0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Let U be the subgroup of unipotent matrices in GL(6, F) and $U_A = U \cap H_A$. Clearly, we have $U_A \simeq U_1 \times U_2$ where U_1 and U_2 are the upper triangular unipotent subgroups of GL(3, F). For $k = 1, 2$, let $\mu_k : U_k \to \mathbb{C}^\times$ be the non-degenerate character of U_k given by

$$\mu_k\left(\begin{bmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{bmatrix}\right) = \psi_0(x_{12} + x_{23}).$$

Let $\mu : U_A \to \mathbb{C}^\times$ be the character of U_A given by

$$\mu(u) = \mu_1(u_1)\mu_2(u_2)$$

where $u = \begin{bmatrix} u_1 & 0 \\ 0 & u_2 \end{bmatrix}$.

Theorem 1.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A} \mu$$

as M_{ψ_A} modules.

We establish the above isomorphism by explicitly calculating the characters of π_{N,ψ_A} and $\theta|_{F^\times} \otimes \mathrm{ind}_{U_A}^{H_A}(\mu)$, and showing that they are equal at any arbitrary element of M_{ψ_A}.

2. Preliminaries

In this section, we mention some preliminary results that we need in our paper.

2.1. Character of a cuspidal representation. Let F be the finite field of order q and $G = \mathrm{GL}(m, F)$. Let F_m be the unique field extension of F of degree m. A character θ of $F_m^\times$ is called a "regular" character, if under the action of the Galois group of F_m over F, θ gives rise to m distinct characters of $F_m^\times$. It is a well known fact that the cuspidal representations of $\mathrm{GL}(m, F)$ are parametrized by the regular characters of $F_m^\times$. To avoid introducing more notation, we mention below only the relevant statements on computing the character values that we have used. We refer the reader to Section 6 in [3] for more precise statements on computing character values.

Theorem 2.1. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_θ be its character. If $g \in \mathrm{GL}(m, F)$ is such that the characteristic polynomial of g is not a power of a polynomial irreducible over F. Then, we have*

$$\Theta_\theta(g) = 0.$$

Theorem 2.2. *Let θ be a regular character of $F_m^\times$. Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of $\mathrm{GL}(m, F)$ associated to θ. Let Θ_θ be its character. Suppose that $g = s.u$ is the Jordan decomposition of an element g in $\mathrm{GL}(m, F)$. If $\Theta_\theta(g) \neq 0$, then the semisimple element s must come from $F_m^\times$. Suppose that s comes from $F_m^\times$. Let z be an eigenvalue of s in F_m and let t be the dimension of the kernel of $g - z$ over F_m. Then*

$$\Theta_\theta(g) = (-1)^{m-1}\Big[\sum_{\alpha=0}^{d-1} \theta(z^{q^\alpha})\Big](1 - q^d)(1 - (q^d)^2) \cdots (1 - (q^d)^{t-1}).$$

where q^d is the cardinality of the field generated by z over F, and the summation is over the distinct Galois conjugates of z.

See Theorem 2 in [4] for this version.

2.2. Twisted Jacquet module. In this section, we recall the character and the dimension formula of the twisted Jacquet module of a representation π.

Let $G = \mathrm{GL}(k, F)$ and $P = MN$ be a parabolic subgroup of G. Let ψ be a character of N. For $m \in M$, let ψ^m be the character of N defined by $\psi^m(n) = \psi(mnm^{-1})$. Let

$$V(N, \psi) = \mathrm{Span}_{\mathbb{C}}\{\pi(n)v - \psi(n)v \mid n \in N, v \in V\}$$

and

$$M_\psi = \{m \in M \mid \psi^m(n) = \psi(n), \forall n \in N\}.$$

Clearly, M_ψ is a subgroup of M and it is easy to see that $V(N,\psi)$ is an M_ψ-invariant subspace of V. Hence, we get a representation $(\pi_{N,\psi}, V/V(N,\psi))$ of M_ψ. We call $(\pi_{N,\psi}, V/V(N,\psi))$ the twisted Jacquet module of π with respect to ψ. We write $\Theta_{N,\psi}$ for the character of $\pi_{N,\psi}$.

Proposition 2.3. *Let (π, V) be a representation of* $\mathrm{GL}(k,F)$ *and Θ_π be the character of π. We have*

$$\Theta_{N,\psi}(m) = \frac{1}{|N|}\sum_{n\in N}\Theta_\pi(mn)\overline{\psi(n)}.$$

We refer the reader to Proposition 2.3 in [1] for a proof.

Remark 2.4. Taking $m = 1$, we get the dimension of $\pi_{N,\psi}$. To be precise, we have

$$\dim_{\mathbb{C}}(\pi_{N,\psi}) = \frac{1}{|N|}\sum_{n\in N}\Theta_\pi(n)\overline{\psi(n)}.$$

2.3. Character of the induced representation. In this section, we recall the character formula for the induced representation of a group G. For a proof, we refer the reader to Chapter 3, Theorem 12 in [5].

Proposition 2.5. *Let G be a finite group and H be a subgroup of G. Let (π, V) be a representation of H and χ_π be the character of π. Then for each $s \in G$, the character of* $\mathrm{ind}_H^G(\pi)$ *is given by*

$$\chi_{\mathrm{ind}_H^G(\pi)}(s) = \frac{1}{|H|}\sum_{\substack{t\in G\\ t^{-1}st\in H}}\chi_\pi(t^{-1}st).$$

3. Dimension of the twisted Jacquet module

Let $\pi = \pi_\theta$ be an irreducible cuspidal representation of G corresponding to the regular character θ of $F_6^\times$ and Θ_θ be its character. Throughout, we write $\mathrm{M}(n,m,r,q)$ for the set of $n\times m$ matrices of rank r over the finite field $F = F_q$. In this section, we calculate the dimension of π_{N,ψ_A}. Before we continue, we record some preliminary lemmas that we need.

Lemma 3.1. *Let $r \in \{0,1,2,3\}$ and $X \in \mathrm{M}(3,3,r,q)$. We have*

$$\Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right) = \begin{cases} (q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1), & \text{if } r=0\\ -(q-1)(q^2-1)(q^3-1)(q^4-1), & \text{if } r=1\\ (q-1)(q^2-1)(q^3-1), & \text{if } r=2\\ -(q-1)(q^2-1), & \text{if } r=3\end{cases}$$

Proof. The result follows from Theorem 2.2 above. □

Let

$$X = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, AX = \begin{bmatrix} a & d & g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\alpha \in F$ and $r \in \{0, 1, 2, 3\}$, consider the subset $Y^{\alpha}_{3,r}$ of $\mathrm{M}(3, F)$ given by

$$Y^{\alpha}_{3,r} = \{X \in \mathrm{M}(3, F) \mid \mathrm{Rank}(X) = r, \mathrm{Tr}(AX) = \alpha\}.$$

Lemma 3.2. *Let $r \in \{1, 2, 3\}$ and $\alpha, \beta \in F^{\times}$. Then we have*

$$\#Y^{\alpha}_{3,r} = \#Y^{\beta}_{3,r}.$$

Proof. Consider the map $\phi : Y^{\alpha}_{3,r} \to Y^{\beta}_{3,r}$ given by

$$\phi(X) = \alpha^{-1}\beta X.$$

Suppose that $\phi(X) = \phi(Y)$. Since $\alpha^{-1}\beta \neq 0$, it follows that ϕ is injective. For $Y \in Y^{\beta}_{3,r}$, let $X = \alpha\beta^{-1}Y$. Clearly, we have $\mathrm{Tr}(AX) = \alpha$ and $\mathrm{Rank}(X) = \mathrm{Rank}(Y) = r$. Thus, ϕ is surjective and hence the result. □

Theorem 3.3. *Let θ be a regular character of $F_6^{\times}$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of* $\mathrm{GL}(6, F)$. *We have*

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = (q-1)^2(q^2-1)^2.$$

Proof. It is easy to see that the dimension of π_{N,ψ_A} is given by

$$\dim_{\mathbb{C}}(\pi_{N,\psi_A}) = \frac{1}{q^9} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))}. \tag{3.1}$$

We calculate the following sums

$$(a) \qquad S_1 = \sum_{X \in \mathrm{M}(3,3,0,q)} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))}$$

$$(b) \qquad S_2 = \sum_{\substack{X \in \mathrm{M}(3,3,1,q) \\ \mathrm{Tr}(AX)=0}} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X \in \mathrm{M}(3,3,1,q) \\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))}$$

$$(c) \qquad S_3 = \sum_{\substack{X \in \mathrm{M}(3,3,2,q) \\ \mathrm{Tr}(AX)=0}} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X \in \mathrm{M}(3,3,2,q) \\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta\left(\begin{bmatrix} 1 & X \\ 0 & 1 \end{bmatrix}\right) \overline{\psi_0(\mathrm{Tr}(AX))}$$

$$(d) \qquad S_4 = \sum_{\substack{X\in \mathrm{M}(3,3,3,q)\\ \mathrm{Tr}(AX)=0}} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))} + \sum_{\substack{X\in \mathrm{M}(3,3,3,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(\mathrm{Tr}(AX))}$$

separately to compute the dimension of π_{N,ψ_A}.

For a fixed $r \in \{0,1,2,3\}$ and $\alpha \in \{0,1\}$, we find a partition of $Y^{\alpha}_{3,r}$ into certain subsets, and compute the cardinality of each of these subsets to find the cardinality of $Y^{\alpha}_{3,r}$. We record the necessary information in the tables below.

For (a), we clearly have

$$S_1 = \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\overline{\psi_0(0)} = (q-1)(q^2-1)(q^3-1)(q^4-1)(q^5-1).$$

For (b), we have

TABLE 1. Rank$(X) = 1$

Partition of $Y^0_{3,1}$	Cardinality	Partition of $Y^1_{3,1}$	Cardinality
$\left\{\begin{bmatrix}0 & 0 & 0\\ b & \lambda b & \beta b\\ c & \lambda c & \beta c\end{bmatrix}\right\}$	$(q^2-1)q^2$	$\left\{\begin{bmatrix}1 & \lambda & \beta\\ b & \lambda b & \beta b\\ c & \lambda c & \beta c\end{bmatrix}\right\}$	q^4
$\left\{\begin{bmatrix}0 & d & \lambda d\\ 0 & e & \lambda e\\ 0 & f & \lambda f\end{bmatrix}\right\}$	$(q^3-1)q$	-	-
$\left\{\begin{bmatrix}0 & 0 & g\\ 0 & 0 & h\\ 0 & 0 & k\end{bmatrix}\right\}$	q^3-1	-	-

Thus,

$$S_2 = \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(3,3,1,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} + \sum_{\substack{X\in \mathrm{M}(3,3,1,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\#Y^0_{3,1} - \#Y^1_{3,1}\right)$$

$$= -(q-1)(q^2-1)(q^3-1)(q^4-1)((q^2-1)q^2 + (q^3-1)q + (q^3-1) - q^4)$$

$$= -(q-1)^2(q^2-1)^2(q^8 + 2q^7 + 2q^6 + q^5 - 2q^4 - 3q^3 - 4q^2 - 2q - 1).$$

For (d), we have

TABLE 2. Rank$(X) = 3$

Partition of $Y^0_{3,3}$	Cardinality	Partition of $Y^1_{3,3}$	Cardinality
$\left\{\begin{bmatrix}0 & d & g\\ b & e & h\\ c & f & k\end{bmatrix}\right\}$	$(q^2-1)(q^3-q)(q^3-q^2)$	$\left\{\begin{bmatrix}1 & d & g\\ b & e & h\\ c & f & k\end{bmatrix}\right\}$	$q^2(q^3-q)(q^3-q^2)$

Thus,

$$S_4 = \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(3,3,3,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} + \sum_{\substack{X\in \mathrm{M}(3,3,3,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right)$$

$$= \Theta_\theta\left(\begin{bmatrix}1 & X\\0 & 1\end{bmatrix}\right)\left(\#Y^0_{3,3} - \#Y^1_{3,3}\right)$$

$$= -(q-1)(q^2-1)((q^2-1)(q^3-q)(q^3-q^2) - q^2(q^3-q)(q^3-q^2))$$

$$= (q-1)^2(q^2-1)^2q^3.$$

For (c), we let $X' = \begin{bmatrix}e & h\\ f & k\end{bmatrix}$. For $\alpha \in \{0,1\}$, we partition the set $Y^\alpha_{3,2}$ according to the rank of X' and count the cardinalities of each of these subsets. For Rank$(X') \in \{0,1,2\}$ and $\alpha \in \{0,1\}$, we record the cardinality of such subsets of $Y^\alpha_{3,2}$ in the following tables.

TABLE 3. $\mathrm{Rank}(X) = 2, \mathrm{Rank}(X') = 0$

Partition of $Y^0_{3,2}$	Cardinality	Partition of $Y^1_{3,2}$	Cardinality
$B_1 = \left\{ \begin{bmatrix} 0 & d & g \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \middle\vert (d,g) \neq (0,0) \right\}$	$(q^2-1)^2$	$B_2 = \left\{ \begin{bmatrix} 1 & d & g \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix} \middle\vert (d,g) \neq (0,0) \right\}$	$(q^2-1)^2$

TABLE 4. $\mathrm{Rank}(X) = 2, \mathrm{Rank}(X') = 2$

Partition of $Y^0_{3,2}$	Cardinality	Partition of $Y^1_{3,2}$	Cardinality
$C_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ \lambda e + \beta h & e & h \\ \lambda f + \beta k & f & k \end{bmatrix} \right\}$	$(q^2-1)(q^2-q)(q^2)$	$C_4 = \left\{ \begin{bmatrix} 1 & d & 0 \\ d^{-1}e + \beta h & e & h \\ d^{-1}f + \beta k & f & k \end{bmatrix}, \begin{bmatrix} 1 & 0 & g \\ \lambda e + g^{-1}h & e & h \\ \lambda f + g^{-1}k & f & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)^2$
$C_2 = \left\{ \begin{bmatrix} 0 & d & 0 \\ \beta h & e & h \\ \beta k & f & k \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ \beta e & e & h \\ \beta f & f & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)^2$	$C_5 = \left\{ \begin{bmatrix} 1 & d & g \\ d^{-1}(1-\beta g)e + \beta h & e & h \\ d^{-1}(1-\beta g)f + \beta k & f & k \end{bmatrix} \right\}$	$(q^2-1)(q^3-q^2)(q-1)^2$
$C_3 = \left\{ \begin{bmatrix} 0 & d & g \\ \beta(-gd^{-1}e + h) & e & h \\ \beta(-gd^{-1}f + k) & f & k \end{bmatrix} \right\}$	$(q^2-1)(q^2-q)^2(q-1)$	-	-

TABLE 5. $\mathrm{Rank}(X) = 2, \mathrm{Rank}(X') = 1$

Partition of $Y^0_{3,2}$	Cardinality	Partition of $Y^1_{3,2}$	Cardinality
$E_1 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ b & 0 & h \\ c & 0 & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)$	$F_1 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ b & 0 & h \\ c & 0 & k \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix} \right\}$	$2(q^2-1)q^2$
$E_2 = \left\{ \begin{bmatrix} 0 & d & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ b & 0 & h \\ c & 0 & k \end{bmatrix} \right\}$	$2(q^2-1)^2(q-1)$	$F_2 = \left\{ \begin{bmatrix} 1 & d & 0 \\ \beta h & 0 & h \\ \beta k & 0 & k \end{bmatrix}, \begin{bmatrix} 1 & 0 & g \\ \beta e & e & 0 \\ \beta f & f & 0 \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)$
$E_3 = \left\{ \begin{bmatrix} 0 & 0 & g \\ \lambda e & e & 0 \\ \lambda f & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & d & 0 \\ \lambda h & 0 & h \\ \lambda k & 0 & k \end{bmatrix} \right\}$	$2(q^2-1)(q^2-q)$	$F_3 = \left\{ \begin{bmatrix} 1 & 0 & g \\ b & 0 & h \\ c & 0 & k \end{bmatrix} \middle\vert (b,c) \neq (g^{-1}h, g^{-1}k) \right\} \cup \left\{ \begin{bmatrix} 1 & d & 0 \\ b & e & 0 \\ c & f & 0 \end{bmatrix} \middle\vert (b,c) \neq (d^{-1}e, d^{-1}f) \right\}$	$2(q^2-1)^2(q-1)$
$E_4 = \left\{ \begin{bmatrix} 0 & d & g \\ -d^{-1}\beta ge & e & 0 \\ -d^{-1}\beta gf & f & 0 \end{bmatrix}, \begin{bmatrix} 0 & d & g \\ -g^{-1}\beta dh & 0 & h \\ -g^{-1}\beta dk & 0 & k \end{bmatrix} \right\}$	$2(q^3-q)(q-1)^2$	$F_4 = \left\{ \begin{bmatrix} 1 & d & g \\ \beta h & 0 & h \\ \beta k & 0 & k \end{bmatrix}, \begin{bmatrix} 1 & d & g \\ \beta e & e & 0 \\ \beta f & f & 0 \end{bmatrix} \right\}$	$2q(q^2-1)(q-1)^2$
$E_5 = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ b & \lambda h & h \\ c & \lambda k & k \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$(q^2-1)(q-1)(q^2-q)$	$F_5 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ b & e & \lambda e \\ c & f & \lambda f \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$(q^2-1)q^2(q-1)$
$E_6 = \left\{ \begin{bmatrix} 0 & d & 0 \\ \beta h & \lambda h & h \\ \beta k & \lambda k & k \end{bmatrix}, \begin{bmatrix} 0 & 0 & g \\ \beta h & \lambda h & h \\ \beta k & \lambda k & k \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$2q(q^2-1)(q-1)^2$	$F_6 = \left\{ \begin{bmatrix} 1 & d & 0 \\ d^{-1}e + \beta\lambda e & e & \lambda e \\ d^{-1}f + \beta\lambda f & f & \lambda f \end{bmatrix}, \begin{bmatrix} 1 & 0 & g \\ \delta e + g^{-1}\lambda e & e & \lambda e \\ \delta f + g^{-1}\lambda f & f & \lambda f \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$2q(q^2-1)(q-1)^2$
$E_7 = \left\{ \begin{bmatrix} 0 & d & g \\ -d^{-1}\beta g\lambda h + \beta h & \lambda h & h \\ -d^{-1}\beta g\lambda k + \beta k & \lambda k & k \end{bmatrix} \mid d \neq \lambda g, \lambda \neq 0 \right\}$	$(q^2-1)(q-1)^2(q^2-2q)$	$F_7 = \left\{ \begin{bmatrix} 1 & d & g \\ d^{-1}(1-\beta g)e + \beta\lambda e & e & \lambda e \\ d^{-1}(1-\beta g)f + \beta\lambda f & f & \lambda f \end{bmatrix} \middle\vert g \neq \lambda d, \lambda \neq 0 \right\}$	$(q^2-1)(q-1)^2(q^2-2q)$
$E_8 = \left\{ \begin{bmatrix} 0 & \lambda g & g \\ b & \lambda h & h \\ c & \lambda k & k \end{bmatrix} \lambda \neq 0 \right\}$	$(q^2-1)^2(q-1)^2$	$F_8 = \left\{ \begin{bmatrix} 1 & d & \lambda d \\ b & e & \lambda e \\ c & f & \lambda f \end{bmatrix} \middle\vert \lambda \neq 0 \right\}$	$(q^2-1)^2(q-1)^2$

We have,

$$Y^0_{3,2} = B_1 \bigsqcup_{i=1}^{3} C_i \bigsqcup_{j=1}^{8} E_j$$

and

$$Y^1_{3,2} = B_2 \bigsqcup_{i=4}^{5} C_i \bigsqcup_{j=1}^{8} F_j.$$

Thus,

$$\begin{aligned} S_3 &= \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\left(\sum_{\substack{X\in \mathrm{M}(3,3,2,q)\\ \mathrm{Tr}(AX)=0}} \overline{\psi_0(0)} + \sum_{\substack{X\in \mathrm{M}(3,3,2,q)\\ \mathrm{Tr}(AX)=\alpha\neq 0}} \overline{\psi_0(\alpha)}\right) \\ &= \Theta_\theta\left(\begin{bmatrix}1 & X\\ 0 & 1\end{bmatrix}\right)\left(\#Y^0_{3,2} - \#Y^1_{3,2}\right) \\ &= (q-1)(q^2-1)(q^3-1)(q^6-q^5-2q^4+q^2+q) \\ &= (q-1)^2(q^2-1)^2(q^6-q^4-3q^3-2q^2-q). \end{aligned}$$

From equation (3.1), it follows that

$$\begin{aligned} \dim_{\mathbb{C}}(\pi_{N,\psi_A}) &= \frac{1}{q^9}\{S_1+S_2+S_3+S_4\} \\ &= \frac{1}{q^9}(q-1)^2(q^2-1)^2q^9 \\ &= (q-1)^2(q^2-1)^2. \end{aligned}$$

□

Remark 3.4. Suppose that $B = Aw_0$. It is easy to see that $\Theta_{N,\psi_A}\left(\begin{bmatrix}m_1 & 0\\ 0 & m_2\end{bmatrix}\right) = \Theta_{N,\psi_B}\left(\begin{bmatrix}w_0m_1w_0 & 0\\ 0 & m_2\end{bmatrix}\right)$. Thus, we have that $\dim(\pi_{N,\psi_A}) = \dim(\pi_{N,\psi_B})$.

4. Main theorem

In this section, we prove the main result of this paper. Hereafter, we take

$$A = \begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

For the sake of completeness, we recall the statement below.

Theorem 4.1. *Let θ be a regular character of $F_6^\times$ and $\pi = \pi_\theta$ be an irreducible cuspidal representation of G. Then*

$$\pi_{N,\psi_A} \simeq \theta|_{F^\times} \otimes \operatorname{ind}_{U_A}^{H_A} \mu$$

as M_{ψ_A} modules.

The key idea of the proof is to compute the characters of the representations $\rho = \theta|_{F^\times} \otimes \operatorname{ind}_{U_A}^{H_A} \mu$ and π_{N,ψ_A} and show that they are equal at any arbitrary element in M_{ψ_A}. Before we continue, we set up some notation and record a few lemmas that we need.

Lemma 4.2. *Let $M_{\psi_A} = \{m \in M \mid \psi_A^m(n) = \psi_A(n), \forall n \in N\}$. Then we have*

$$M_{\psi_A} = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} & & & \\ a_{21} & a_{22} & a_{23} & & & \\ 0 & 0 & a & & & \\ & & & a & y_{12} & y_{13} \\ & & & 0 & y_{22} & y_{23} \\ & & & 0 & y_{32} & y_{33} \end{bmatrix} \mid a \in F^\times \right\}.$$

Proof. Let $g = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix} \in M$. Then $g \in M_{\psi_A}$ if and only if $Ag_1 = g_2A$. It follows that $g \in M_{\psi_A}$ if and only if $g_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & a \end{bmatrix}$ and $g_2 = \begin{bmatrix} a & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix}$. □

Lemma 4.3. *Let $Z = Z(G)$ be the center of G. Then, we have*

$$M_{\psi_A} \simeq Z \times H_A.$$

Proof. Trivial. □

4.1. Character calculation for ρ. Let $\rho_1 = \operatorname{ind}_{U_1}^{M_1} \mu_1$ and $\rho_2 = \operatorname{ind}_{U_2}^{M_2} \mu_2$. In this section, we calculate the character of the representation

$$\rho = \theta|_{F^\times} \otimes \operatorname{ind}_{U_A}^{H_A} \mu \simeq \theta|_{F^\times} \otimes (\rho_1 \otimes \rho_2).$$

4.1.1. Character computation of ρ_1. Let μ_1 be same as above. Consider the representation

$$\rho_1 = \operatorname{ind}_{U_1}^{M_1} \mu_1$$

of M_1. Let χ_{ρ_1} be the character of ρ_1. Let

$$S_1 = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a, b \in F^\times \right\} \text{ and } S_2 = \left\{ \begin{bmatrix} p & q & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid p \in F, q, r \in F^\times \right\}.$$

It is easy to see that $S = S_1 \cup S_2$ is a set of left coset representatives of U_1 in M_1.

Lemma 4.4. *Let*

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1 \text{ and } t = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_1.$$

Then, $t^{-1}mt \in U_1$ if and only if $a_{11} = a_{22} = 1$ and $a_{21} = 0$. In particular, for $m \in M_1$ with $a_{11} = a_{22} = 1$ and $a_{21} = 0$, we have

$$\sum_{t \in S_1} \mu_1(t^{-1}mt) = \sum_{a,b \in F^\times} \psi_0(a^{-1}ba_{12} + b^{-1}a_{23}).$$

Proof. For $m \in M_1$ and $t \in S_1$, we have

$$t^{-1}mt = \begin{bmatrix} a_{11} & a^{-1}ba_{12} & a^{-1}a_{13} \\ b^{-1}aa_{21} & a_{22} & b^{-1}a_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, it follows that $t^{-1}mt \in U_1$ if and only if $a_{11} = a_{22} = 1$ and $a_{21} = 0$. Clearly, we have

$$\sum_{t \in S_1} \mu_1(t^{-1}mt) = \sum_{a,b \in F^\times} \psi_0(a^{-1}ba_{12} + b^{-1}a_{23}),$$

hence the result. □

Lemma 4.5. *Let*

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1 \text{ and } t = \begin{bmatrix} p & q & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_2.$$

Suppose that $a_{21} = 0$. Then,

$$t^{-1}mt \in U_1 \text{ if and only if } a_{11} = a_{22} = 1 \text{ and } a_{12} = 0.$$

In particular, for $m \in M_1$ with $a_{11} = a_{22} = 1$ and $a_{21} = a_{12} = 0$, we have

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = \sum_{\substack{p \in F \\ r,q \in F^\times}} \psi_0(-pq^{-1}r^{-1}a_{23} + q^{-1}a_{13}).$$

Proof. Let $m \in M_1$ and $t \in S_2$. If $a_{21} = 0$, we have,

$$t^{-1}mt = \begin{bmatrix} a_{22} & 0 & r^{-1}a_{23} \\ pq^{-1}(a_{11} - a_{22}) + rq^{-1}a_{12} & a_{11} & q^{-1}a_{13} - pq^{-1}r^{-1}a_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, $t^{-1}mt \in U_1$ if and only if $a_{22} = a_{11} = 1$ and $a_{12} = 0$. Clearly, we have

$$\sum_{t \in S_2} \mu_1(t^{-1}mt) = \sum_{\substack{p \in F \\ r,q \in F^\times}} \psi_0(-pq^{-1}r^{-1}a_{23} + q^{-1}a_{13}).$$

Hence the result. □

Lemma 4.6. *Let*

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1 \text{ and } t = \begin{bmatrix} p & q & 0 \\ r & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in S_2.$$

Suppose that $a_{21} \neq 0$.

a) *If* $p = 0$, *then* $t^{-1}mt \in U_1$ *if and only if* $a_{11} = a_{22} = 1$ *and* $a_{12} = 0$. *In particular, we have*

$$\sum_{t\in S_2} \mu_1(t^{-1}mt) = \sum_{r,q\in F^\times} \psi_0(qr^{-1}a_{21} + q^{-1}a_{13}).$$

b) *If* $p \neq 0$, *then* $t^{-1}mt \in U_1$ *if and only if* $a_{11} + a_{22} = 2$, $a_{12} = \frac{-(a_{11}-1)^2}{a_{21}}$ *and* $r = (\frac{pa_{21}}{a_{11}-1})$. *In particular, we have*

$$\sum_{t\in S_2} \mu_1(t^{-1}mt) = -\sum_{q\in F^\times} \psi_0(q^{-1}(-\delta + a_{13})),$$

where $\delta = a_{21}^{-1}a_{23}(a_{11} - 1)$.

Proof. Let

$$m = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 1 \end{bmatrix} \in M_1$$

and suppose that $a_{21} \neq 0$. In case a), since $p = 0$, we have

$$t^{-1}mt = \begin{bmatrix} a_{22} & qr^{-1}a_{21} & r^{-1}a_{23} \\ rq^{-1}a_{12} & a_{11} & q^{-1}a_{13} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, it follows that $t^{-1}mt \in U_1$ if and only if $a_{22} = a_{11} = 1$ and $a_{12} = 0$. In particular, we have

$$\sum_{t\in S_2} \mu_1(t^{-1}mt) = \sum_{r,q\in F^\times} \psi_0(qr^{-1}a_{21} + q^{-1}a_{13}).$$

In case b), since $p \neq 0$, we see that $t^{-1}mt$ is equal to

$$\begin{bmatrix} a_{22} + pr^{-1}a_{21} & qr^{-1}a_{21} & r^{-1}a_{23} \\ pq^{-1}(a_{11} - a_{22}) + rq^{-1}a_{12} - p^2q^{-1}r^{-1}a_{21} & a_{11} - pr^{-1}a_{21} & q^{-1}a_{13} - pq^{-1}r^{-1}a_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, we have $t^{-1}mt \in U_1$ if and only if $a_{11} = 1 + pr^{-1}a_{21}$, $a_{22} = 1 - pr^{-1}a_{21}$ and $pq^{-1}(a_{11} - a_{22}) + rq^{-1}a_{12} - p^2q^{-1}r^{-1}a_{21} = 0$. Using $a_{11} - a_{22} = 2pr^{-1}a_{21}$, $a_{11} + a_{22} = 2$ and $\det(t^{-1}mt) = 1$, it follows that $a_{12} = \frac{-(a_{11}-1)^2}{a_{21}}$ and $r = (\frac{pa_{21}}{a_{11}-1})$. In particular, taking $\delta = a_{21}^{-1}a_{23}(a_{11} - 1)$ we have

$$\sum_{t\in S_2} \mu_1(t^{-1}mt) = -\sum_{q\in F^\times} \psi_0(q^{-1}(-a_{21}{}^{-1}a_{23}(a_{11} - 1) + a_{13}))$$

$$= -\sum_{q \in F^\times} \psi_0(q^{-1}(-\delta + a_{13})),$$

hence the result. □

We summarize the character values of ρ_1 in the table below.

TABLE 6. Character of ρ_1

Type of m	m	$\chi_{\rho_1}(m)$	Type of m	m	$\chi_{\rho_1}(m)$
Type-1	$\left\{\begin{bmatrix}1 & a_{12} & a_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{12} \in F^\times\right\}$	$(1-q)$	Type-6	$\left\{\begin{bmatrix}1 & 0 & a_{13}\\ 0 & 1 & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{13}, a_{23} \in F^\times\right\}$	$(1-q)$
Type-2	$\left\{\begin{bmatrix}1 & a_{12} & a_{13}\\ 0 & 1 & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{12}, a_{23} \in F^\times\right\}$	1	Type-7	$\left\{\begin{bmatrix}1 & 0 & 0\\ a_{21} & 1 & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{21} \in F^\times\right\}$	$(1-q)$
Type-3	$\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{23} \in F^\times\right\}$	$(1-q)$	Type-8	$\left\{\begin{bmatrix}1 & 0 & a_{13}\\ a_{21} & 1 & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{21}, a_{13} \in F^\times\right\}$	1
Type-4	$\left\{\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\right\}$	$(1-q)(1-q^2)$	Type-9	$\left\{\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert \begin{matrix} a_{21} \in F^\times, a_{13} = \delta \\ a_{12} = -a_{21}^{-1}(a_{11}-1)^2 \end{matrix}\right\}$	$(1-q)$
Type-5	$\left\{\begin{bmatrix}1 & 0 & a_{13}\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \middle\vert a_{13} \in F^\times\right\}$	$(1-q)$	Type-10	$\left\{\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & 0 & 1\end{bmatrix} \middle\vert \begin{matrix} a_{21} \in F^\times, a_{13} \neq \delta \\ a_{12} = -a_{21}^{-1}(a_{11}-1)^2 \end{matrix}\right\}$	1

If $m \in M_1$ is not one of the types mentioned in Table 6, then $\chi_{\rho_1}(m) = 0$.

4.1.2. Character computation of ρ_2. Let μ_2 be same as above. Consider the representation

$$\rho_2 = \operatorname{ind}_{U_2}^{M_2} \mu_2$$

of M_2. Let χ_{ρ_2} be the character of ρ_2. Let

$$S_3 = \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & a & 0\\ 0 & 0 & b\end{bmatrix} \mid a, b \in F^\times\right\}$$

and

$$S_4 = \left\{\begin{bmatrix}1 & 0 & 0\\ 0 & p & q\\ 0 & r & 0\end{bmatrix} \mid p \in F, q, r \in F^\times\right\}.$$

It is easy to see that $S = S_3 \cup S_4$ is a set of left coset representatives of U_2 in M_2.

Lemma 4.7. *Let*

$$m = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \in M_2, t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix} \in S_3.$$

Then, $t^{-1}mt \in U_2$ if and only if $y_{22} = y_{33} = 1$ and $y_{32} = 0$. In particular, for $m \in M_2$ with $y_{22} = y_{33} = 1$ and $y_{32} = 0$, we have

$$\sum_{t \in S_3} \mu_2(t^{-1}mt) = \sum_{a,b \in F^\times} \psi_0(ay_{12} + ba^{-1}y_{23}).$$

Proof. Similar to Lemma 4.4. □

Lemma 4.8. *Let*

$$m = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \in M_2 \text{ and } t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & 0 \end{bmatrix} \in S_4.$$

Suppose that $y_{32} = 0$. Then,

$$t^{-1}mt \in U_2 \text{ if and only if } y_{22} = y_{33} = 1 \text{ and } y_{23} = 0.$$

In particular, for $m \in M_2$ with $y_{22} = y_{33} = 1$ and $y_{32} = y_{23} = 0$, we have

$$\sum_{t \in S_4} \mu_2(t^{-1}mt) = \sum_{\substack{p \in F \\ r,q \in F^\times}} \psi_0(py_{12} + ry_{13}).$$

Proof. Similar to Lemma 4.5. □

Lemma 4.9. *Let*

$$m = \begin{bmatrix} 1 & y_{12} & y_{13} \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \in M_2 \text{ and } t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & q \\ 0 & r & 0 \end{bmatrix} \in S_4.$$

Suppose that $y_{32} \neq 0$.

a) *If $p = 0$, then $t^{-1}mt \in U_2$ if and only if $y_{22} = y_{33} = 1$ and $y_{23} = 0$. In particular, we have*

$$\sum_{t \in S_4} \mu_2(t^{-1}mt) = \sum_{r,q \in F^\times} \psi_0(qr^{-1}y_{32} + ry_{13}).$$

b) *If $p \neq 0$, then $t^{-1}mt \in U_2$ if and only if $y_{22} + y_{33} = 2$, $y_{23} = -\frac{(y_{22}-1)^2}{y_{32}}$ and $r = \left(\frac{y_{32}p}{y_{22}-1}\right)$. In particular, we have*

$$\sum_{t \in S_4} \mu_2(t^{-1}mt) = -\sum_{p \in F^\times} \psi_0(p(\gamma + y_{12})),$$

where $\gamma = y_{32}y_{13}(y_{22} - 1)^{-1}$.

Proof. Similar to Lemma 4.6. □

We record the character values of ρ_2 in the following table.

TABLE 7. Character of ρ_2

Type of m	m	$\chi_{\rho_2}(m)$	Type of m	m	$\chi_{\rho_2}(m)$
Type-1	$\left\{\begin{bmatrix}1&0&y_{13}\\0&1&y_{23}\\0&0&1\end{bmatrix} \middle\| y_{23}\in F^\times\right\}$	$(1-q)$	Type-6	$\left\{\begin{bmatrix}1&y_{12}&y_{13}\\0&1&0\\0&0&1\end{bmatrix} \middle\| y_{12},y_{13}\in F^\times\right\}$	$(1-q)$
Type-2	$\left\{\begin{bmatrix}1&y_{12}&y_{13}\\0&1&y_{23}\\0&0&1\end{bmatrix} \middle\| y_{12},y_{23}\in F^\times\right\}$	1	Type-7	$\left\{\begin{bmatrix}1&y_{12}&0\\0&1&0\\0&y_{32}&1\end{bmatrix} \middle\| y_{32}\in F^\times\right\}$	$(1-q)$
Type-3	$\left\{\begin{bmatrix}1&y_{12}&0\\0&1&0\\0&0&1\end{bmatrix} \middle\| y_{12}\in F^\times\right\}$	$(1-q)$	Type-8	$\left\{\begin{bmatrix}1&y_{12}&y_{13}\\0&1&0\\0&y_{32}&1\end{bmatrix} \middle\| y_{13},y_{32}\in F^\times\right\}$	1
Type-4	$\left\{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right\}$	$(1-q)(1-q^2)$	Type-9	$\left\{\begin{bmatrix}1&y_{12}&y_{13}\\0&y_{22}&y_{23}\\0&y_{32}&y_{33}\end{bmatrix} \middle\| \substack{y_{32}\in F^\times,\, y_{12}=-\gamma,\\ y_{23}=-y_{32}^{-1}(y_{22}-1)^2}\right\}$	$(1-q)$
Type-5	$\left\{\begin{bmatrix}1&0&y_{13}\\0&1&0\\0&0&1\end{bmatrix} \middle\| y_{13}\in F^\times\right\}$	$(1-q)$	Type-10	$\left\{\begin{bmatrix}1&y_{12}&y_{13}\\0&y_{22}&y_{23}\\0&y_{32}&y_{33}\end{bmatrix} \middle\| \substack{y_{32}\in F^\times,\, y_{12}\neq-\gamma,\\ y_{23}=-y_{32}^{-1}(y_{22}-1)^2}\right\}$	1

If $m \in M_2$ is not one of the types mentioned above, we have $\chi_{\rho_2}(m) = 0$.

For $1 \leq i, j \leq 10$, we let

$$T(i,j) = \{k = (m_1, m_2) \in H_A \mid m_1 \in \text{Type}-i, m_2 \in \text{Type}-j\}.$$

Theorem 4.10. *Let $\rho = \theta|_{F^\times} \otimes \rho_1 \otimes \rho_2$. Let χ_ρ be the character of ρ. For $m = (a, m_1, m_2) \in Z \times M_1 \times M_2$, we have*

$$\chi_\rho(m) = \theta(a)\chi_{\rho_1}(m_1)\chi_{\rho_2}(m_2)$$

where $(m_1, m_2) \in T(i,j)$, $i, j \in \{1, \ldots, 10\}$. Otherwise, $\chi_\rho(m) = 0$.

Proof. We summarize the results from Table (6) and Table (7). □

TABLE 8. Character of ρ

	Type-1	Type-2	Type-3	Type-4	Type-5	Type-6	Type-7	Type-8	Type-9	Type-10
Type-1	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-2	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)$
Type-3	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-4	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)(1-q^2)$
Type-5	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-6	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-7	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-8	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)$
Type-9	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2(1-q^2)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$	$\theta(a)(1-q)^2$	$\theta(a)(1-q)$
Type-10	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)(1-q)(1-q^2)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)(1-q)$	$\theta(a)$	$\theta(a)(1-q)$	$\theta(a)$

4.2. Character calculation for π_{N,ψ_A}.

Lemma 4.11. *Let $m = ah \in M_{\psi_A}$, where $a \in Z$ and $h \in H_A$. Then,*

$$\Theta_{N,\psi_A}(m) = \theta(a)\Theta_{N,\psi_A}(h).$$

Proof. We have

$$\begin{aligned}
\Theta_{N,\psi_A}(m) &= \Theta_{N,\psi_A}(ah) \\
&= \frac{1}{|N|}\sum_{n\in N} \Theta_\theta(ahn)\overline{\psi_A(n)} \\
&= \frac{1}{|N|}\sum_{n\in N} \mathrm{Tr}(\pi(ahn))\overline{\psi_A(n)} \\
&= \frac{1}{|N|}\sum_{n\in N} \mathrm{Tr}(\pi(a)\pi(hn))\overline{\psi_A(n)} \\
&= \omega_\pi(a)\frac{1}{|N|}\sum_{n\in N} \mathrm{Tr}(\pi(hn))\overline{\psi_A(n)} \\
&= \omega_\pi(a)\Theta_{N,\psi_A}(h)
\end{aligned}$$

where ω_π is the central character of π. Explicitly, we have

$$\Theta_\theta(a) = \mathrm{Tr}(\pi(a)) = \mathrm{Tr}(\omega_\pi(a)) = \omega_\pi(a)\dim(\pi).$$

Using Theorem 2.2, it is easy to see that

$$\Theta_\theta(a) = \theta(a)\dim(\pi).$$

Thus, we have $\omega_\pi(a) = \theta(a)$ and the result follows. □

Lemma 4.12. *Let $\tau = \begin{bmatrix} 0 & w_0 \\ w_0 & 0 \end{bmatrix}$. For $1 \le i, j \le 10$, we have*

$$T(j,i) = \tau T(i,j)^{\top}\tau^{-1}.$$

Proof. Trivial. □

Theorem 4.13. *Let $m' \in T(j,i)$. Then there exists $m \in T(i,j)$ such that*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m' = \begin{bmatrix} m_1' & 0 \\ 0 & m_2' \end{bmatrix} \in T(j,i)$. Since $T(j,i) = \tau T(i,j)^\top \tau^{-1}$, it follows that,

$$m' = \tau m^\top \tau^{-1}$$

for some $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,j)$. Thus we have $m_1' = w_0 {m_2}^\top {w_0}^{-1}$ and $m_2' = w_0 {m_1}^\top {w_0}^{-1}$. Since $m_1' \in M_1$, clearly $\psi_A(X) = \psi_A((w_0 {m_2}^\top w_0)^{-1} X)$. We have

$$\begin{aligned}
\Theta_{N,\psi_A}(m') &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} w_0 {m_2}^\top {w_0}^{-1} & X \\ 0 & w_0 {m_1}^\top {w_0}^{-1} \end{bmatrix} \overline{\psi_A(X)} \\
&= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \left(\begin{bmatrix} w_0 & 0 \\ 0 & w_0 \end{bmatrix} \begin{bmatrix} {m_2}^\top & {w_0}^{-1} X w_0 \\ 0 & {m_1}^\top \end{bmatrix} \begin{bmatrix} w_0 & 0 \\ 0 & w_0 \end{bmatrix} \right) \overline{\psi_A(X)} \\
&= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \left(\begin{bmatrix} {m_2}^\top & {w_0}^{-1} X w_0 \\ 0 & {m_1}^\top \end{bmatrix} \right) \overline{\psi_A(X)}.
\end{aligned}$$

On the other hand, using $\mathrm{Tr}(A({w_0}^{-1} X^\top w_0)) = \mathrm{Tr}(AX)$ we have

$$\begin{aligned}
\Theta_{N,\psi_A}(m) &= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X)} \\
&= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X^\top \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X^\top)} \\
&= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & {w_0}^{-1} X^\top w_0 \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({w_0}^{-1} X^\top w_0)}. \\
&= \frac{1}{|N|} \sum_{X \in \mathrm{M}(3,F)} \Theta_\theta \begin{bmatrix} m_1 & {w_0}^{-1} X^\top w_0 \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(X)}.
\end{aligned}$$

Since

$$\begin{aligned}
\mathrm{Rank}\left(\begin{bmatrix} {m_2}^\top - 1 & w_0^{-1} X w_0 \\ 0 & {m_1}^\top - 1 \end{bmatrix} \right) &= \mathrm{Rank}\left(\begin{bmatrix} m_2 - 1 & 0 \\ w_0^{-1} X^\top w_0 & m_1 - 1 \end{bmatrix} \right) \\
&= \mathrm{Rank}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} m_2 - 1 & 0 \\ w_0^{-1} X^\top w_0 & m_1 - 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \\
&= \mathrm{Rank}\left(\begin{bmatrix} m_1 - 1 & w_0^{-1} X^\top w_0 \\ 0 & m_2 - 1 \end{bmatrix} \right)
\end{aligned}$$

we have,

$$\dim(\ker\left(\begin{bmatrix} {m_2}^\top - 1 & w_0^{-1} X w_0 \\ 0 & {m_1}^\top - 1 \end{bmatrix}\right) = \dim(\ker\left(\begin{bmatrix} m_1 - 1 & w_0^{-1} X^\top w_0 \\ 0 & m_2 - 1 \end{bmatrix}\right).$$

Hence,

$$\Theta_\theta \begin{bmatrix} {m_2}^\top & w_0^{-1} X w_0 \\ 0 & {m_1}^\top \end{bmatrix} = \Theta_\theta \begin{bmatrix} m_1 & w_0^{-1} X^\top w_0 \\ 0 & m_2 \end{bmatrix}$$

and the result follows. □

Remark 4.14. We have used the fact that $\mathrm{Rank}(M) = \mathrm{Rank}(M^\top)$ and $\mathrm{Rank}(NMP) = \mathrm{Rank}(M)$ if N and P are invertible matrices.

Let $m = (m_1, m_2) \in M_1 \times M_2 = H_A$. Suppose also that m_1, m_2 are unipotent. To calculate $\Theta_{N,\psi_A}(m)$, we need to compute $\Theta_\theta(h)$, where $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$. Using Theorem 2.2, it suffices to compute $\dim \mathrm{Ker}(h-1)$. We note that the following proposition is valid even when H_A is a subgroup of $\mathrm{GL}(2n, F)$.

Proposition 4.15. *Let* $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \in \mathrm{GL}(2n, F)$*, where* $(m_1, m_2) \in H_A$*. Suppose that* m_1 *and* m_2 *are also unipotent. Let* $W' = \mathrm{Ker}(m_2 - 1)$*. Then, we have*

$$\dim \mathrm{Ker}(h-1) = \dim \mathrm{Ker}(m_1 - 1) + \dim \mathrm{Ker}(m_2 - 1) \\ - \dim(XW') + \dim\{XW' \cap \mathrm{Im}(m_1 - 1)\}.$$

Proof. Let V be an n-dimensional vector space over F and let m_1, m_2, X be linear operators on V. Suppose that $\{e_1, \dots, e_m\}$ is a basis for $\mathrm{Ker}(m_1 - 1)$ and $\{f_1, \dots, f_k\}$ is a basis for $\mathrm{Ker}(m_2 - 1)$. Extending the basis of $\mathrm{Ker}(m_1 - 1)$ and $\mathrm{Ker}(m_2 - 1)$ we get ordered bases $\beta = \{e_1, \dots, e_n\}$ and $\beta' = \{f_1, \dots, f_n\}$ of V. Consider the ordered basis $\tilde{\beta} = \{(e_1, 0), \dots, (e_n, 0), (0, f_1), \dots, (0, f_n)\}$ of $V \oplus V$. We let h to be the linear operator on $V \oplus V$ defined as follows. For $1 \le i, j \le n$,

$$h((e_i, 0)) = (m_1, 0)(e_i, 0) = (m_1(e_i), 0)$$

and

$$h((0, f_j)) = (X, m_2)(0, f_j) = (X(f_j), m_2(f_j)).$$

Then,

$$[h]_{\tilde{\beta}} = \begin{bmatrix} [m_1]_\beta & [X]_{\beta'}^{\beta} \\ 0 & [m_2]_{\beta'} \end{bmatrix}$$

where

$$[X]_{\beta'}^{\beta} = [X_1\, X_2\, \cdots\, X_n].$$

Let

$$W_1 = \mathrm{Span}\{(m_1 - 1, 0)(e_{m+1}, 0), \dots, (m_1 - 1, 0)(e_n, 0)\} = \mathrm{Im}(m_1 - 1),$$
$$W_2 = \mathrm{Span}\{(X, m_2 - 1)(0, f_1), \dots, (X, m_2 - 1)(0, f_k)\} = XW'$$

and

$$W_3 = \text{Span}\{(Xf_{k+1}, (m_2-1)f_{k+1}), \ldots, (Xf_n, (m_2-1)f_n)\}.$$

Clearly,

$$\text{Im}(h-1) = W_1 + W_2 + W_3.$$

It is easy to see that

$$W_2 \cap W_3 = \{0\} = W_1 \cap W_3.$$

Since $\dim(\text{Ker}(m_2-1)) = k$, we have that

$$\dim(W_3) = \dim(\text{Im}(m_2-1)).$$

Therefore,

$$\begin{aligned}\dim(\text{Im}(h-1)) = \dim(\text{Im}(m_1-1)) + \dim(\text{Im}(m_2-1)) \\ + \dim(XW') - \dim(XW' \cap \text{Im}(m_1-1)),\end{aligned}$$

hence the result. □

Remark 4.16. Let h be as in Proposition 4.15. We note that

$$XW' = \text{Span}\{X_1, X_2, \ldots, X_k\}.$$

We will continue to use this in our character calculations at several instances to follow.

Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in H_A$, where $m_1 \in M_1$, $m_2 \in M_2$. Throughout we write $W' = \text{Ker}(m_2-1)$. For $X \in \text{M}(3,F)$, we let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$. For $\beta \in F$, we define

$$S(\beta) = \{X \in M(3,F) \mid \text{Tr}(A{m_1}^{-1}X) = \text{Tr}(AX) = \beta\}.$$

Let

$$E = \bigcup_{\substack{i \le j \\ i,j \in \{1,2,4\}}} T(i,j).$$

We call E to be the fundamental set. To determine $\Theta_{N,\psi_A}(m)$ for $m \in T(i,j)$, it is enough to compute $\Theta_{N,\psi_A}(m)$ for $m \in E$.

Theorem 4.17. *Let $m \in T(1,1)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Note that $\dim \text{Ker}(m_1-1) = 2$, $\text{Im}(m_1-1) = \text{Span}\,\{e_1\}$ and $\text{Ker}(m_2-1) = \text{Span}\,\{e_1, e_2\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(A_1 + A_2)$$

where

$$A_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}$$

and

$$A_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

For simplicity, we let $t = \dim(\mathrm{Ker}(h-1))$. To compute A_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

TABLE 9. A_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1 - 1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)(a)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix}\right\}$	0	0	4	q^3
1)(b)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times\right\}$	1	0	3	$(q-1)q^3$
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & l \end{bmatrix} \middle\vert ad - bc \neq 0\right\}$	2	1	3	$(q^2-1)(q-1)q^4$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert c, k \in F^\times, ad - bc \neq 0\right\}$	2	0	2	$(q-1)^3q^5$
2)(c)	$\left\{\begin{bmatrix} a & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times, ad \neq 0\right\}$	2	1	3	$(q-1)^3q^4$
3)(a)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & 0 & l \end{bmatrix} \middle\vert c \in F^\times, \gamma \in F\right\}$	1	0	3	$(q-1)q^5$
3)(b)	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix} \middle\vert a \in F^\times, \gamma \in F\right\}$	1	1	4	$(q-1)q^4$
3)(c)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & k & l \end{bmatrix} \middle\vert c, k \in F^\times, \gamma \in F\right\}$	2	0	2	$(q-1)^2q^5$
3)(d)	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times, \gamma \in F\right\}$	2	1	3	$(q-1)^2q^4$
4)(a)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & 0 & l \end{bmatrix} \middle\vert d \in F^\times\right\}$	1	0	3	$(q-1)q^4$
4)(b)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix} \middle\vert b \in F^\times\right\}$	1	1	4	$(q-1)q^3$
4)(c)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times\right\}$	1	0	3	$(q^2-1)(q-1)q^3$

Hence,

$$A_1 = K_1 + K_2 + K_3$$

where

a) $K_1 = \sum_{\substack{X \in S(0) \\ t=4}} \Theta_\theta \left(\begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \right) \overline{\psi_A(m_1{}^{-1}X)} = -q^5(1-q)(1-q^2)(1-q^3).$

b) $K_2 = \sum_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^5(2q+1)(q-1)^2(q^2-1).$

c) $K_3 = \sum_{\substack{X \in S(0) \\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^6(q-1)^3.$

It follows that

$$A_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^8(q-1)^3. \tag{4.1}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute A_2. We record the details in the table 10.

Hence, $A_2 = K_4 + K_5$, where

a) $K_4 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^7(q-1)(q^2-1).$

b) $K_5 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^7(q-1)^2.$

It follows that

$$A_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^8(q-1)^2. \tag{4.2}$$

From (4.1) and (4.2), it follows that

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

TABLE 10. A_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1 - 1))$	$t = \dim(\mathrm{Ker}(h - 1))$	Cardinality
1)(a)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & 0 & l \end{bmatrix}\right\}$	1	0	3	q^3
1)(b)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert k \in F^\times\right\}$	1	0	3	$(q-1)q^3$
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & 0 & l \end{bmatrix} \middle\vert d \in F^\times, ad - bc \neq 0\right\}$	2	0	2	$(q-1)^2q^5$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ c & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert bc \neq 0\right\}$	2	1	3	$(q-1)^2q^4$
2)(c)	$\left\{\begin{bmatrix} a & b & e \\ 0 & d & f \\ \beta & k & l \end{bmatrix} \middle\vert k \in F^\times, ad \neq 0\right\}$	2	0	2	$(q-1)^2(q^2-q)q^3$
2)(d)	$\left\{\begin{bmatrix} a & b & e \\ c & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert k \in F^\times, bc \neq 0\right\}$	2	0	2	$(q-1)^2(q^2-q)q^3$
2)(d)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix} \middle\vert c, d, k \in F^\times, ad - bc \neq 0, d = \beta^{-1}ck\right\}$	2	1	3	$(q-1)^2(q^2-q)q^3$
2)(e)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix} \middle\vert c, d, k \in F^\times, ad - bc \neq 0, d \neq \beta^{-1}ck\right\}$	2	0	2	$(q-1)^3(q-2)q^4$
3)(a)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & 0 & l \end{bmatrix} \middle\vert c, \gamma \in F^\times\right\}$	2	0	2	$(q-1)^2q^4$
3)(b)	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert a, \gamma \in F^\times\right\}$	2	1	3	$(q-1)^2q^3$
3)(c)	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert (a, c) \neq 0\right\}$	1	0	3	$(q^2-1)q^3$
3)(d)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert \gamma \in F^\times, k = \gamma\beta\right\}$	1	0	3	$(q^2-1)(q-1)q^3$
3)(e)	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert k, c \in F^\times\right\}$	2	0	2	$(q-1)^2q^4$
3)(f)	$\left\{\begin{bmatrix} a & 0 & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert k \in F^\times\right\}$	2	1	3	$(q-1)^2q^3$
3)(g)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert c, k, \gamma \in F^\times, k \neq \gamma\beta\right\}$	2	0	2	$(q-1)^2(q-2)q^4$
3)(h)	$\left\{\begin{bmatrix} a & \gamma a & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert a, k, \gamma \in F^\times, k \neq \gamma\beta\right\}$	2	1	3	$(q-1)^2(q-2)q^3$
4)(a)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ \beta & k & l \end{bmatrix} \middle\vert d \in F^\times\right\}$	2	0	2	$(q-1)q^5$
4)(b)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert b \in F^\times\right\}$	2	1	3	$(q-1)q^4$

□

Theorem 4.18. *Let $m \in T(1,2)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

Note that $\dim \operatorname{Ker}(m_1 - 1) = 2$, $\operatorname{Im}(m_1 - 1) = \operatorname{Span}\{e_1\}$ and $\operatorname{Ker}(m_2 - 1) = \operatorname{Span}\{e_1\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(B_1 + B_2)$$

where

$$B_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}$$

and

$$B_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)}.$$

For simplicity, we let $t = \dim(\operatorname{Ker}(h-1))$. To compute B_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

TABLE 11. B_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1-1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix}\right\}$	0	0	3	q^6
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert\, c \in F^\times\right\}$	1	0	2	$(q-1)q^7$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \middle\vert\, a \in F^\times\right\}$	1	1	3	$(q-1)q^6$

Hence,

$$B_1 = K_1 + K_2$$

where

a) $$K_1 = \sum_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = -q^7(1-q)(1-q^2).$$

b) $K_2 = \sum_{\substack{X\in S(0)\\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^7(1-q)^2.$

It follows that

$$B_1 = \sum_{X\in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^8(q-1)^2. \tag{4.3}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute B_2. We record the details in the table below.

TABLE 12. B_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1-1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix}\right\}$	1	0	2	q^8

Hence,

$$B_2 = \sum_{\beta\in F^\times} \sum_{\substack{X\in S(\beta)\\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^8(1-q). \tag{4.4}$$

From (4.3) and (4.4), it follows that

$$\Theta_{N,\psi_A}(m) = (1-q).$$

□

Theorem 4.19. *Let $m \in T(4,1)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X\in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Note that $\dim \operatorname{Ker}(m_1-1) = 3$, $\operatorname{Im}(m_1-1) = \{0\}$ and $\operatorname{Ker}(m_2-1) = \operatorname{Span}\{e_1, e_2\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(C_1 + C_2)$$

where we have

$$C_1 = \sum_{X\in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}$$

and

$$C_2 = \sum_{\beta\in F^\times} \sum_{X\in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

For simplicity, we let $t = \dim(\mathrm{Ker}(h-1))$. To compute C_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the following table.

TABLE 13. C_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1 - 1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)(a)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & 0 & l \end{bmatrix}\right\}$	0	0	5	q^3
1)(b)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times\right\}$	1	0	4	$(q-1)q^3$
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & l \end{bmatrix} \middle\vert ad - bc \neq 0\right\}$	2	0	3	$(q^2-1)(q^2-q)q^3$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times, ad - bc \neq 0\right\}$	2	0	3	$(q^2-1)(q-1)^2q^4$
3)(a)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & 0 & l \end{bmatrix} \middle\vert \gamma \in F\right\}$	1	0	4	$(q^2-1)q^4$
3)(b)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times, \gamma \in F\right\}$	2	0	3	$(q^2-1)(q-1)q^4$
4)(a)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & 0 & l \end{bmatrix}\right\}$	1	0	4	$(q^2-1)q^3$
4)(b)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \middle\vert k \in F^\times\right\}$	1	0	4	$(q^2-1)(q-1)q^3$

Hence,

$$C_1 = K_1 + K_2 + K_3$$

where

a) $K_1 = \sum_{\substack{X \in S(0) \\ t=5}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^3(1-q)(1-q^2)(1-q^3)(1-q^4)$.

b) $K_2 = \sum_{\substack{X \in S(0) \\ t=4}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^3(1-q)^2(1-q^2)(1-q^3)(2q^2+2q+1)$.

c) $K_3 = \sum_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^4(q^2-1)^2(1-q)(1-q^2).$

It follows that

$$C_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^8(1-q)^3(1-q^2). \tag{4.5}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute C_2. We record the details in the table below.

TABLE 14. C_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \operatorname{Im}(m_1-1))$	$t = \dim(\operatorname{Ker}(h-1))$	Cardinality
1)(a)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & 0 & l \end{bmatrix}\right\}$	1	0	4	q^3
1)(b)	$\left\{\begin{bmatrix} 0 & 0 & e \\ 0 & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k \in F^\times\right\}$	1	0	4	$(q-1)q^3$
2)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, ad-bc \neq 0\right\}$	2	0	3	$(q^2-1)(q^2-q)q^3$
2)(b)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k \in F^\times, ad-bc \neq 0\right\}$	2	0	3	$(q^2-1)(q-1)^2q^4$
3)(a)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, \gamma \in F^\times\right\}$	2	0	3	$(q^2-1)(q-1)q^3$
3)(b)	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, (a,c) \neq 0\right\}$	1	0	4	$(q^2-1)q^3$
3)(c)	$\left\{\begin{bmatrix} a & 0 & e \\ c & 0 & f \\ \beta & k & l \end{bmatrix} \middle\vert\, (a,c) \neq 0, k \in F^\times\right\}$	2	0	3	$(q^2-1)(q-1)q^3$
3)(d)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert\, \gamma \in F^\times, k = \gamma\beta\right\}$	1	0	4	$(q^2-1)(q-1)q^3$
3)(e)	$\left\{\begin{bmatrix} a & \gamma a & e \\ c & \gamma c & f \\ \beta & k & l \end{bmatrix} \middle\vert\, k, \gamma \in F^\times, k \neq \gamma\beta\right\}$	2	0	3	$(q^2-1)(q-1)(q-2)q^3$
4)(a)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ \beta & 0 & l \end{bmatrix} \middle\vert\, (b,d) \neq 0\right\}$	2	0	3	$(q^2-1)q^3$
4)(b)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ \beta & k & l \end{bmatrix} \middle\vert\, (b,d) \neq 0, k \in F^\times\right\}$	2	0	3	$(q^2-1)(q-1)q^3$

We have

$$C_2 = K_4 + K_5$$

where

a) $K_4 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=4}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^6(1-q)(1-q^2)(1-q^3).$

b) $K_5 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^6(1-q)(1-q^2)^2.$

It follows that

$$C_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^8(1-q)^2(1-q^2). \tag{4.6}$$

From (4.5) and (4.6), we have

$$\Theta_{N,\psi_A}(m) = (1-q^2)(1-q)^2.$$

□

Remark 4.20. Let $m \in T(1,4)$. Since

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m')$$

for some $m' \in T(4,1)$, it is enough to compute $\Theta_{N,\psi_A}(m')$ for $m' \in T(4,1)$ to obtain the character value $\Theta_{N,\psi_A}(m)$.

Theorem 4.21. *Let $m \in T(2,2)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = 1.$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Note that $\dim \operatorname{Ker}(m_1 - 1) = 1$, $\operatorname{Im}(m_1 - 1) = \operatorname{Span}\{e_1, e_2\}$ and $\operatorname{Ker}(m_2 - 1) = \operatorname{Span}\{e_1\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(D_1 + D_2)$$

where

$$D_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}$$

and

$$D_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Let $t = \dim(\operatorname{Ker}(h-1))$. To compute D_1 we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

TABLE 15. D_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)(a)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \mid (a,c) \neq 0\right\}$	1	1	2	$(q^2-1)q^6$
1)(b)	$\left\{\begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix}\right\}$	0	0	2	q^6

Hence,

$$D_1 = \sum_{\substack{X \in S(0) \\ t=2}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^8(1-q). \tag{4.7}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute D_2. We record the details in the following table.

TABLE 16. D_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix}\right\}$	1	0	1	q^8

Thus, we have

$$D_2 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=1}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^8. \tag{4.8}$$

From (4.7) and (4.8), it follows that

$$\Theta_{N,\psi_A}(m) = 1.$$

□

Theorem 4.22. *Let $m \in T(4,2)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)(1-q^2).$$

Proof. We have

$$\Theta_{N,\psi_A}(m) = \frac{1}{|N|} \sum_{X \in M(3,F)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Note that $\dim \mathrm{Ker}(m_1 - 1) = 3$, $\mathrm{Im}(m_1 - 1) = \{0\}$ and $\mathrm{Ker}(m_2 - 1) = \mathrm{Span}\{e_1\}$. To calculate the character value, we write

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(H_1 + H_2)$$

where

$$H_1 = \sum_{X \in S(0)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}$$

and

$$H_2 = \sum_{\beta \in F^\times} \sum_{X \in S(\beta)} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)}.$$

Let $t = \dim(\mathrm{Ker}(h-1))$. To compute H_1, we find a partition of $S(0)$ according to the value of t and compute the respective cardinalities. We record the details in the table below.

TABLE 17. H_1

	Partition of $S(0)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1 - 1))$	$t = \dim(\mathrm{Ker}(h-1))$	Cardinality
1)(a)	$\left\{ \begin{bmatrix} a & b & e \\ c & d & f \\ 0 & k & l \end{bmatrix} \middle\vert (a,c) \neq 0 \right\}$	1	0	3	$(q^2-1)q^6$
1)(b)	$\left\{ \begin{bmatrix} 0 & b & e \\ 0 & d & f \\ 0 & k & l \end{bmatrix} \right\}$	0	0	4	q^6

Hence,

$$H_1 = K_1 + K_2$$

where

a) $K_1 = \sum_{\substack{X \in S(0) \\ t=4}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^6(1-q)(1-q^2)(1-q^3).$

b) $K_2 = \sum_{\substack{X \in S(0) \\ t=3}} \Theta_\theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A(m_1{}^{-1}X)} = q^6(1-q)(1-q^2)^2.$

It follows that

$$H_1 = \sum_{X \in S(0)} \Theta_\theta \begin{pmatrix} m_1 & X \\ 0 & m_2 \end{pmatrix} \overline{\psi_A(m_1{}^{-1}X)} = -q^8(1-q)^2(1-q^2). \tag{4.9}$$

Proceeding in a similar way, we find a partition of $S(\beta)$ to compute H_2. We record the details in the following table.

TABLE 18. H_2

	Partition of $S(\beta)$	$\dim(XW')$	$\dim(XW' \cap \mathrm{Im}(m_1-1))$	$t=\dim(\mathrm{Ker}(h-1))$	Cardinality
1)	$\left\{\begin{bmatrix} a & b & e \\ c & d & f \\ \beta & k & l \end{bmatrix}\right\}$	1	0	3	q^8

Thus, we have

$$H_2 = \sum_{\beta \in F^\times} \sum_{\substack{X \in S(\beta) \\ t=3}} \Theta \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix} \overline{\psi_A({m_1}^{-1}X)} = q^8(1-q)(1-q^2). \qquad (4.10)$$

From (4.9) and (4.10), it follows that

$$\Theta_{N,\psi_A} = (1-q)(1-q^2).$$

□

Remark 4.23. Let $m \in T(2,4)$. Since

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m')$$

for some $m' \in T(4,2)$, it is enough to compute $\Theta_{N,\psi_A}(m')$ for $m' \in T(4,2)$ to obtain $\Theta_{N,\psi_A}(m)$.

Theorem 4.24. *Let $m \in T(4,4)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2)^2.$$

Proof. Since $m \in T(4,4)$, we have $m=1$, and the result follows from Theorem 3.3. To be precise, we have

$$\Theta_{N,\psi_A}(m) = \dim_{\mathbb{C}}(\pi_{N,\psi_A}) = (1-q)^2(1-q^2)^2.$$

□

Theorem 4.25. *Let $1 \leq i \leq 10$. Suppose that $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,5)$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix} \in T(i,1)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$ and $h' = \begin{bmatrix} m_1 & X \\ 0 & m_2' \end{bmatrix}$ for $X \in \mathrm{M}(3,F)$. Let

$$M^{d,\beta}_{m_1,m_2} = \{X \in S(\beta) \mid \dim(\mathrm{Ker}(h-1)) = d\}$$

for $\beta \in F$. Clearly,

$$\mathrm{Ker}(m_2 - 1) = \mathrm{Ker}(m_2' - 1).$$

Hence for any $X \in \mathrm{M}(3,F)$,

$$X\,\mathrm{Ker}(m_2 - 1) = X\,\mathrm{Ker}(m_2' - 1)$$

and

$$X\operatorname{Ker}(m_2-1)\cap\operatorname{Im}(m_1-1)=X\operatorname{Ker}(m_2'-1)\cap\operatorname{Im}(m_1-1).$$

In particular, for any $\beta\in F$, we have that

$$M^{d,\beta}_{m_1,m_2}=M^{d,\beta}_{m_1,m_2'}. \tag{4.11}$$

We have,

$$\Theta_{N,\psi_A}(m)=\frac{1}{q^9}(R_1+R_2)$$

where

$$\begin{aligned}R_1&=\sum_{d=1}^{6}\sum_{X\in M^{d,0}_{m_1,m_2}}\Theta_\theta(h)\overline{\psi_A(m_1{}^{-1}X)}\\&=\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\#M^{d,0}_{m_1,m_2})\overline{\psi_0(0)}\end{aligned}$$

and

$$\begin{aligned}R_2&=\sum_{\beta\in F^\times}\sum_{d=1}^{6}\sum_{X\in M^{d,\beta}_{m_1,m_2}}\Theta_\theta(h)\overline{\psi_A(m_1{}^{-1}X)}\\&=\sum_{\beta\in F^\times}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\#M^{d,\beta}_{m_1,m_2})\overline{\psi_0(\beta)}\end{aligned}$$

Thus,

$$\Theta_{N,\psi_A}(m)=\frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\#M^{d,0}_{m_1,m_2}+\sum_{\beta\in F^\times}\#M^{d,\beta}_{m_1,m_2}\overline{\psi_0(\beta)}).$$

Similarly,

$$\Theta_{N,\psi_A}(m')=\frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\#M^{d,0}_{m_1,m_2'}+\sum_{\beta\in F^\times}\#M^{d,\beta}_{m_1,m_2'}\overline{\psi_0(\beta)}).$$

Hence, it follows from equation (4.11) that

$$\Theta_{N,\psi_A}(m)=\Theta_{N,\psi_A}(m').$$

□

Proposition 4.26. *Let* $1\le i\le 10$ *and* $m\in T(i,3)$*. Then, there exists some* $m'\in T(i,5)$ *such that*

$$\Theta_{N,\psi_A}(m)=\Theta_{N,\psi_A}(m').$$

Proof. Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,3)$, and $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Let $m_2' = wm_2w^{-1}$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix}$. Clearly, we have $m = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} m' \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$ and $m' \in T(i,5)$, hence the result. □

Corollary 4.27. *Let $1 \le i \le 10$ and $m \in T(i,3)$. Then,*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m')$$

for some $m' \in T(i,1)$.

Proof. Using Proposition 4.26 and Theorem 4.25, the result follows. □

Proposition 4.28. *Let $1 \le i \le 10$ and $m \in T(i,7)$. Then, there exists some $m' \in T(i,1)$ such that*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,7)$, and $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Let $m_2' = wm_2w^{-1}$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix}$. Clearly, we have $m = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} m' \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$ and $m' \in T(i,1)$, hence the result. □

Proposition 4.29. *Let $1 \le i \le 10$ and $m \in T(i,8)$. Then, there exists some $m' \in T(i,2)$ such that*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,8)$, and $w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Let $m_2' = wm_2w^{-1}$ and $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix}$. Clearly, we have $m = \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix} m' \begin{bmatrix} 1 & 0 \\ 0 & w \end{bmatrix}^{-1}$ and $m' \in T(i,2)$, hence the result. □

Theorem 4.30. *Let $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(1,6)$ or $T(1,9)$. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. Note that $\dim \operatorname{Ker}(m_1 - 1) = 2$, $\operatorname{Im}(m_1 - 1) = \operatorname{Span}\{e_1\}$, $\dim \operatorname{Ker}(m_2 - 1) = 2$. From Remark 4.16, it follows that whenever

$$h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix},$$

we have

$$XW' = \operatorname{Span}\{X_1, X_2\}.$$

Thus, proceeding in a similar fashion as in Theorem 4.17, we get that

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

□

Theorem 4.31. *Let* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(2,6)$ *or* $T(2,9)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q).$$

Proof. Note that $\dim \mathrm{Ker}(m_1-1) = 1$, $\mathrm{Im}(m_1-1) = \{e_1, e_2\}$, $\dim \mathrm{Ker}(m_2-1) = 2$. From Remark 4.16, it follows that whenever

$$h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix},$$

we have

$$XW' = \mathrm{Span}\{X_1, X_2\}.$$

Thus, the computations are similar to the case where $m \in T(2,1)$. The result follows from Theorem 4.13 and Theorem 4.18. □

Theorem 4.32. *Let* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(4,6)$ *or* $T(4,9)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2).$$

Proof. Note that $\dim \mathrm{Ker}(m_1-1) = 3$, $\mathrm{Im}(m_1-1) = \{0\}$, $\dim \mathrm{Ker}(m_2-1) = 2$. From Remark 4.16, it follows that whenever

$$h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix},$$

we have

$$XW' = \mathrm{Span}\{X_1, X_2\}.$$

Thus, proceeding in a similar fashion as in Theorem 4.19, we get that

$$\Theta_{N,\psi_A}(m) = (1-q)^2(1-q^2).$$

□

Theorem 4.33. *Let* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(6,6)$ *or* $T(6,9)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. Note that $\dim \mathrm{Ker}(m_1-1) = 2$, $\dim \mathrm{Ker}(m_2-1) = 2$, $\mathrm{Im}(m_1-1) = \mathrm{Span}\{\eta e_1 + e_2\}$ for $\eta \in F^\times$. From Remark 4.16, it follows that computing $\Theta_{N,\psi_A}(m)$ for $m \in T(6,6)$ or $m \in T(6,9)$ is the same as computing $\Theta_{N,\psi_A}(m')$ for $m' \in T(6,1)$. Using Theorem 4.13 and Theorem 4.30, it follows that

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

□

Theorem 4.34. *Let* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(9,9)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

Proof. The proof is similar to Theorem 4.33. □

Theorem 4.35. *Let* $1 \le i \le 10$*. Suppose* $m = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \in T(i,10)$ *and* $m' = \begin{bmatrix} m_1 & 0 \\ 0 & m_2' \end{bmatrix} \in T(i,2)$*. Then, we have*

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

Proof. Let $h = \begin{bmatrix} m_1 & X \\ 0 & m_2 \end{bmatrix}$ and $h' = \begin{bmatrix} m_1 & X \\ 0 & m_2' \end{bmatrix}$ for $X \in \mathrm{M}(3,F)$. Let

$$M^{d,\beta}_{m_1,m_2} = \{X \in S(\beta) \mid \dim(\mathrm{Ker}(h-1)) = d\}$$

for $\beta \in F$. Clearly,

$$\mathrm{Ker}(m_2 - 1) = \mathrm{Ker}(m_2' - 1).$$

Hence for any $X \in \mathrm{M}(3,F)$,

$$X\,\mathrm{Ker}(m_2 - 1) = X\,\mathrm{Ker}(m_2' - 1)$$

and

$$X\,\mathrm{Ker}(m_2-1) \cap \mathrm{Im}(m_1-1) = X\,\mathrm{Ker}(m_2'-1) \cap \mathrm{Im}(m_1-1).$$

In particular, for any $\beta \in F$, we have that

$$M^{d,\beta}_{m_1,m_2} = M^{d,\beta}_{m_1,m_2'}. \tag{4.12}$$

Therefore,

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}(R_1 + R_2)$$

where

$$\begin{aligned} R_1 &= \sum_{d=1}^{6} \sum_{X \in M^{d,0}_{m_1,m_2}} \Theta_\theta(h)\overline{\psi_A({m_1}^{-1}X)} \\ &= \sum_{d=1}^{6} (-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,0}_{m_1,m_2})\overline{\psi_0(0)} \end{aligned}$$

and

$$\begin{aligned} R_2 &= \sum_{\beta\in F^\times} \sum_{d=1}^{6} \sum_{X \in M^{d,\beta}_{m_1,m_2}} \Theta_\theta(h)\overline{\psi_A({m_1}^{-1}X)} \\ &= \sum_{\beta\in F^\times} \sum_{d=1}^{6} (-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\# M^{d,\beta}_{m_1,m_2})\overline{\psi_0(\beta)} \end{aligned}$$

Thus,

$$\Theta_{N,\psi_A}(m) = \frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\#M^{d,0}_{m_1,m_2} + \sum_{\beta\in F^\times}\#M^{d,\beta}_{m_1,m_2}\overline{\psi_0(\beta)}).$$

Similarly,

$$\Theta_{N,\psi_A}(m') = \frac{1}{q^9}\sum_{d=1}^{6}(-1)^{6-1}(1-q)\cdots(1-q^{d-1})(\#M^{d,0}_{m_1,m'_2} + \sum_{\beta\in F^\times}\#M^{d,\beta}_{m_1,m'_2}\overline{\psi_0(\beta)}).$$

Hence, it follows from equation (4.12) that

$$\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(m').$$

□

Remark 4.36. Let $1 \leq i \leq j \leq 10$. To determine $\Theta_{N,\psi_A}(m)$ for $m \in T(i,j)$, it is enough to compute $\Theta_{N,\psi_A}(m)$ for $m \in E$. We illustrate this by an example.

Suppose that we want to compute the character value $\Theta_{N,\psi_A}(m)$ for $m \in T(3,7)$. From Proposition 4.28, it follows that $\Theta_{N,\psi_A}(m) = \Theta_{N,\psi_A}(k)$ for some $k \in T(3,1)$. By Theorem 4.13, we have $\Theta_{N,\psi_A}(k) = \Theta_{N,\psi_A}(x)$ for some $x \in T(1,3)$. Using Theorem 4.26, we have, $\Theta_{N,\psi_A}(x) = \Theta_{N,\psi_A}(y)$ for some $y \in T(1,1)$. Thus, using Theorem 4.17 we have

$$\Theta_{N,\psi_A}(m) = (1-q)^2.$$

For clarity, we give an example to illustrate the chain of computations used to determine the character value of an element in $T(i,j)$ using the character value of an element in the fundamental set E.

$$(3,7) \to (3,1) \to (1,3) \to (1,1).$$

We summarize the sequence of computations used to calculate $\Theta_{N,\psi_A}(m)$ for $m \in T(i,j)$, $1 \leq i \leq j \leq 5$ in Table 19.

TABLE 19. Sequence of computations for $\Theta_{N,\psi_A}(m)$

	Type-1	Type-2	Type-3	Type-4	Type-5
Type-1	$(1,1)$	$(1,2)$	$(1,3)\to(1,1)$	$(1,4)$	$(1,5)\to(1,1)$
Type-2	-	$(2,2)$	$(2,3)\to(2,1)\to(1,2)$	$(2,4)$	$(2,5)\to(2,1)\to(1,2)$
Type-3	-	-	$(3,3)\to(3,1)\to(1,3)\to(1,1)$	$(4,3)\to(4,1)\to(1,4)$	$(3,5)\to(3,1)\to(1,3)\to(1,1)$
Type-4	-	-	-	$(4,4)$	$(4,5)\to(4,1)\to(1,4)$
Type-5	-	-	-	-	$(5,5)\to(5,1)\to(1,5)\to(1,1)$

Table 20 summarizes the sequence of computations used to calculate $\Theta_{N,\psi_A}(m)$ for $m \in T(i,j)$, $1 \leq i \leq 5, 6 \leq j \leq 10, i \leq j$.

TABLE 20. Sequence of computations for $\Theta_{N,\psi_A}(m)$

	Type-6	Type-7	Type-8	Type-9	Type-10
Type-1	$(1,6) \to (1,1)$	$(1,7) \to (1,1)$	$(1,8) \to (1,2)$	$(1,9) \to (1,1)$	$(1,10) \to (1,2)$
Type-2	$(2,6) \to (2,1) \to (1,2)$	$(2,7) \to (2,1) \to (1,2)$	$(2,8) \to (2,2)$	$(2,9) \to (2,1) \to (1,2)$	$(2,10) \to (2,2)$
Type-3	$(3,6) \to (6,3) \to (6,1) \to (1,6) \to (1,1)$	$(3,7) \to (3,1) \to (1,3) \to (1,1)$	$(3,8) \to (3,2) \to (2,3) \to (2,1) \to (1,2)$	$(3,9) \to (9,3) \to (9,1) \to (1,9) \to (1,1)$	$(3,10) \to (3,2) \to (2,3) \to (2,1) \to (1,2)$
Type-4	$(4,6) \to (4,1) \to (1,4)$	$(4,7) \to (4,1) \to (1,4)$	$(4,8) \to (4,2) \to (2,4)$	$(4,9) \to (4,1) \to (1,4)$	$(4,10) \to (4,2) \to (2,4)$
Type-5	$(5,6) \to (6,5) \to (6,1) \to (1,6) \to (1,1)$	$(5,7) \to (5,1) \to (1,5) \to (1,1)$	$(5,8) \to (5,2) \to (2,5) \to (2,1) \to (1,2)$	$(5,9) \to (9,5) \to (9,1) \to (1,9) \to (1,1)$	$(5,10) \to (5,2) \to (2,5) \to (2,1) \to (1,2)$

Table 21 summarizes the sequence of computations used to calculate $\Theta_{N,\psi_A}(m)$ for $m \in T(i, j)$, $6 \leq i \leq 10, 6 \leq j \leq 10, i \leq j$.

TABLE 21. Sequence of computations for $\Theta_{N,\psi_A}(m)$

	Type-6	Type-7	Type-8	Type-9	Type-10
Type-6	$(6,6) \to (6,1) \to (1,6) \to (1,1)$	$(6,7) \to (6,1) \to (1,6) \to (1,1)$	$(6,8) \to (6,2) \to (2,6) \to (2,1) \to (1,2)$	$(6,9) \to (6,1) \to (1,6) \to (1,1)$	$(6,10) \to (6,2) \to (2,6) \to (2,1) \to (1,2)$
Type-7	-	$(7,7) \to (7,1) \to (1,7) \to (1,1)$	$(7,8) \to (7,2) \to (2,7) \to (2,1) \to (1,2)$	$(7,9) \to (9,7) \to (9,1) \to (1,9) \to (1,1)$	$(7,10) \to (7,2) \to (2,7) \to (2,1) \to (1,2)$
Type-8	-	-	$(8,8) \to (8,2) \to (2,8) \to (2,2)$	$(8,9) \to (9,8) \to (9,2) \to (2,9) \to (2,1) \to (1,2)$	$(8,10) \to (8,2) \to (2,8) \to (2,2)$
Type-9	-	-	-	$(9,9) \to (9,1) \to (1,9) \to (1,1)$	$(9,10) \to (9,2) \to (2,9) \to (2,1) \to (1,2)$
Type-10	-	-	-	-	$(10,10) \to (10,2) \to (2,10) \to (2,2)$

Acknowledgements

We thank Professor Dipendra Prasad for suggesting this problem and for some helpful discussions. Research of Kumar Balasubramanian is supported by the SERB grant: MTR/2019/000358.

References

[1] BALASUBRAMANIAN, KUMAR; KHURANA, HIMANSHI. A certain twisted Jacquet module of GL(4) over a finite field. *J. Pure Appl. Algebra* **226** (2022), no. 5, Paper No. 106932, 16 pp. MR4328653, Zbl 7455909, doi: 10.1016/j.jpaa.2021.106932. 874, 875, 877

[2] BERNŠTEĬN, JOSEPH N.; ZELEVINSKIĬ, ANDREY V. Representations of the group GL(n, F), where F is a local non-Archimedean field. *Uspehi Mat. Nauk* **31** (1976), no. 3(189), 5–70. MR0425030, Zbl 0342.43017, doi: 10.1070/RM1976v031n03ABEH001532. 875

[3] GELFAND, SERGEI I. Representations of the full linear group over a finite field, *Mat. Sb. (N. S.)* **83 (125)** (1970), 15–41. MR0272916, Zbl 0252.20034, doi: 10.1070/SM1970v012n01ABEH000907. 876

[4] PRASAD, DIPENDRA, The space of degenerate Whittaker models for general linear groups over a finite field. *Internat. Math. Res. Notices* (2000), no. 11, 579–595. MR1763857, Zbl 0983.22014, doi: https://doi.org/10.1155/S1073792800000313. 874, 876

[5] SERRE, JEAN-PIERRE. Linear representations of finite groups. Graduate Texts in Mathematics, 42. *Springer-Verlag, New York-Heidelberg*, 1977. x+170 pp. ISBN:0-387-90190-6. MR0450380, Zbl 0355.20006, doi: 10.1007/978-1-4684-9458-7. 877

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This paper is available via http://nyjm.albany.edu/j/2023/29-34.html.