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Fourier-theoretic inequalities for inclusions of simple C^*-algebras

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ABSTRACT. This paper originates from a naive attempt to establish various non-commutative Fourier-theoretic inequalities for an inclusion of simple C^*-algebras equipped with a conditional expectation of index-finite type. In this setting, we discuss the Hausdorff-Young inequality and Young's inequality. As a consequence, we prove the Hirschman–Beckner uncertainty principle and Donoho–Stark uncertainty principle. Our results generalize some of the results of Jiang, Liu and Wu [Noncommutative uncertainty principle, J. Funct. Anal., 270(1): 264–311, 2016].

CONTENTS

1. Introduction

Jones had discovered a notion of index $[M : N]$ for an inclusion of type II_1 factors $N \subset M$ in [10], which is now an active area of research in operator algebra having applications in various other fields of mathematics and mathematical physics. Subsequently, Kosaki [15] introduced a notion of index for an inclusion of type III factors. Both type II_1 and (σ-finite) type III factors are particular cases of more general objects, called simple C^*-algebras (i.e., C^*-algebras having no proper closed ideals). Thus, inclusion of simple C^*-algebras encompasses both type II_1 and type III subfactor theory. As a generalization

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of both Jones' and Kosaki's indices, Watatani [21] discussed index for an inclusion of C^*-algebras with a conditional expectation having a 'quasi-basis', a generalization of Pimsner-Popa basis [18]. Given a subfactor $N \subset M$ with $[M : N] < \infty$, Jones crucially observed that one can obtain another type II_1 factor M_1, called the basic construction, so that $[M_1 : M] = [M : N]$ and furthermore, this operation can be iterated to obtain a tower of basic constructions: $N \subset M \subset M_1 \subset \cdots \subset M_k \subset \cdots$. This was the key observation in establishing the famous 'Jones' index rigidity' result in [10]. It is well-known from the early days of subfactor theory that the higher relative commutants $N' \cap M_k$ and $M' \cap M_k$ have incredibly rich structures. Using the relative commutants, Popa had associated a 'standard invariant' to a subfactor : the λ-lattice [19], which is arguably the most powerful invariant of a subfactor. Subsequently, Jones discovered a pictorial description of the standard invariant what he called 'planar algebra' [11] and it becomes an indispensable tool in subfactor theory. In another direction, Ocneanu introduced a (fundamental) notion of Fourier transform $\mathcal{F}$ from $N' \cap M_1$ onto $M' \cap M_2$ (see also [3, 4] for details). This generalizes the classical notion of Fourier transform for a finite abelian group. Furthermore, using the Fourier transform, one can associate a new multiplication structure on $N' \cap M_1$ that generalizes the classical convolution. Fourier transform on the relative commutants plays a major role in the abstract subfactor theory; for example, the Fourier transform and 'rotation operators' are instrumental in the formalism of Jones' planar algebra, Ocneanu's paragroups and Popa's λ-lattice. Fourier transform also appeared naturally in Bisch's biprojection theory (see [2]) which is an indispensable tool in the theory of intermediate subfactors. In Jones' planar algebraic language, the Fourier transform, convolution and rotation operator have beautiful pictorial descriptions (see [11, 4]). Exploiting these pictorial formulations, in a recent paper [8] Jiang, Liu and Wu provided a non-commutative version of the Hausdorff-Young inequality, the Young's inequality and uncertainty principles for a subfactor $N \subset M$ with $[M : N] < \infty$ and $N' \cap M = \mathbb{C}$ (such a subfactor is called irreducible). Moreover, for any extremal subfactor which is not necessarily irreducible, these inequalities were proved in the Section 7 of [16] using planar algebraic techniques. However, the proofs use sphericality of the planar algebra and so are no longer valid for the non-extremal subfactors. We also mention some related works for Kac algebras and locally compact quantum groups, see for instance [9, 17].

Analogous to Jones' subfactor theory, given a unital inclusion of simple C^*-algebras $B \subset A$, recently in [1], one of the authors and Gupta have systematically developed a Fourier theory on the relative commutants $B' \cap A_k$ and $A' \cap A_k$, where $B \subset A \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$ is Watatani's tower of C^*-basic constructions (this notion parallels to the Jones' tower of basic construction in the C^*-world). More precisely, we have a notion of the Fourier transform $\mathcal{F} : B' \cap A_1 \to A' \cap A_2$, rotation map $\rho_+ : B' \cap A_1 \to B' \cap A_1$ and the convolution product $*\colon B' \cap A_1 \to B' \cap A_1$. The crucial ingredient that was used repeatedly is that of minimal conditional expectation and the minimal (Watatani) index $[A : B]_0$. In particular, for a subfactor $N \subset M$ with $[M : N] < \infty$ we have another notion of index $[M : N]_0$ and it is a fact that $N \subset M$ is 'extremal' if and only if $[M : N] = [M : N]_0$. In this paper, we prove a non-commutative version of the Hausdorff-Young inequality, Young's inequality and a couple of uncertainty principles in the setting of inclusion of simple C^*-algebras and in particular, for a subfactor (of both type II_1 factors and (σ-finite) type III-factors) with finite index that is not-necessarily extremal. Unlike [8], in the C^*-setting the main difficulty lies in the fact that it is still not known whether there are pictorial descriptions (similar to planar algebra) of the Fourier transform and convolution. However, we have found a way around using the relationship between quasi-basis and minimal conditional expectations as exploited in [1]. Our proofs are inspired by the corresponding proofs in [8].

2. Preliminaries

In this section, we fix the notations and briefly recall a few key ingredients which we repeatedly use in the sequel. For more details the readers are requested to see [21, 14, 1].

2.1. Non-commutative conditional expectation. Analogous to the classical case, in the theory of operator algebras we have a well-studied notion of conditional expectation. Suppose we have an inclusion of C^*-algebras $B \subset A$. All C^*-algebras considered in this article will be unital, and all inclusion $B \subset A$ of C^*-algebras will be considered as unital inclusion. A conditional expectation $E : A \to B$ is a linear surjective map satisfying

$$E(ba) = bE(a) \quad , \quad E(ab) = E(a)b \quad \text{and} \quad E(b) = b$$

for all $b \in B$ and $a \in A$. In particular, E is a norm one projection (see for instance in [13]). A C^*-inclusion may not have any conditional expectation. However, if we consider an inclusion of von Neumann algebras $N \subset M$ with M having a tracial state tr (i.e., a σ- weak-operator-topology(WOT) continuous linear functional $\mathrm{tr} : M \to \mathbb{C}$ satisfying $\mathrm{tr}(xy) = \mathrm{tr}(yx)$ for all $x, y \in M$ and $\mathrm{tr}(1) = 1$), there always exists a unique 'trace preserving' conditional expectation, denoted by E^M_N. More precisely, E^M_N is characterized by $\mathrm{tr}(nE^M_N(m)) = \mathrm{tr}(nm)$ for all $n \in N$ and $m \in M$.

Let us recall the very useful Kadison-Schwarz inequality involving conditional expectation which we shall use later.

Lemma 2.1 ([13]). *Suppose that N and M are von Neumann algebras acting on a Hilbert space $\mathcal{H}$, $N \subset M$, and $E : M \to N$ is a conditional expectation. Then, for all $x \in M$ one has $E(x)^*E(x) \leq E(x^*x)$.*

We remark that all conditional expectations in this paper are assumed to be faithful.

2.2. A quick look at C^*-index theory. Motivated by the Jones' index theory, Watatani developed a theory of index for inclusion of C^*-algebras. Given a pair $B \subset A$ of C^*-algebras, a conditional expectation $E : A \to B$ is said to be of *index-finite type* if there exists a finite set $\{\lambda_1, \dots, \lambda_n\} \subset A$ such that

$$x = \sum_{i=1}^n E(x\lambda_i)\lambda_i^* = \sum_{i=1}^n \lambda_i E(\lambda_i^* x)$$

for every $x \in A$. The set $\{\lambda_1, \dots, \lambda_n\}$ is called a *quasi-basis* for E (see [21]). The Watatani index of E is defined by

$$\mathrm{Ind}_w(E) = \sum_{i=1}^n \lambda_i\lambda_i^*,$$

and is independent of a quasi-basis. In general, $\mathrm{Ind}_w(E)$ is not a scalar but an invertible positive element in the center $\mathcal{Z}(A)$ of A. In particular, if A is a simple C^*-algebra, the index is scalar-valued. We denote by $\mathcal{E}_0(A, B)$ the set of all index-finite type conditional expectations from A onto B. A conditional expectation $F \in \mathcal{E}_0(A, B)$ is said to be minimal if it satisfies $\mathrm{Ind}_w(F) \leq \mathrm{Ind}_w(E)$ for all $E \in \mathcal{E}_0(A, B)$ (see [21] and the references therein). For inclusion of simple C^*-algebras, we have a privileged minimal conditional expectation as mentioned below.

Theorem 2.2. [21, Theorem 2.12.3] *Let $B \subset A$ be an inclusion of simple C^*-algebras such that $\mathcal{E}_0(A, B) \neq \emptyset$. Then, there exists a unique minimal conditional expectation E_0 from A onto B.*

For inclusion of simple C^*-algebras $B \subset A$, the minimal index is defined as

$$[A : B]_0 := \mathrm{Ind}_w(E_0).$$

As is customary, we shall denote $[A : B]_0$ by δ^2. We point out here that if $N \subset M$ is a subfactor with finite Jones index $[M : N]$ and is irreducible (i.e., $N' \cap M = \mathbb{C}$), then the trace preserving conditional expectation E_N^M is the minimal conditional expectation with $[M : N] = [M : N]_0$. In general, the minimal index and Jones index need not coincide. Indeed, a subfactor is extremal if and only if $[M : N] = [M : N]_0$. We also remark that irreducibility of a subfactor automatically implies extremality.

In the subfactor theory, Jones' basic construction plays a pivotal role. Using the language of the Hilbert C^*-module, Watatani proposed a parallel notion of basic construction in the C^*-world, the so-called C^*-basic construction. For the convenience of the reader we briefly recall it here and the details can be found in [21]. Let $B \subset A$ be an inclusion of C^*-algebras and $E_B \in \mathcal{E}_0(A, B)$. Then, A is a Hilbert B-module with respect to the B-valued inner product given by

$$\langle x, y\rangle_B = E_B(x^*y) \quad \text{for all } x, y \in A. \tag{2.1}$$

Recall that the space $\mathcal{L}_B(A)$ consisting of adjointable B-linear maps on A is a C^*-algebra. For each $a \in A$, consider $\lambda(a) \in \mathcal{L}_B(A)$ given by $\lambda(a)(x) = ax$ for $x \in A$. For $x \in A$, the association $x \mapsto E_B(x)$ is an adjointable projection on A, and is denoted by $e_B \in \mathcal{L}_B(A)$. The projection e_B is called the Jones projection for the pair $B \subset A$. The C^*-basic construction $C^*\langle A, e_B\rangle$ is defined to be the C^*-subalgebra generated by $\{\lambda(A), e_B\}$ in $\mathcal{L}_B(A)$. It turns out that $C^*\langle A, e_B\rangle$ equals the closure of the linear span of $\{\lambda(x)e_B\lambda(y) : x, y \in A\}$ in the C^*-algebra $\mathcal{L}_B(A)$; λ is an injective $*$-homomorphism, and thus we may consider A as a C^*-subalgebra of $C^*\langle A, e_B\rangle$. It is customary to denote the basic construction by A_1 if the conditional expectation is understood from the context, and it is a fact that A_1 is simple whenever $B \subset A$ is an inclusion of simple C^*-algebras. There exists a unique finite index conditional expectation $\widetilde{E}_B : A_1 \to A$ called the 'dual conditional expectation' of E_B. Furthermore, $\mathrm{Ind}_w(E_B) = \mathrm{Ind}_w(\widetilde{E}_B)$.

2.3. Tracial states on the relative commutants. Let $B \subset A$ be an inclusion of simple C^*-algebras and $E \in \mathcal{E}_0(A, B)$. Let E_0 be the unique minimal conditional expectation from A onto B. Suppose $B \subset A \subset A_1$ is the basic construction corresponding to E_0. Note that $[A_1 : A]_0 = [A : B]_0$. The dual conditional expectation $\widetilde{E}_0$ is also minimal [21]. We put $E_1 = \widetilde{E}_0$. Iterating the tower of C^*-basic construction for the inclusion $B \subset A$, we obtain

$$B \subset A \subset A_1 \subset A_2 \subset \cdots \subset A_k \subset \cdots$$

with unique (dual) minimal conditional expectations $E_k : A_k \to A_{k-1}$, $k \geq 0$, with the convention that $A_{-1} := B$ and $A_0 := A$. For each $k \geq 0$, let e_k be the Jones projection in A_k. The following extremely useful lemma is the C^*-analogue of the 'push-down lemma' in subfactor theory [18].

Lemma 2.3 ([1]). *If $x_1 \in A_1$, then there exists a unique $x_0 \in A$ such that $x_1 e_1 = x_0 e_1$, where $x_0 = [A : B]_0 E_1(x_1 e_1)$.*

Let $B' \cap A_k := \{x \in A_k : xb = bx \text{ for all } b \in B\}$ be the relative commutants of B in A_k. It is known that for each k, the relative commutants are finite dimensional [21]. On each $B' \cap A_k$, using the minimal conditional expectations, one obtains a consistent 'Markov type trace' (Proposition 2.21 in [1]). More precisely, for each $k \geq 0$, the map $\mathrm{tr}_k : B' \cap A_k \to \mathbb{C}$ defined by $\mathrm{tr}_k = (E_0 \circ E_1 \circ \cdots \circ E_k)_{|_{B' \cap A_k}}$ becomes a faithful traical state on $B' \cap A_k$.

Proposition 2.4 ([1]). *For each $k \geq 0$, $B' \cap A_k$ admits a faithful tracial state* tr_k *such that*

$$\mathrm{tr}_k(xe_k) = \delta^{-2}\mathrm{tr}_{k-1}(x) \ \textit{ for all } x \in B' \cap A_{k-1} \tag{2.2}$$

and $\mathrm{tr}_k|_{B' \cap A_{k-1}} = \mathrm{tr}_{k-1}$ *for all $k \geq 1$.*

We shall sometimes drop k and denote tr_k simply by tr for notational brevity. The following lemma is very useful.

Lemma 2.5 ([1]). *Let $\{\lambda_i : 1 \leq i \leq n\} \subset A$ be a quasi-basis for the minimal conditional expectation E_0. Then, the* tr*-preserving conditional expectation from $B' \cap A_k$ onto $A' \cap A_k$ is given by the following,*

$$E^{B' \cap A_k}_{A' \cap A_k}(x) = \frac{1}{[A : B]_0} \sum_i \lambda_i x \lambda_i^*, \quad x \in B' \cap A_k.$$

Recall that if $\{\lambda_i : 1 \leq i \leq n\} \subset A$ is a quasi-basis for E_0, then we have (see [21])

$$\sum_i \lambda_i e_1 \lambda_i^* = 1. \tag{2.3}$$

Corollary 2.6 ([1]). *Following the notation in Lemma* 2.5*, we have* $E^{B'\cap A_1}_{A'\cap A_1}(e_1) = \delta^{-2}$.

Remark 2.7. The reader should note that in [8] the authors have taken unnormalized trace Tr_k on $N' \cap M_k$, and thus $Tr_2(e_1) = 1$. More precisely, $\mathrm{tr}_k(x) = \delta^{-(k+1)} Tr_k(x)$ for any $x \in B' \cap A_k$.

2.4. Some useful inequalities. For the convenience of the reader, we recall a few important inequalities as mentioned in [8].

Definition 2.8. *For $x \in B' \cap A_1$, we define the p-norm of x for $1 \leq p < \infty$ as follows :*

$$||x||_p = (\mathrm{tr}(|x|^p))^{\frac{1}{p}};$$

and for $p = \infty$

$$||x||_\infty = ||x||,$$

where $||.||$ denotes the operator norm and tr *denotes the Markov type trace on $B' \cap A_1$ as in $Proposition$* 2.4*.*

Proposition 2.9 (Hölder's Inequality). [22] *For any x, y, z in $B' \cap A_1$, we have the following,*

(i) $|\mathrm{tr}(xy)| \leq ||x||_p ||y||_q$, *where* $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$;

(ii) $|\mathrm{tr}(xyz)| \leq ||x||_p ||y||_q ||z||_r$, *where* $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$;

(iii) $||xy||_r \leq ||x||_p ||y||_q$, *where* $1 \leq p, q, r \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$.

Proposition 2.10 ([22]). *For any x in $B' \cap A_1$ and $1 \leq p < \infty$, we have*

$$||x||_p = \sup\{|\mathrm{tr}(xy)| \, : \, y \in B' \cap A_1, ||y||_q \leq 1\},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 2.11 ([15]). *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Suppose that $T \, : \, \mathcal{M} \to \mathcal{M}$ is a linear map. If*

$$||Tx||_{p_1} \leq K_1 ||x||_{q_1} \quad \text{and} \quad ||Tx||_{p_2} \leq K_2 ||x||_{q_2},$$

then for any $\theta \in [0, 1]$,

$$||Tx||_{p_\theta} \leq K_1^{1-\theta} K_2^{\theta} \, ||x||_{q_\theta}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$ *and* $\frac{1}{q_\theta} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

3. Revisit of non-commutative Fourier theory

Analogous to subfactor theory, in [1] the authors have provided a Fourier theory using Watatani's notions of index and C^*-basic construction of certain inclusions of C^*-algebras. In this section we further investigate this Fourier theory and its properties which we shall use in the sequel. Throughout this section let $B \subset A$ denote an inclusion of simple C^*-algebras with a conditional expectation of finite Watatani index.

3.1. The Rotation maps. The Fourier transform of paragroups for a finite depth subfactor was first introduced by Ocneanu and as already mentioned in the introduction, it plays a major role in the development of the subfactor theory. More generally, for any extremal subfactor an explicit formula for the Fourier transform on the higher relative commutants was given by Bisch in (Def. 2.16, [3]) (see also [4] for many interesting results involving the Fourier transforms). The subtle difference between the Fourier theory for C^*-inclusion as in [1] and that of subfactor theory lies in the fact that, unlike for finite factors, we neither have a tracial state on the C^*-algebra to begin with nor the 'modular conjugation operator'.

Definition 3.1. *For each $k \geq 0$, the Fourier transform $\mathcal{F}_k : B' \cap A_k \longrightarrow A' \cap A_{k+1}$ is defined by the following,*

$$\mathcal{F}_k(x) = \delta^{k+2}\, E^{B'\cap A_{k+1}}_{A'\cap A_{k+1}}(xe_{k+1}e_k \cdots e_2e_1)\,.$$

The inverse Fourier transform $\mathcal{F}_k^{-1} : A' \cap A_{k+1} \longrightarrow B' \cap A_k$ is defined by the following,

$$\mathcal{F}_k^{-1}(y) = \delta^{k+2}\, E_{k+1}(ye_1e_2 \cdots e_ke_{k+1})\,.$$

The meaning of "inverse" in the preceding definition is justified by the fact that $\mathcal{F}_k \circ \mathcal{F}_k^{-1} = \mathrm{id}_{A'\cap A_{k+1}}$ and $\mathcal{F}_k^{-1} \circ \mathcal{F}_k = \mathrm{id}_{B'\cap A_k}$ for all $k \geq 0$ (Proposition 3.2 in [1]). In this paper, we will be mainly interested in the case of $k = 1$ and hence for simplicity, we denote the Fourier transform $\mathcal{F}_1$ by $\mathcal{F}$. Recall that both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are isometries with respect to the norm given by $\|x\|_2 = (\mathrm{tr}(x^*x))^{\frac{1}{2}}$ (Theorem 3.5 in [1]).

We now revisit the rotation map defined in (Definition 3.7 in [1]) and derive a few more properties of it. Recall the rotation map $\rho_+ : B' \cap A_1 \longrightarrow B' \cap A_1$ defined by,

$$\rho_+(x) = (\mathcal{F}^{-1}((\mathcal{F}(x))^*))^*\,.$$

It is known that ρ_+ is a unital involutive $*$-preserving anti-automorphism (Remark 3.11 and Theorem 3.16 in [1]), and hence $\|x\|_\infty = \|\rho_+(x)\|_\infty$. It is also shown in [1] that if the inclusion $B \subset A$ is irreducible, then ρ_+ is tr-preserving.

Analogous to ρ_+, we can define a rotation operator $\rho_- : A' \cap A_2 \longrightarrow A' \cap A_2$ by the following,

$$\rho_-(w) = (\mathcal{F}((\mathcal{F}^{-1}(w))^*))^*.$$

Lemma 3.2. *We have $\rho_- = \mathcal{F} \circ \rho_+ \circ \mathcal{F}^{-1}$. In other words, the diagram in Figure 1 commutes.*

Proof: Since ρ_+ is $*$-preserving, we have the following,

$$\mathcal{F}(x)^* = \mathcal{F} \circ \rho_+(x^*). \tag{3.1}$$

Now again from the definition,

$$\rho_-(\mathcal{F}(x)) = (\mathcal{F}(x^*))^* = \mathcal{F} \circ \rho_+(x), \tag{3.2}$$

where the last equality follows from Equation (3.1). Thus, $\rho_- \circ \mathcal{F} = \mathcal{F} \circ \rho_+$. $\square$

$$\begin{array}{ccc} B' \cap A_1 & \xrightarrow{\mathcal{F}} & A' \cap A_2 \\ {\scriptstyle\rho_+}\big\downarrow & & \big\downarrow{\scriptstyle\rho_-} \\ B' \cap A_1 & \xrightarrow{\mathcal{F}} & A' \cap A_2 \end{array}$$

FIGURE 1. Relation between ρ_+ and ρ_-

Next we show that ρ_- also satisfies properties similar to ρ_+.

Proposition 3.3. *The rotation operator ρ_- is a unital involutive $*$-preserving anti-automorphism.*

Proof: First we show that ρ_- is $*$-preserving. To see this, for any $w \in A' \cap A_2$ we observe the following,

$$\begin{aligned} w^* &= (\mathcal{F}(\mathcal{F}^{-1}(w)))^* \\ &= \mathcal{F} \circ \rho_+((\mathcal{F}^{-1}(w))^*) \end{aligned} \tag{3.3}$$

using Equation (3.1). Applying $\mathcal{F}^{-1}$ on both sides of the Equation (3.3) we get the following,

$$\mathcal{F}^{-1}(w^*)) = \rho_+((\mathcal{F}^{-1}(w))^*). \tag{3.4}$$

Now apply ρ_+ on both sides of Equation (3.4) and use the fact that $\rho_+^2 = \mathrm{id}$ to get the following,

$$\begin{aligned} (\mathcal{F}^{-1}(w))^* &= \rho_+ \circ \mathcal{F}^{-1}(w^*) \\ &= \mathcal{F}^{-1} \circ \rho_-(w^*) \end{aligned} \tag{3.5}$$

using Lemma 3.2. Finally, apply $\mathcal{F}$ on both sides of Equation (3.5) and use the definition of ρ_- to conclude that ρ_- is $*$-preserving. The fact that $\rho_-^2 = \mathrm{id}$ is an easy consequence of $\rho_+^2 = \mathrm{id}$ and Lemma 3.2.

It remains to show that ρ_- is a unital anti-homomorphism. If $\{\lambda_i : i \in I\}$ is a quasi-basis for E_0, then by Lemma 2.5 we obtain the following,

$$\begin{aligned} \rho_-(w) &= (\mathcal{F}((\mathcal{F}^{-1}(w))^*))^* \\ &= \delta^3 E^{B' \cap A_2}_{A' \cap A_2}(e_1 e_2 \mathcal{F}^{-1}(w)) \\ &= \delta \sum_i \lambda_i e_1 e_2 \mathcal{F}^{-1}(w) \lambda_i^* \\ &= \delta^4 \sum_i \lambda_i e_1 e_2 E_2(w e_1 e_2) \lambda_i^* \, . \end{aligned} \tag{3.6}$$

Since, $e_1 e_2 e_1 = \delta^{-2} e_1$ and $E_2(e_2) = \delta^{-2}$, we have

$$\rho_-(1) = \delta^2 \textstyle\sum_i \lambda_i e_1 E_2(e_2) \lambda_i^* = \sum_i \lambda_i e_1 \lambda_i^* = 1.$$

This last equality follows from Equation (2.3). As ρ_- is $*$-preserving, to show that it is an anti-homomorphism, it is enough to show that, $\rho_-(w_1)\rho_-(w_2)^* =$

$\rho_-(w_2^*w_1)$, for any $w_1, w_2 \in A' \cap A_2$. By Equation (3.6) and Lemma 2.3 we finally get the following,

$$\begin{aligned}
\rho_-(w_1)\rho_-(w_2)^* &= \delta^8 \sum_{i,j} \lambda_i e_1 e_2 E_2(w_1 e_1 e_2)\lambda_i^* \lambda_j E_2(e_2 e_1 w_2^*) e_2 e_1 \lambda_j^* \\
&= \delta^6 \sum_{i,j} \lambda_i e_1 E_1 \circ E_2(w_1 e_1 \lambda_i^* \lambda_j e_2 e_1 w_2^*) e_2 e_1 \lambda_j^* \\
&= \delta^6 \sum_{i,j} \lambda_i e_1 E_1 \circ E_2(e_1 \lambda_i^* \lambda_j e_2 e_1 w_2^* w_1) e_2 e_1 \lambda_j^* \text{ (Lemma 3.11 in [14])} \\
&= \delta^6 \sum_{i,j} \lambda_i e_1 E_1\big(e_1 E_2(\lambda_i^* \lambda_j e_2 e_1 w_2^* w_1)\big) e_2 e_1 \lambda_j^* \\
&= \delta^4 \sum_{i,j} \lambda_i e_1 E_2(\lambda_i^* \lambda_j e_2 e_1 w_2^* w_1) e_2 e_1 \lambda_j^* \text{ (by Lemma 2.3)} \\
&= \delta^4 \sum_{i,j} \lambda_i e_1 \lambda_i^* \lambda_j E_2(e_2 e_1 w_2^* w_1) e_2 e_1 \lambda_j^* \\
&= \delta^4 \sum_{j} \lambda_j E_2(e_2 e_1 w_2^* w_1) e_2 e_1 \lambda_j^* \\
&= \rho_-(w_1^* w_2)^* \\
&= \rho_-(w_2^* w_1)
\end{aligned}$$

and this completes the proof. □

Proposition 3.4. *If $B \subset A$ is irreducible, then ρ_- is a* tr*-preserving map on $A' \cap A_2$, where* tr *on $A' \cap A_2$ is the restriction of* tr *on $B' \cap A_2$.*

Proof: For $w \in A' \cap A_2$, there exists a unique $x \in B' \cap A_1$ such that $w = \mathcal{F}(x)$. By Lemma 3.2, we have

$$\mathrm{tr}(\rho_-(w)) = \mathrm{tr}(\rho_- \circ \mathcal{F}(x)) = \mathrm{tr}(\mathcal{F} \circ \rho_+(x)) = \delta^3\, \mathrm{tr}\big(E^{B'\cap A_2}_{A'\cap A_2}(\rho_+(x) e_2 e_1)\big).$$

Since $E^{B'\cap A_2}_{A'\cap A_2}$ is tr-preserving, we get by Proposition 2.4,

$$\mathrm{tr}(\rho_-(w)) = \delta^3\, \mathrm{tr}(\rho_+(x) e_2 e_1) = \delta\, \mathrm{tr}(e_1 \rho_+(x)) = \delta\, \mathrm{tr}(\rho_+(x e_1)).$$

Here, the last equality follows from the fact that $\rho_+(e_1) = e_1$ and ρ_+ is an anti-homomorphism. Since ρ_+ is tr-preserving, we immediately obtain the following,

$$\begin{aligned}
\mathrm{tr}(\rho_+(x e_1)) = \mathrm{tr}((x e_1)) &= \delta^2\, \mathrm{tr}(x e_2 e_1) = \delta^2\, \mathrm{tr}(E^{B'\cap A_2}_{A'\cap A_2}(x e_2 e_1)) \\
&= \delta^{-1} \mathrm{tr}(\mathcal{F}(x)) = \delta^{-1} \mathrm{tr}(w).
\end{aligned}$$

Therefore, we have $\mathrm{tr}(\rho_-(w)) = \mathrm{tr}(w)$ as desired. □

Corollary 3.5. *Let $1 \le p \le \infty$. For an irreducible inclusion $B \subset A$, we have*

$$\|x\|_p = \|\rho_+(x)\|_p \qquad \textit{and} \qquad \|w\|_p = \|\rho_-(w)\|_p$$

for $x \in B' \cap A_1$ and $w \in A' \cap A_2$.

Remark 3.6. Note that Corollary 3.5 need not be true for non-irreducible simple C^*-inclusions for $p \neq \infty$. For extremal II_1 factors, not necessarily irreducible, an easy pictorial calculation shows that ρ_+ (resp. ρ_-) being tr-preserving is equivalent to Figure 2, which in turn is equivalent to the sphericality, and hence Corollary 3.5 holds.

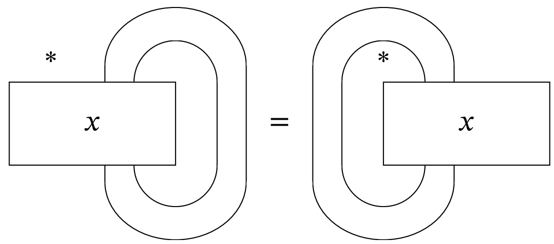

FIGURE 2. $\mathrm{tr}(x) = \mathrm{tr}\circ\rho_+(x)$

3.2. Convolution. Using the Fourier transform, we can introduce a new multiplication structure on the relative commutant $B' \cap A_1$ (resp. $A' \cap A_2$), which we call the *convolution* product. This is defined formally below.

Definition 3.7. [1] *The convolution product of two elements x and y in $B' \cap A_1$, denoted by $x * y$, is defined as*

$$x * y = \mathcal{F}^{-1}\big(\mathcal{F}(y)\mathcal{F}(x)\big).$$

Similarly, for any two elements $w, z \in A' \cap A_2$ we define

$$w * z = \mathcal{F}\big(\mathcal{F}^{-1}(z)\mathcal{F}^{-1}(w)\big).$$

Recall that (Lemma 3.20, [1]) the convolution * is associative. We now prove that it is well behaved with the adjoint operation.

Proposition 3.8. *For $x, y \in B' \cap A_1$, we have $(x * y)^* = (x^*) * (y^*)$. Similarly, for $w, z \in A' \cap A_2$, we have $(w * z)^* = (w^*) * (z^*)$.*

Proof: Let $x, y \in B' \cap A_1$. Using Proposition 3.3, Equations (3.5 , 3.1) and Lemma 3.2 we observe the following,

$$\begin{aligned}
(x * y)^* &= (\mathcal{F}^{-1}(\mathcal{F}(y)\mathcal{F}(x)))^* \\
&= \mathcal{F}^{-1}(\rho_-((\mathcal{F}(y)\mathcal{F}(x))^*)) \\
&= \mathcal{F}^{-1}(\rho_-((\mathcal{F}(x))^*(\mathcal{F}(y))^*)) \\
&= \mathcal{F}^{-1}(\rho_-((\mathcal{F}(y))^*)\rho_-((\mathcal{F}(x))^*)) \\
&= \mathcal{F}^{-1}(\rho_-\circ\mathcal{F}\circ\rho_+(y^*)\rho_-\circ\mathcal{F}\circ\rho_+(x^*)) \\
&= \mathcal{F}^{-1}(\mathcal{F}(y^*)\mathcal{F}(x^*)) \\
&= x^* * y^*,
\end{aligned}$$

which proves the first assertion, and the second assertion follows similarly. $\square$

We finally show that ρ_+ and ρ_- are anti-multiplicative with respect to the convolution.

Proposition 3.9. *For $x, y \in B' \cap A_1$, we have $\rho_+(x * y) = \rho_+(y) * \rho_+(x)$. Similarly, for $w, z \in A' \cap A_2$ one has $\rho_-(w * z) = \rho_-(z) * \rho_+(w)$.*

Proof: Observe that by Equations (3.2 , 3.4) and Proposition 3.8, we get

$$\begin{aligned}
\rho_+(y) * \rho_+(x) &= \mathcal{F}^{-1}(\mathcal{F}\circ\rho_+(x)\mathcal{F}\circ\rho_+(y)) \\
&= \mathcal{F}^{-1}((\mathcal{F}(x^*))^*(\mathcal{F}(y^*))^*) \\
&= \mathcal{F}^{-1}((\mathcal{F}(y^*)\mathcal{F}(x^*))^*) \\
&= \rho_+((\mathcal{F}^{-1}((\mathcal{F}(y^*)\mathcal{F}(x^*))^*) \\
&= \rho_+((x^* * y^*)^*) \\
&= \rho_+(x * y)\,,
\end{aligned}$$

proving that ρ_+ is anti-multiplicative. A similar computation proves the result for ρ_-. □

4. Fourier-theoretic inequalities

In this section we prove the Hausdorff-Young inequality and Young's inequality for inclusion $B \subset A$ of simple C^*-algebras which is not necessarily irreducible. We also provide various uncertainty principles on the second relative commutant of such an inclusion. In the non-commutative world, these Fourier-theoretic inequalities were first established in [8] for a finite index irreducible subfactor. We prove the non-commutative version of these inequalities for inclusion of simple C^*-algebras with a conditional expectation of index-finite type by finding the correct constants.

Notation: To avoid notational difficulty, for $x \in B' \cap A_1$ and $w \in A' \cap A_2$, we denote $\rho_+(x)$ and $\rho_-(w)$ by $\overline{x}$ and $\overline{w}$ respectively. In the case of inclusion of extremal II_1 factors, ρ_+ coincides with the 2-click rotation in the anti-clockwise direction, and hence it is consistent with the notations in [8].

Define

$$\varkappa_0^+ = \min\{\mathrm{tr}(p) : p \in \mathcal{P}(B' \cap A)\},$$

$$\varkappa_0^- = \min\{\mathrm{tr}(q) : q \in \mathcal{P}(A' \cap A_1)\}$$

and

$$\varkappa_0 = \sqrt{\varkappa_0^+\varkappa_0^-}.$$

Recall that, $\delta = \sqrt{[A : B]_0}$.

4.1. Hausdorff-Young inequality. Goal of this subsection is to prove the following non-commutative analogue of the classical Hausdorff-Young inequality for inclusion of simple C^*-algebras $B \subset A$ with a conditional expectation of index-finite type.

Theorem 4.1 (Hausdorff-Young inequality)**.** *Let $B \subset A$ be an inclusion of simple C^*-algebras with a conditional expectation of index-finite type. For any $x \in B' \cap A_1$,*

$$\|x\|_q \leq \|\mathcal{F}(x)\|_p \leq \left(\frac{\delta}{\kappa_0}\right)^{1-\frac{2}{p}} \|x\|_q$$

where, $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The main point is Proposition 4.6 which is instrumental in proving Theorem 4.1. To begin with, we prove a few useful lemmas.

Lemma 4.2. *For $x \in B' \cap A_1$, we have $\mathcal{F}(x)\mathcal{F}(x)^* = \delta^2 E^{B'\cap A_2}_{A'\cap A_2}(xe_2x^*)$.*

Proof: Let $\{\lambda_1, \cdots, \lambda_n\} \subset A$ be a *quasi-basis* for the minimal conditional expectation E_0. Using Lemma 2.5, we observe that for $x \in B' \cap A_1$ the following holds :

$$\begin{aligned}
\mathcal{F}(x)\mathcal{F}(x)^* &= \delta^6 E^{B'\cap A_2}_{A'\cap A_2}(xe_2e_1)E^{B'\cap A_2}_{A'\cap A_2}(e_1e_2x^*)\\
&= \delta^6 E^{B'\cap A_2}_{A'\cap A_2}\Big(xe_2e_1E^{B'\cap A_2}_{A'\cap A_2}(e_1e_2x^*)\Big)\\
&= \delta^4 E^{B'\cap A_2}_{A'\cap A_2}\big(\sum_i xe_2e_1\lambda_ie_1e_2x^*\lambda_i^*\big)\\
&= \delta^4 E^{B'\cap A_2}_{A'\cap A_2}\big(\sum_i xE_1(E_0(\lambda_i)e_1)e_2x^*\lambda_i^*\big)\\
&= \delta^2 E^{B'\cap A_2}_{A'\cap A_2}\big(\sum_i xE_0(\lambda_i)e_2x^*\lambda_i^*\big)\\
&= \delta^2 E^{B'\cap A_2}_{A'\cap A_2}\Big(xe_2x^*\big(\sum_i E_0(\lambda_i)\lambda_i^*\big)\Big)\\
&= \delta^2 E^{B'\cap A_2}_{A'\cap A_2}(xe_2x^*)
\end{aligned}$$

which completes the proof. □

Recall the following well-known result from the basic von Neumann algebra theory (see item 2.17 in [20], for instance).

Lemma 4.3. *If $0 \leq a \leq 1$ and p is a projection, then $0 \leq a \leq p$ if and only if $a = ap$.*

Lemma 4.4. *Suppose that $v \in B' \cap A_1$ is a non-zero partial isometry. Then, we have the following.*

(*i*) $E_1(v^*v) \leq \frac{1}{\kappa_0^+}\|v\|_1$.

(ii) $E^{B'\cap A_1}_{A'\cap A_1}(vv^*) \leq \dfrac{1}{\kappa_0^-}\|v\|_1$.

Proof: First note that $p = v^*v$ is a projection in $B' \cap A_1$ and so,

$$\|v\|_1 = \mathrm{tr}|v| = \mathrm{tr}((v^*v)^{\frac{1}{2}}) = \mathrm{tr}(v^*v).$$

Take minimal projections $\{e_k\}_k \subset B' \cap A$ such that $\sum_k e_k = 1$. Thus, there are scalars $\alpha_k \geq 0$ such that $E_1(v^*v) = \sum_k \alpha_k e_k$. It follows that

$$tr(v^*v) = \textstyle\sum_k \alpha_k \mathrm{tr}(e_k) \geq \kappa_0^+ \sum_k \alpha_k \geq \kappa_0^+ \sum_k \alpha_k e_k = \kappa_0^+ E_1(v^*v).$$

This completes the proof for the first part. The proof for the other part is similar and we omit it. □

Lemma 4.5. *Suppose that $v \in B' \cap A_1$ is a non-zero partial isometry. Then, we have*

$$ve_2v^* \leq \frac{\|v\|_1}{\kappa_0^+}\, vv^*.$$

Proof: Let $p = vv^*$ be the range projection of v. Now, using Lemma 4.4 we get

$$ve_2v^*.ve_2v^* = vE_1(v^*v)e_2v^* \leq \tfrac{\|v\|_1}{\kappa_0^+} ve_2v^*.$$

After taking ∞-norm on both sides of the above inequality we have

$$\left\|\frac{\kappa_0^+}{\|v\|_1} ve_2v^*\right\|_\infty \leq 1$$

and hence, $0 \leq \frac{\kappa_0^+}{\|v\|_1} ve_2v^* \leq 1$. Since v is a partial isometry, we have

$$\Big(\frac{\kappa_0^+}{\|v\|_1} ve_2v^*\Big)\big(vv^*\big) = \frac{\kappa_0^+}{\|v\|_1} ve_2v^*.$$

Thus the proof follows by Lemma 4.3 with $a = \frac{\kappa_0^+}{\|v\|_1} ve_2v^*$ and $p = vv^*$. □

Proposition 4.6. *For $x \in B' \cap A_1$, we have*

$$\|x\|_1 \leq \|\mathcal{F}(x)\|_\infty \leq \frac{\delta}{\kappa_0}\|x\|_1.$$

Proof: Note that by the Kadison-Schwarz inequality we have $\|x\|_1 \leq \|x\|_2$, and we also know that $\|x\|_2 \leq \|x\|_\infty$. Since $\|\mathcal{F}(x)\|_2 = \|x\|_2$, it follows that $\|x\|_1 \leq \|\mathcal{F}(x)\|_\infty$. It remains to prove $\|\mathcal{F}(x)\|_\infty \leq \dfrac{\delta}{\kappa_0}\|x\|_1$. We first prove the inequality for a partial isometry and then appealing to rank-one decomposition as depicted in [8] proves the result for a general $x \in B' \cap A_1$. Let v be a partial isometry in $B' \cap A_1$. By Lemma 4.2 and Lemma 4.5 we have the following,

$$\mathcal{F}(v)\mathcal{F}(v)^* = \delta^2 E^{B'\cap A_2}_{A'\cap A_2}(ve_2v^*) \leq \frac{\|v\|_1}{\kappa_0^+}\delta^2 E^{B'\cap A_1}_{A'\cap A_1}(vv^*).$$

Now, using Lemma 4.4(ii), we obtain the following,

$$\mathcal{F}(\nu)\mathcal{F}(\nu)^* \leq \frac{\delta^2}{\kappa_0^2}\|\nu\|_1^2.$$

Applying $\| \, . \, \|_\infty$ on both sides and taking square root finishes the proof for ν. For an arbitrary $x \in B' \cap A_1$, let $x = \sum_k \alpha_k \nu_k$ be the rank- one decomposition of x. Then, $\|x\|_1 = \sum_k \alpha_k \|\nu_k\|_1$ and hence we have the following,

$$\|\mathcal{F}(x)\|_\infty \leq \sum_k \alpha_k \|\mathcal{F}(\nu_k)\|_\infty \leq \frac{\delta}{\kappa_0} \sum_k \alpha_k \|\nu_k\|_1 = \frac{\delta}{\kappa_0}\|x\|_1 \, ,$$

and this completes the proof. □

Proof of Theorem 4.1: Using Proposition 4.6 and the fact that $\mathcal{F}$ and $\mathcal{F}^{-1}$ both are isometries with respect to the norm given by $\|x\|_2 = (\mathrm{tr}(x^*x))^{\frac{1}{2}}$, we have the following,

$$\|\mathcal{F}(x)\|_\infty \leq \frac{\delta}{\kappa_0}\|x\|_1 \quad \text{ and } \quad \|\mathcal{F}(x)\|_2 = \|x\|_2 \, .$$

The proof of the inequality $\|\mathcal{F}(x)\|_p \leq \left(\frac{\delta}{\kappa_0}\right)^{1-\frac{2}{p}} \|x\|_q$ is now clear from Proposition 2.11 with $p_1 = \infty$, $q_1 = 1$, $p_2 = 2$, $q_2 = 2$, $K_1 = \frac{\delta}{\kappa_0}$, $K_2 = 1$ and $\theta = \frac{2}{p}$. The proof of $\|x\|_q \leq \|\mathcal{F}(x)\|_p$ is similar. □

As a corollary of Theorem 4.1, we obtain the following Hausdorff-Young inequality for a finite index subfactor not necessarily irreducible and in particular, in the extremal case we recover the result in (Theorem 7.3, [8]).

Corollary 4.7. *Let $N \subset M$ be a subfactor with finite Jones index. Then, for any $x \in N' \cap M_1$,*

$$\|x\|_q \leq \|\mathcal{F}(x)\|_p \leq \left(\frac{\sqrt{[M:N]_0}}{\kappa_0}\right)^{1-\frac{2}{p}} \|x\|_q,$$

where, $2 \leq p \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

4.2. Non-commutative uncertainty principles. Motivated by [8], we prove the Donoho-Stark uncertainty principle and Hirschman-Beckner uncertainty principle for inclusion of simple C^*-algebras $B \subset A$ with a conditional expectation of index-finite type. Our proofs are essentially applications of Section 4.1 and marginal modification of the proofs in [8] with revised constants.

Recall that for $x \in B' \cap A_1$, the range projection x is the smallest projection $l(x) \in B' \cap A_1$ such that $l(x)x = x$. Now if $x = \sum_j \lambda_j \nu_j$ is the rank-one decomposition of x, then it is easy to see the following,

$$l(x) = \sum_j \nu_j \nu_j^*. \tag{4.1}$$

For $x \in B' \cap A_1$, we denote $\mathcal{S}(x) = \mathrm{tr}(l(x))$.

Theorem 4.8 (Donoho-Stark uncertainty principle)**.** *Consider an inclusion of simple C^*-algebras $B \subset A$ with a conditional expectation of index-finite type. For any non zero $x \in B' \cap A_1$, we have*

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \frac{\kappa_0^2}{[A:B]_0}.$$

In particular, if $N \subset M$ is a subfactor with finite Jones index, then for any non zero $x \in N' \cap M_1$ we have

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geq \frac{\kappa_0^2}{[M:N]_0}.$$

Proof: The proof is inspired by the proof of Theorem 5.2 in [8]. Let $x \in B' \cap A_1$ and $\mathcal{F}(x) = \sum_j \lambda_j \nu_j$ be the rank one decomposition of $\mathcal{F}(x)$. It is easy to see that $\mathcal{S}(\mathcal{F}(x)) = \sum_j \|\nu_j\|_1$. By Proposition 2.9 and Proposition 4.6 we have the following,

$$\begin{aligned}
\sup_j \lambda_j = \|\mathcal{F}(x)\|_\infty &\leq \frac{\delta}{\kappa_0}\|x\|_1 \\
&= \frac{\delta}{\kappa_0}\|l(x)x\|_1 \\
&= \frac{\delta}{\kappa_0}\|x\|_2\|l(x)\|_2 \\
&= \frac{\delta}{\kappa_0}\|\mathcal{F}(x)\|_2\|l(x)\|_2 \\
&= \frac{\delta}{\kappa_0}\|\mathcal{F}(x)\|_2(\mathcal{S}(x))^{\frac{1}{2}}. \qquad (4.2)
\end{aligned}$$

It is clear from the rank one decomposition of $\mathcal{F}(x)$ that $\|\mathcal{F}(x)\|_2 = (\sum_j \lambda_j^2 \|\nu_j\|_1)^{\frac{1}{2}}$. Hence, Equation (4.2) becomes,

$$\begin{aligned}
\sup_j \lambda_j &\leq \frac{\delta}{\kappa_0}\big(\sum_j \lambda_j^2\|\nu_j\|_1\big)^{\frac{1}{2}}(\mathcal{S}(x))^{\frac{1}{2}} \\
&\leq \frac{\delta}{\kappa_0}(\sup_j \lambda_j)\big(\sum_j \|\nu_j\|_1\big)^{\frac{1}{2}}(\mathcal{S}(x))^{\frac{1}{2}} \\
&= \frac{\delta}{\kappa_0}(\sup_j \lambda_j)(\mathcal{S}(\mathcal{F}(x)))^{\frac{1}{2}}(\mathcal{S}(x))^{\frac{1}{2}}
\end{aligned}$$

which completes the proof. □

Consider the continuous function $\eta : [0, \infty) \longrightarrow \mathbb{R}$ defined by

$$\eta(t) = \begin{cases} -t\log t & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases} \qquad (4.3)$$

Definition 4.9 (von Neumann entropy)**.** *For $x \in B' \cap A_1$, the von Neumann entropy of $|x|^2$ is defined by the following,*

$$H(|x|^2) = \mathrm{tr}(\eta(|x|^2)).$$

Theorem 4.10 (Hirschman-Beckner uncertainty principle)**.** *Let $B \subset A$ be an inclusion of simple C^*-algebras with a conditional expectation of index-finite type. For any $x \in B' \cap A_1$,*

$$\frac{1}{2}\big(H(|\mathcal{F}(x)|^2) + H(|x|^2)\big) \geq -\|x\|_2^2\Big(\log\Big(\frac{\delta}{\kappa_0}\Big) + \log\|x\|_2^2\Big).$$

In particular, if $\|x\|_2 = 1$, then we have

$$\frac{1}{2}\big(H(|\mathcal{F}(x)|^2) + H(|x|^2)\big) \geq -\log\Big(\frac{\delta}{\kappa_0}\Big).$$

Proof: The proof is a consequence of Theorem 4.1 and the standard argument as in Theorem 5.5 in [8]. However, we sketch the proof for completeness. Let $0 \neq x \in B' \cap A_1$ so that $\mathcal{F}(x) \neq 0$. By Theorem 4.1 we have the following,

$$\|\mathcal{F}(x)\|_p \leq \Big(\frac{\delta}{\kappa_0}\Big)^{1-\frac{2}{p}} \|x\|_q \tag{4.4}$$

where, $2 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider the following function,

$$f(p) = \log\|\mathcal{F}(x)\|_p - \log\|x\|_q - \log\Big(\frac{\delta}{\kappa_0}\Big)^{1-\frac{2}{p}}.$$

By Equation (4.4), we have $f(p) \leq 0$. Now, since $\mathcal{F}$ is an isometry with respect to $\|.\|_2$, we have $f(2) = 0$ and hence $f'(2) \leq 0$. Hence, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}p}\Big|_{p=2}\big(\|\mathcal{F}(x)\|_p^p\big) = -\frac{1}{2}H\big(|\mathcal{F}(x)|^2\big)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}p}\Big|_{p=2}\big(\log\|\mathcal{F}(x)\|_p\big) = -\frac{1}{4}\log\|\mathcal{F}(x)\|_2^2 - \frac{H(|\mathcal{F}(x)|^2)}{4\|\mathcal{F}(x)\|_2^2}.$$

Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}p}\Big|_{p=2}\big(\log\|x\|_q\big) = \frac{1}{4}\log\|x\|_2^2 + \frac{H(|x|^2)}{4\|x\|_2^2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}p}\Big|_{p=2}\Big(\log\Big(\frac{\delta}{\kappa_0}\Big)^{1-\frac{2}{p}}\Big) = \frac{1}{2}\log\Big(\frac{\delta}{\kappa_0}\Big).$$

Now, the above equations together with the facts $f'(2) \leq 0$ and $\|\mathcal{F}(x)\|_2 = \|x\|_2$ implies the following,

$$-\frac{1}{4}\log\|x\|_2^2 - \frac{1}{4}\frac{H(|\mathcal{F}(x)|^2)}{\|\mathcal{F}(x)\|_2^2} - \frac{1}{4}\log\|x\|_2^2 - \frac{1}{4}\frac{H(|x|^2)}{\|x\|_2^2} - \frac{1}{2}\log\frac{\delta}{\kappa_0} \leq 0. \tag{4.5}$$

A rearrangement of Equation (4.5) completes the proof. □

Corollary 4.11. *Let $N \subset M$ be a subfactor with finite Jones index. Then, for any $x \in N' \cap M_1$ we have*

$$\frac{1}{2}\big(H(|\mathcal{F}(x)|^2) + H(|x|^2)\big) \geq -\|x\|_2^2 \Big(\log\Big(\frac{\sqrt{[M:N]_0}}{\kappa_0}\Big) + \log \|x\|_2^2\Big).$$

In particular, if $\|x\|_2 = 1$, then we have

$$\frac{1}{2}\big(H(|\mathcal{F}(x)|^2) + H(|x|^2)\big) \geq -\log\Big(\frac{\sqrt{[M:N]_0}}{\kappa_0}\Big).$$

4.3. Young's inequality. Goal of this subsection is to prove the Young's inequality. Throughout this subsection we fix an (not necessarily irreducible) inclusion of simple C^*-algebras $B \subset A$ with a conditional expectation of finite Watatani index.

Theorem 4.12 (Young's Inequality)**.** *Suppose $B \subset A$ is an inclusion of simple C^*-algebras with a conditional expectation of index-finite type. Then, for any $x, y \in B' \cap A_1$, we have*

$$\|x * y\|_r \leq \frac{\delta}{\kappa_0^+}\Big(\frac{\|y\|_1}{\|\overline{y}\|_1}\Big)^{\frac{1}{r}} \|x\|_p \|\overline{y}\|_q$$

where, $1 \leq p, q, r \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

As a corollary, we prove Young's inequality for a subfactor which is not necessarily extremal. Recall that, a subfactor is extremal if and only if $[M:N] = [M:N]_0$ and furthermore, in the extremal case for any $y \in N' \cap M_1$ we have $\|y\|_1 = \|\overline{y}\|_1$ (see Remark 3.6). We would like to mention that in the extremal case we recover the following Young's inequality for spherical planar algebras as in (Theorem 7.6, [8]).

Corollary 4.13 ([8])**.** *If $N \subset M$ is an extremal subfactor with $[M:N] < \infty$, then for any $x, y \in N' \cap M_1$ we have*

$$\|x * y\|_r \leq \frac{\sqrt{[M:N]}}{\kappa_0^+} \|x\|_p \|y\|_q$$

where, $1 \leq p, q, r \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

To prove Theorem 4.12 we start with proving a few results involving the Fourier transform $\mathcal{F}$ which will be crucially used.

Lemma 4.14. *For $x \in B' \cap A_1$, we have $\mathcal{F}(x)e_1\mathcal{F}(x)^* = xe_2x^*$.*

Proof: Let $\{\lambda_i : i \in I\}$ be a quasi-basis for E_0. Using Lemma 2.5 we observe the following,

$$\mathcal{F}(x)e_1\mathcal{F}(x)^* = \delta^6 \, E^{B'\cap A_2}_{A'\cap A_2}(xe_2e_1)e_1E^{B'\cap A_2}_{A'\cap A_2}(e_1e_2x^*)$$

$$
\begin{aligned}
&= \delta^2 \sum_{i,j} \lambda_i x e_2 e_1 \lambda_i^* e_1 \lambda_j e_1 e_2 x^* \lambda_j^* \\
&= \delta^2 \sum_{i,j} \lambda_i x E_0(\lambda_i^*) E_0(\lambda_j) e_2 e_1 e_2 x^* \lambda_j^* \\
&= \sum_j \Big(\sum_i \lambda_i E_0(\lambda_i^*) \Big) E_0(\lambda_j) x e_2 x^* \lambda_j^* \\
&= \sum_j E_0(\lambda_j) x e_2 x^* \lambda_j^* \\
&= x e_2 x^* \Big(\sum_j E_0(\lambda_j) \lambda_j^* \Big) \\
&= x e_2 x^*
\end{aligned}
$$

which finishes the proof. $\square$

Lemma 4.15. *For $x, y \in B' \cap A_1$, we have*

$$E_2\big(\mathcal{F}(x) y \mathcal{F}(x)^*\big) = \frac{1}{\delta}\big(y * (xx^*)\big).$$

Proof: Let $\{\lambda_i : i \in I\}$ be a quasi-basis for E_0. Using Lemma 2.5 we observe the following,

$$
\begin{aligned}
E_2(\mathcal{F}(x) y \mathcal{F}(x)^*) &= \delta^6 \, E_2\big(E^{B'\cap A_2}_{A'\cap A_2}(x e_2 e_1) y E^{B'\cap A_2}_{A'\cap A_2}(e_1 e_2 x^*)\big) \\
&= \delta^2 \sum_{i,j} E_2(\lambda_i x e_2 e_1 \lambda_i^* y \lambda_j e_1 e_2 x^* \lambda_j^*) \\
&= \sum_{i,j} \lambda_i x E_1(e_1 \lambda_i^* y \lambda_j e_1) x^* \lambda_j^* \,. \qquad (4.6)
\end{aligned}
$$

Since $y \in B' \cap A_1$, we can write $y = y_0 e_1 y_1$ for some $y_0, y_1 \in A$. Then, from Equation (4.6) and again using Lemma 2.5, we have the following,

$$
\begin{aligned}
E_2(\mathcal{F}(x) y \mathcal{F}(x)^*) &= \sum_{i,j} \lambda_i x E_1(e_1 \lambda_i^* y_0 e_1 y_1 \lambda_j e_1) x^* \lambda_j^* \\
&= \sum_{i,j} \lambda_i E_0(\lambda_i^* y_0) x x^* E_1(e_1) E_0(y_1 \lambda_j) \lambda_j^* \\
&= \sum_i \lambda_i x x^* E_1(E_0(\lambda_i^* y_0) e_1 y_1) \\
&= \sum_i \lambda_i x x^* E_1(e_1 \lambda_i^* y) \\
&= \sum_{i,j} \lambda_i x x^* E_1(e_1 \lambda_i^* \lambda_j E_0(\lambda_j^*) y) \\
&= \delta^2 \sum_{i,j} \lambda_i x x^* E_1(e_1 \lambda_i^* \lambda_j y) E_0(\lambda_j^*) E_2(e_2) \\
&= \delta^4 \sum_{i,j} E_2(\lambda_i x x^* E_1(e_1 \lambda_i^* \lambda_j y) e_2 E_0(\lambda_j^*) e_1 e_2)
\end{aligned}
$$

$$
\begin{aligned}
&= \delta^4 \sum_{i,j} E_2(\lambda_i x x^* e_2 e_1 \lambda_i^* \lambda_j y e_2 e_1 \lambda_j^* e_1 e_2)\\
&= \delta^8\, E_2(E^{B'\cap A_2}_{A'\cap A_2}(xx^* e_2 e_1) E^{B'\cap A_2}_{A'\cap A_2}(y e_2 e_1) e_1 e_2)\\
&= \frac{1}{\delta}\, \mathcal{F}^{-1}(\mathcal{F}(xx^*)\mathcal{F}(y))\\
&= \frac{1}{\delta}\, y * (xx^*)
\end{aligned}
$$

which completes the proof. □

As a corollary, we prove the following Schur product theorem. In the planar algebraic language this was first noticed by Liu (Theorem 4.1 in [16]).

Corollary 4.16. *(Schur product theorem) If $x, y \in B' \cap A_1$ are positive, then $x * y$ is positive.*

Proof: Let $x = aa^*$, $y = bb^*$ for some $a, b \in B' \cap A_1$. Then, by Lemma 4.15 and in view of the fact that E_2 is a positive map it is now easy to see that $x * y$ is positive. Indeed,

$$
\begin{aligned}
x * y = (aa^*) * (bb^*) &= \delta\, E_2\big(\mathcal{F}(b) a a^* \mathcal{F}(b)\big)\\
&= \delta\, E_2\big(\mathcal{F}(b) a (\mathcal{F}(b) a)^*\big)\\
&\geq 0\,.
\end{aligned}
$$

□

Remark 4.17. We would like to remark that as a consequence of Lemma 4.15, we can prove that the 'coproduct' on $B' \cap A_1$ is well behaved with adjoints. However, at present we are not sure whether these two notions are equivalent. To see this, it is enough to take $x, y \in B' \cap A_1$ such that $y \geq 0$. We write $y = bb^*$ for some $b \in B' \cap A_1$. Then by Lemma 4.15 and since E_2 is $*$ - preserving, we have:

$$
\begin{aligned}
(x * y)^* &= (x * (bb^*))^*\\
&= \delta^{-1} E_2(\mathcal{F}(b) x \mathcal{F}(b)^*)^*\\
&= \delta^{-1} E_2(\mathcal{F}(b) x^* f(b)^*)\\
&= (x^* * (bb^*))\\
&= (x^* * y^*).
\end{aligned}
$$

Next we prove a Frobenius reciprocity type result as follows.

Corollary 4.18. *For $x, y, z \in B' \cap A_1$, we have*

$$\operatorname{tr}((x * y) z) = \operatorname{tr}(x (z * \overline{y})$$

Proof: First we claim that $\operatorname{tr} \circ E_2 = \operatorname{tr}_2$, where tr_2 is the Markov type trace on $B' \cap A_2$ (see Proposition 2.4). Observe the following for $x \in B' \cap A_2$:

$$\operatorname{tr}_1 \circ E_2(x) = E_0 \circ E_1|_{B'\cap A_1}(E_2(x))$$

$$\begin{aligned}&= E_0 \circ E_1 \circ E_2|_{B' \cap A_2}(x) \\ &= \mathrm{tr}_2(x)\end{aligned}$$

which finishes the proof of the claim.

Recall that by Equation (3.1), we have $(\mathcal{F}(x))^* = \mathcal{F}(\overline{x^*})$. Now to prove the statement, assume that y is positive so that $y = bb^*$ for some $b \in B' \cap A_1$. Then, by Lemma 4.15 we have the following,

$$\begin{aligned}\mathrm{tr}(x * y)z) &= \mathrm{tr}((x * (bb^*))z) \\ &= \delta\, \mathrm{tr}(E_2(\mathcal{F}(b)x\mathcal{F}(b)^* z) \\ &= \delta\, \mathrm{tr}_2(\mathcal{F}(b)x\mathcal{F}(b)^* z) \\ &= \delta\, \mathrm{tr}_2(x\mathcal{F}(b)^* z\mathcal{F}(b)) \\ &= \delta\, \mathrm{tr}(xE_2(\mathcal{F}(\overline{b^*})z(\mathcal{F}(\overline{b^*}))^*)) \\ &= \mathrm{tr}(x(z * (\overline{b^*}(\overline{b^*})^*))) \\ &= \mathrm{tr}(x(z * \overline{y}))\end{aligned}$$

where, the last equation follows from Proposition 3.3. □

The following lemma is crucial in proving the Young's inequality.

Lemma 4.19. *For any $x, y \in B' \cap A_1$, we have*

$$\|x * y\|_\infty \le \frac{\delta}{\kappa_0^+} \|x\|_\infty \|\overline{y}\|_1 \quad \textit{and} \quad \|y * x\|_\infty \le \frac{\delta}{\kappa_0^+} \|x\|_\infty \|y\|_1 .$$

Proof: First we prove that for $w \in A' \cap A_2$ we have the following,

$$\mathcal{F}^{-1}(w)\mathcal{F}^{-1}(w)^* = \delta^2 E_2(we_1w^*) . \tag{4.7}$$

To see this, using Lemma 2.3 we observe the following,

$$\begin{aligned}\mathcal{F}^{-1}(w)\mathcal{F}^{-1}(w)^* &= \delta^6 E_2(we_1e_2)E_2(e_2e_1w^*) \\ &= \delta^6 E_2(we_1e_2E_2(e_2e_1w^*)) \\ &= \delta^4 E_2(we_1e_2e_1w^*) \\ &= \delta^2 E_2(we_1w^*) .\end{aligned}$$

Next, suppose $v \in B' \cap A_1$ is a partial isometry. For $x \in B' \cap A_1$, using Equation (4.7) we observe the following,

$$\begin{aligned}(v * x)(v * x)^* &= \mathcal{F}^{-1}(\mathcal{F}(x)\mathcal{F}(v))(\mathcal{F}^{-1}(\mathcal{F}(x)\mathcal{F}(v)))^* \\ &= \delta^2 E_2(\mathcal{F}(x)\mathcal{F}(v)e_1(\mathcal{F}(v))^*(\mathcal{F}(x))^*) \\ &= \delta^2 E_2(\mathcal{F}(x)ve_2v^*\mathcal{F}(x)^*) .\end{aligned}$$

Now, by Lemma 4.5 we have $\nu e_2 \nu^* \leq \frac{\|\nu\|_1}{\kappa_0^+}\nu\nu^*$. Then, using Lemma 4.15, we observe that

$$\begin{aligned}(\nu * x)(\nu * x)^* &\leq \delta^2 \frac{\|\nu\|_1}{\kappa_0^+} E_2(\mathcal{F}(x)\nu\nu^*\mathcal{F}(x)^*)\\ &= \delta \frac{\|\nu\|_1}{\kappa_0^+}(\nu\nu^*) * (xx^*)\\ &\leq \delta \frac{\|\nu\|_1}{\kappa_0^+}\|x\|_\infty^2 (\nu\nu^*) * 1. \end{aligned} \tag{4.8}$$

On the other hand, we note that

$$\begin{aligned}(\nu\nu^*) * 1 &= \mathcal{F}^{-1}(\mathcal{F}(1)\mathcal{F}(\nu\nu^*))\\ &= \delta^3 E_2(\mathcal{F}(1)\mathcal{F}(\nu\nu^*)e_1e_2)\\ &= \delta^4 E_2(e_2\mathcal{F}(\nu\nu^*)e_1e_2)\\ &= \delta^7 E_2(E^{B'\cap A_2}_{A'\cap A_2}(e_2\nu\nu^* e_2 e_1)e_1e_2)\\ &= \delta^7 E_2(E^{B'\cap A_2}_{A'\cap A_2}(E_1(\nu\nu^*)e_2e_1)e_1e_2)\,. \end{aligned} \tag{4.9}$$

Now, using Corollary 2.6, Lemma 4.4 and Equation (4.9) we conclude that

$$(\nu\nu^*) * 1 \leq \frac{\delta}{\kappa_0^+}\|\nu\|_1.$$

Therefore, from Equation (4.8) it follows that

$$(\nu * x)(\nu * x)^* \leq \left(\frac{\delta}{{\kappa_0}^+}\right)^2 \|\nu\|_1^2 \|x\|_\infty^2.$$

Thus we obtain,

$$\|\nu * x\|_\infty \leq \frac{\delta}{\kappa_0^+}\,\|\nu\|_1\,\|x\|_\infty\,. \tag{4.10}$$

Now, let $y \in B' \cap A_1$ be arbitrary and $y = \sum_k \lambda_k \nu_k$ be the rank one decomposition of y. Then, by Equation (4.10) we have the following,

$$\begin{aligned}\|y * x\|_\infty &\leq \sum_k \lambda_k\,\|\nu * x\|_\infty\\ &\leq \frac{\delta}{\kappa_0^+}\,\sum_k \lambda_k\,\|\nu\|_1\,\|x\|_\infty\\ &= \frac{\delta}{\kappa_0^+}\,\|x\|_\infty\,\|y\|_1, \end{aligned}$$

and by Proposition 3.9 we have the following,

$$\|x * y\|_\infty = \|\overline{x * y}\|_\infty = \|\overline{y} * \overline{x}\|_\infty$$

$$\leq \frac{\delta}{\kappa_0^+} \, ||\overline{x}||_\infty \; ||\overline{y}||_1$$
$$= \frac{\delta}{\kappa_0^+} \, ||x||_\infty \; ||\overline{y}||_1 \, .$$

Here $||x||_\infty = ||\overline{x}||_\infty$, since the map ρ_+ is a unital anti-homomorphsim. This completes the proof. □

Lemma 4.20. *For any $x, y \in B' \cap A_1$, we have*

$$||x * y||_1 \leq \frac{\delta}{\kappa_0^+} ||x||_1 ||y||_1 \, .$$

Proof: For any $x, y \in B' \cap A_1$, using Proposition 2.10, Corollary 4.18, Proposition 2.9 and Lemma 4.19 respectively, we get the following,

$$\begin{aligned} ||x * y||_1 = \sup_{||z||_\infty = 1} |\mathrm{tr}((x * y)z)| &= \sup_{||z||_\infty = 1} |\mathrm{tr}(x(z * \overline{y}))| \\ &\leq ||x||_1 ||z * \overline{y}||_\infty \\ &\leq \frac{\delta}{\kappa_0^+} ||x||_1 ||\overline{\overline{y}}||_1 \\ &= \frac{\delta}{\kappa_0^+} ||x||_1 ||y||_1 \, . \end{aligned}$$

□

Lemma 4.21. *For any $x, y \in B' \cap A_1$, we have*

$$||x * y||_p \leq \frac{\delta}{\kappa_0^+} \Big(\frac{||y||_1}{||\overline{y}||_1} \Big)^{\frac{1}{p}} ||x||_p ||\overline{y}||_1 \quad \textit{and} \quad ||y * x||_p \leq \frac{\delta}{\kappa_0^+} ||x||_p ||y||_1$$

where $1 \leq p \leq \infty$.

Proof: For fixed $y \in B' \cap A_1$, define $T_y : B' \cap A_1 \longrightarrow B' \cap A_1$ by

$$T_y(x) = x * y.$$

Clearly T_y is linear. Now, Lemma 4.19 and Lemma 4.20 respectively implies the following,

$$||T_y(x)||_\infty = ||x * y||_\infty \leq \frac{\delta}{\kappa_0^+} ||x||_\infty ||\overline{y}||_1 \, ,$$

and

$$||T_y(x)||_1 = ||x * y||_1 \leq \frac{\delta}{\kappa_0^+} ||x||_1 ||y||_1 \, .$$

Applying Proposition 2.11 with $p_1 = \infty$, $p_2 = 1$, $q_1 = \infty$, $q_2 = 1$, $\theta = \frac{1}{p}$, $K_1 = \frac{\delta}{\kappa_0^+} ||\overline{y}||_1$ and $K_2 = \frac{\delta}{\kappa_0^+} ||y||_1$ we get the first part. For the second part, for fixed y we define $T_y(x) = y * x$. Then, a similar proof as above implies the result. □

Lemma 4.22. *For any $x, y \in B' \cap A_1$, we have*

$$\|x * y\|_\infty \leq \frac{\delta}{\kappa_0^+} \|x\|_p \|\overline{y}\|_q$$

where $1 \leq p \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: Let $x * y = \sum_k \lambda_k v_k$ be the rank-one decomposition of $x * y$. Then,

$$\|x * y\|_\infty = \sup_k \lambda_k = \sup_k \frac{tr((x * y)v_k^*)}{tr(|v_k|)}. \tag{4.11}$$

Using Corollary 4.18, Proposition 2.9 and Lemma 4.21 respectively we see that

$$\begin{aligned} |\mathrm{tr}((x * y)v_k^*)| &= |\mathrm{tr}(x(v_k^* * \overline{y}))\\ &\leq \|x\|_p \|v_k^* * \overline{y}\|_q \\ &\leq \frac{\delta}{\kappa_0^+} \|x\|_p \|\overline{y}\|_q \|v_k^*\|_1 \\ &\leq \frac{\delta}{\kappa_0^+} \|x\|_p \|\overline{y}\|_q \, \mathrm{tr}(|v_k^*|)\,. \end{aligned} \tag{4.12}$$

The proof is now clear from Equations (4.11 and 4.12). □

Proof of Theorem 4.12 : Fix $x \in B' \cap A_1$ and define $T_x : B' \cap A_1 \to B' \cap A_1$ by $T_x(y) = x * \overline{y}$. For $x, y \in B' \cap A_1$, thanks to Lemma 4.21 and Lemma 4.22, we have the following,

$$\|T_x(\overline{y})\|_p = \|x * y\|_p \leq \frac{\delta}{\kappa_0^+} \|x\|_p \|\overline{y}\|_1^{-\frac{1}{p}} \|y\|_1^{\frac{1}{p}} \|\overline{y}\|_1$$

and

$$\|T_x(\overline{y})\|_\infty = \|x * y\|_\infty \leq \frac{\delta}{\kappa_0^+} \|x\|_p \|\overline{y}\|_{\frac{1}{\frac{1}{q} - \frac{1}{r}}} \,.$$

The proof is now clear by the Proposition 2.11 with $p_1 = p$, $p_2 = \infty$, $q_1 = 1$, $q_2 = \dfrac{1}{\frac{1}{q} - \frac{1}{r}}$, $K_1 = \dfrac{\delta}{\kappa_0^+} \|x\|_p \|\overline{y}\|_1^{-\frac{1}{p}} \|y\|_1^{\frac{1}{p}}$, $K_2 = \dfrac{\delta}{\kappa_0^+} \|x\|_p$ and $\theta = 1 - \dfrac{p}{r}$. □

Remark 4.23. We remark that if $B \subset A$ is an irreducible inclusion of simple C^*-algebras with a conditional expectation of index-finite type, then the quadruple $(B' \cap A_1, \mathrm{tr}, *, \rho_+)$ forms a Frobenius δ-algebra. We refer the reader to [7] for the definition of a Frobenius k-algebra. We also send the reader to [6] for related notions.

5. Appendix

In this appendix, we discuss two examples to illustrate our results. We skip the proofs as they are routine verification. In order to investigate theory concerning Fourier transform for inclusion of simple C^*-algebras, these two examples can be considered as model examples to test new theories.

5.1. Fourier transform for noncommutative torus. Let θ be an irrational number and consider the universal C^*-algebra $\mathscr{A}_\theta$, called the noncommutative torus, generated by two unitary elements U and V satisfying $UV = e^{-2\pi i\theta}VU$. It has a unital dense subalgebra $\mathbb{T}_\theta$ given by the following,

$$\mathbb{T}_\theta := \{a = \sum_{m,n\in\mathbb{Z}} a_{m,n}U^mV^n : \{a_{m,n}\} \in \mathcal{S}(\mathbb{Z}^2)\},$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the space of rapidly decreasing double sequences. The C^*-algebra $\mathscr{A}_\theta$ is equipped with a distinguished faithful *tracial state*, given on the dense subalgebra $\mathbb{T}_\theta$ by $\tau(a) = a_{0,0}$ and extends to $\mathscr{A}_\theta$ by continuity. We refer to (Chapter 6, Section 3 in [5]) for these facts. Moreover, $\mathscr{A}_\theta$ is a simple C^*-algebra since θ is irrational.

Let $k \geq 2$ be any natural number. Let us consider the unital C^*-subalgebra $\mathcal{B}_\theta$ of $\mathscr{A}_\theta$ generated by U^k and V. By the universality and simplicity of $\mathscr{A}_{k\theta}$, it follows that $\mathcal{B}_\theta$ is canonically isomorphic to $\mathscr{A}_{k\theta}$. Assume further that θ is not an algebraic number of degree 2. Then, the Watatani index $[\mathscr{A}_\theta : \mathcal{B}_\theta]_0$ is equal to k (Page 112 in [21]). Observe that $\mathcal{B}_\theta$ is nothing but the fixed point subalgebra of $\mathscr{A}_\theta$ under the $\mathbb{Z}_k$ action given by $\overline{m}.U = e^{2\pi i m/k}U$ and $\overline{m}.V = V$ for all $\overline{m} \in \mathbb{Z}_k$. This says that the inclusion $\mathcal{B}_\theta \subset \mathscr{A}_\theta$ is in fact $\mathscr{A}_\theta^{\mathbb{Z}_k} \subset \mathscr{A}_\theta$, and hence the basic construction is $\mathscr{A}_\theta \rtimes \mathbb{Z}_k$. Since $\mathbb{Z}_k$ is abelian, it then turns out that the Fourier transform $\mathcal{F}$ simply becomes the Fourier transform from the group algebra $\mathbb{C}\mathbb{Z}_k$ onto $\mathbb{C}\widehat{\mathbb{Z}}_k \cong \mathbb{C}\mathbb{Z}_k$, and hence given by the $k\times k$ Fourier matrix.

However, in this example one can take the pedestrian way to find the explicit form of the Fourier transform and the rotation maps without invoking any result involving the crossed product. Observe that (see Proposition (2.2.11, 2.2.12) in [21]) we may use the GNS construction to realize the C^*-basic construction in this case. We only mention the intermediate steps and leave the detail to the interested reader for verification. For notational simplicity, we denote $\mathcal{B} \subset \mathcal{A}$ to mean the inclusion $\mathcal{B}_\theta \subset \mathscr{A}_\theta$. The GNS Hilbert space $L^2(\mathscr{A}_\theta, \tau)$ is isomorphic to $\ell^2(\mathbb{Z}^2)$ via the identification $U^mV^n \longmapsto e_{m,n}$. Define k-many mutually orthogonal projections $p_r \in \mathscr{B}\left(\ell^2(\mathbb{Z}^2)\right)$, $0 \leq r \leq k-1$, by $p_r : e_{m,n} \mapsto e_{m,n}$ if $m \in k\mathbb{Z} + r$ and 0 otherwise. Considering the unital C^*-subalgebra $\mathcal{A}_1$ of $\mathscr{B}\left(\ell^2(\mathbb{Z}^2)\right)$ generated by U, V and p_0 (here $\tau(p_0) = 1/k$) one gets the basic construction $\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_1$. Using the decomposition $\mathcal{A}_1 = \bigoplus_{r=0}^{k-1} p_r\mathscr{A}_\theta$ as inner-product space, the GNS Hilbert space $L^2(\mathcal{A}_1, \tau)$ is isomorphic to $\mathbb{C}^k \otimes \ell^2(\mathbb{Z}^2)$ by the following map,

$$p_0 U^{m_0} V^{n_0} \oplus \dots \dots \oplus p_{k-1} U^{m_{k-1}} V^{n_{k-1}} \longmapsto \frac{1}{\sqrt{k}}(e_{m_0,n_0}, \dots, e_{m_{k-1},n_{k-1}}).$$

The following orthogonal projection $q : \mathbb{C}^k \otimes \ell^2(\mathbb{Z}^2) \longrightarrow \mathbb{C}^k \otimes \ell^2(\mathbb{Z}^2)$

$$q : (e_{m_0,n_0}, \dots, e_{m_{k-1},n_{k-1}}) \longmapsto \frac{1}{k}\left(\sum_{r=0}^{k-1} e_{m_r,n_r}, \dots, \sum_{r=0}^{k-1} e_{m_r,n_r}\right)$$

has range $\ell^2(\mathbb{Z}^2) \cong L^2(\mathcal{A}, \tau)$, and we obtain the basic construction tower of simple C^*-algebras $\mathcal{B} \subset \mathcal{A} \subset^p \mathcal{A}_1 \subset^q \mathcal{A}_2$, where $\mathcal{A}_2 = C^*\{\mathcal{A}_1, q\} \subseteq \mathscr{B}\left(\mathbb{C}^k \otimes \ell^2(\mathbb{Z}^2)\right) = M_k(\mathbb{C}) \otimes \mathscr{B}\left(\ell^2(\mathbb{Z}^2)\right)$. It follows that $\mathcal{A}_2 = M_k(\mathbb{C}) \otimes \mathscr{A}_\theta$. For a subset $S \subseteq M_k(\mathbb{C})$, we denote by $\mathrm{Alg}\{S\}$ the subalgebra generated by S and $S^* = \{x^* : x \in S\}$ in $M_k(\mathbb{C})$. Let C_k denote the permutation matrix $E_{1,k} + \sum_{i=1}^{k-1} E_{i+1,i}$ in $M_k(\mathbb{C})$. Then, $\mathcal{B}' \cap \mathcal{A}_1 = \mathrm{Alg}\{p_0, \dots, p_{k-1}\} \otimes \mathbb{C}$ and $\mathcal{A}' \cap \mathcal{A}_2 = \mathrm{Alg}\{I_k, C_k, \dots, C_k^{k-1}\} \otimes \mathbb{C}$ are subalgebras of $M_k(\mathbb{C}) \otimes \mathscr{B}\left(\ell^2(\mathbb{Z}^2)\right)$. The Fourier and the inverse Fourier transform are given by the following maps,

$$\mathcal{F} : \sum_{r=0}^{k-1} \alpha_r p_r \longmapsto \frac{1}{\sqrt{k}} \sum_{r=0}^{k-1} \alpha_r C_k^r,$$

and

$$\mathcal{F}^{-1} : \sum_{r=0}^{k-1} \gamma_r C_k^r \longmapsto \sqrt{k} \sum_{r=0}^{k-1} \gamma_r p_r,$$

where $\alpha_r, \gamma_r \in \mathbb{C}$. Let us consider the following multiplication on $\mathbb{C}^k$,

$$(\alpha_0, \dots, \alpha_{k-1}) * (\beta_0, \dots, \beta_{k-1}) := (\gamma_0, \dots, \gamma_{k-1}) \tag{5.13}$$

where $\gamma_j = \sum_{r=0}^{k-1} \alpha_r \beta_{k+j-r}$ for $0 \leq j \leq k-1$, with the convention $\beta_{k+j} = \beta_j$ for all j. Then, $\mathrm{Alg}\{I_k, C_k, \dots, C_k^{k-1}\} \cong (\mathbb{C}^k, *)$ as unital algebras and the following map

$$\Phi : (\alpha_0, \dots, \alpha_{k-1}) \longmapsto \frac{1}{\sqrt{k}}\Big(\sum_{r=0}^{k-1} \alpha_r, \sum_{r=0}^{k-1} \omega^r \alpha_r, \sum_{r=0}^{k-1} \omega^{2r} \alpha_r, \dots, \sum_{r=0}^{k-1} \omega^{(k-1)r} \alpha_r\Big),$$

where $\omega = e^{2\pi i/k}$ is a primitive k-th root of unity, implements a unital algebra isomorphism between $(\mathbb{C}^k, *)$ and $\mathbb{C}^k$ equipped with the standard algebra structure. It now follows that $\Phi \circ \mathcal{F} : \mathbb{C}^k \to \mathbb{C}^k$ is equal to the Fourier matrix. Moreover, the rotation maps ρ_+, ρ_- are given by the following,

$$\rho_+ : \sum_{r=0}^{k-1} \alpha_r p_r \longmapsto \sum_{r=1}^{k} \alpha_{k-r} p_r \quad \text{and} \quad \rho_- : \sum_{r=0}^{k-1} \gamma_r C_k^r \longmapsto \sum_{r=1}^{k} \gamma_{k-r} C_k^r,$$

with the convention $p_k = p_0$. Therefore, as an element of $M_k(\mathbb{C})$, both ρ_+ and $\Phi \circ \rho_- \circ \Phi^{-1}$ are equal to the $k \times k$ permutation matrix $E_{11} + \sum_{j=2}^{k} E_{j,k+2-j}$ in $M_k(\mathbb{C})$. It turns out that the convolution product on $Alg\{p_0, \dots, p_{k-1}\} \cong \mathbb{C}^k$ is the product $*$ defined in Equation (5.13), and that on $Alg\{I_k, C_k, \dots, C_k^{k-1}\} \cong (\mathbb{C}^k, *)$ is the usual componentwise multiplication on $\mathbb{C}^k$.

5.2. Fourier transform for matrix algebras. Let us consider the inclusion $\mathbb{C} \subset M_n(\mathbb{C})$. It is well known that the (standard normalized) trace-preserving conditional expectation is of index-finite type and it is the unique minimal one. We shall use the notation $(\alpha_{ij})_{ij}$, $1 \leq i, j \leq n$, to denote a matrix in $M_n(\mathbb{C})$, and the elementary matrices will be denoted by E_{ij}. The unique normalized trace on $M_n(\mathbb{C})$ is denoted by tr. It is known that the basic construction for the unital inclusion $\mathbb{C} \subset M_n(\mathbb{C})$ is of the following form

$$\mathbb{C} \subset M_n(\mathbb{C}) \subset^{e_1} M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \subset^{e_2} M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \subset \cdots\cdots$$

with $e_1 = \frac{1}{n}\sum_{i,j=1}^n E_{ij} \otimes E_{ij}$ and $e_2 = \frac{1}{n}\sum_{i,j=1}^n E_{ij} \otimes E_{ij} \otimes I_n$ (see [12]). Thus, in accordance with the notations used in earlier sections, we have in this situation the inclusion $\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}_1 \subset \mathcal{A}_2$ where,

$$\mathcal{B} = \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \quad , \quad \mathcal{A} = \mathbb{C} \otimes \mathbb{C} \otimes M_n(\mathbb{C}),$$

$$\mathcal{A}_1 = \mathbb{C} \otimes M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \quad , \quad \mathcal{A}_2 = M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes M_n(\mathbb{C}).$$

Clearly, $\mathcal{B}' \cap \mathcal{A}_1 = \mathcal{A}_1$ and $\mathcal{A}' \cap \mathcal{A}_2 = M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \otimes \mathbb{C}$. Then, it is clear that the conditional expectation $E^{\mathcal{A}_2}_{\mathcal{A}' \cap \mathcal{A}_2}$ is given by id $\otimes$ id $\otimes$ tr and the conditional expectation $E^{\mathcal{A}_2}_{\mathcal{A}_1}$ is given by tr $\otimes$ id $\otimes$ id. We shall use the standard convention $E_{(i,p)(j,q)} := E_{ij} \otimes E_{pq}$ for the matrix units in $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ (Sec. 6, Page 97 in [18]).

Proposition 5.1. *The Fourier and the inverse Fourier transform on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ are given by the following,*

$$\mathcal{F} : E_{k\ell} \otimes E_{pq} \longmapsto E_{\ell q} \otimes E_{kp},$$
$$\mathcal{F}^{-1} : E_{k\ell} \otimes E_{pq} \longmapsto E_{pk} \otimes E_{q\ell}.$$

We now find convolution on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ defined by the formula

$$x * y := \mathcal{F}^{-1}(\mathcal{F}(y)\mathcal{F}(x)).$$

Given two elements $A, D \in M_n(\mathbb{C})$, let $A \odot D$ be their Schur product. Since the projection $\frac{1}{n}J_n \in M_n(\mathbb{C})$, with $J_n = \sum_{i,j=1}^n E_{ij}$, is a minimal projection, we have $\frac{1}{n^2}J_n(A \odot D)J_n$ is a scalar multiple of $\frac{1}{n}J_n$. Denote this scalar by $\alpha_{A,D}$. That is,

$$\left(\frac{1}{n}J_n\right)(A \odot D)\left(\frac{1}{n}J_n\right) = \alpha_{A,D}\left(\frac{1}{n}J_n\right) \tag{5.14}$$

with $\alpha_{A,D} \in \mathbb{C}$.

Proposition 5.2. *The convolution on $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ implemented by the Fourier and the inverse Fourier transform is the given by the following,*

$$(A \otimes B) * (C \otimes D) = n\,\alpha_{A,D}(C \otimes B)$$

*where $\alpha_{A,D} \in \mathbb{C}$ is as defined in Equation (*5.14*).*

Proposition 5.3. *The rotation maps $\rho_+, \rho_- : M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ coincide, i,e. $\rho_+ = \rho_-$, and they are given by the following,*

$$E_{ij} \otimes E_{k\ell} \longmapsto E_{\ell k} \otimes E_{ji}.$$

Remark 5.4. It is easy to see that ρ_+ is trace preserving, i,e., $\mathrm{tr}(\rho_+(x)) = \mathrm{tr}(x)$ for all $x \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. Therefore, the Young's inequality in Theorem 4.12 becomes the following,

$$\|x * y\|_r \leq n^2 \|x\|_p \|y\|_q$$

for $x, y \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, where $1 \leq p, q, r \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$.

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