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Revisiting the Manin–Peyre conjecture for the split del Pezzo surface of degree 5

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Abstract. An improved asymptotic formula is established for the number of rational points of bounded height on the split smooth del Pezzo surface of degree 5. The proof uses the five conic bundle structures on the surface.

Contents

1. Introduction

This paper concerns the arithmetic of smooth del Pezzo surfaces X of degree 5 over $\mathbb{Q}$. Such surfaces can be realised as the blow-up of $\mathbb{P}^2$ along 4 points, no 3 of which are collinear. We will focus here on the split del Pezzo surface of degree 5, for which the 4 points in $\mathbb{P}^2$ are all defined over $\mathbb{Q}$. There are 10 lines contained in X and we let $U \subset X$ be the Zariski open subset obtained by deleting them. The Manin–Peyre conjecture makes a precise prediction for the asymptotic behaviour of the counting function

$$N_{U,H}(P) = \#\{x \in U(\mathbb{Q}) : H(x) \leqslant P\},$$

as $P \to \infty$, where H is the anticanonical height function. The conjecture first appears in work of Franke, Manin and Tschinkel [7], and applies more broadly to arbitrary smooth Fano varieties over $\mathbb{Q}$, with a prediction for the leading constant worked out by Peyre [12].

The split del Pezzo surface has proved a popular testing ground for this conjecture, with Manin and Tschinkel [10] establishing linear growth, proving that $N_{U,H}(P) = O(P(\log P)^6)$. In a very influential 1993 lecture "Counting rational

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points on del Pezzo surfaces of degree 5" in Bern, Salberger demonstrated how a passage to the universal torsor can lead to the expected upper bound $N_{U,H}(P) = O(P(\log P)^4)$, which agrees with the Manin conjecture, since $\mathrm{Pic}(X) \cong \mathbb{Z}^5$. In this setting, the universal torsor above X is an open subset of the Grassmannian $\mathbb{G}(1,4)$ parametrising lines in $\mathbb{P}^4$. This signalled the first use of universal torsors to count rational points of bounded height, a point of view that has led to industrial scale activity for singular del Pezzo surfaces, as summarised in [4], for example.

For quintic del Pezzo surfaces, a breakthrough was achieved in the work of de la Bretèche [1], who succeeded in establishing that

$$N_{U,H}(P) = c_U P(\log P)^4\Big(1 + O\Big(\frac{1}{\log\log P}\Big)\Big), \tag{1.1}$$

where

$$c_U = \frac{\pi^2}{72}\prod_p \Big(1-\frac{1}{p}\Big)^5\Big(1+\frac{5}{p}+\frac{1}{p^2}\Big) \tag{1.2}$$

is the constant predicted by Peyre [12]. De la Bretèche makes essential use of the universal torsor approach discovered by Salberger. The purpose of this paper is to explore an alternative approach to the Manin conjecture for the split del Pezzo surface of degree 5, which is based on five conic bundle structures $X \to \mathbb{P}^1$ that can be associated to X. This mechanism lies at the heart of the resolution of the Manin conjecture for a smooth del Pezzo surface of degree 4, in work of Browning and de la Bretèche [3]. The situation is simpler here and will yield the following sharpening of (1.1).

Theorem 1.1. *We have*

$$N_{U,H}(P) = c_U P(\log P)^4 + O\big(P(\log P)^3(\log\log P)^3\big),$$

where c_U is the constant (1.2) predicted by Peyre.

It would be interesting to determine whether a conic bundle approach could be used to handle some non-split del Pezzo surfaces of degree 5. The starting point for the proof of Theorem 1.1 is a passage to the universal torsor, following Salberger, followed by a translation of the problem from counting integral points on the universal torsor to one that profits from the conic bundle structures on X. This is achieved in §2. For positive integers $A, B \leqslant P^{\frac{2}{5}}$, it leads to counting rational points on the conics

$$C_{A,B}: \quad AX^2 + BY^2 = (A+B)Z^2,$$

subject to certain conditions. Observing that $[1,1,-1] \in C_{A,B}(\mathbb{Q})$, for any choice of A, B, we may carry out a uniform parametrisation of the conics. This is the subject of §3. The outcome is a complicated lattice point counting problem, which needs to be accomplished with a sufficient degree of uniformity in A and B. This is achieved in §4. Finally, in §5 the main term in our lattice point counting argument is evaluated asymptotically. In the present setting this essentially boils down to a sum of the type

$$\sum_{(A,B)\in\mathbb{Z}^2\cap\mathscr{R}} \tau(AB(A+B)),$$

where $\tau(n)$ is the divisor function and $\mathscr{R}\subset\mathbb{R}^2$ is a suitable expanding region. Problems of this sort were first studied by Greaves [8] when $AB(A+B)$ is replaced by an arbitrary irreducible binary cubic form $F\in\mathbb{Z}[A,B]$. In §5 we shall argue differently, using an alternative geometry of numbers approach worked out by Browning [5].

The approach in this paper covers the rational points of height B on the split del Pezzo surface of degree 5 by $O(P^{\frac{4}{5}})$ conics $C_{A,B}\subset\mathbb{P}^2$. Although it represents a formidable technical challenge, it would be intriguing to determine whether a version of Theorem 1.1 can be proved with a full power saving in the error term.

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2. The conic bundle structures

Let X be a split del Pezzo surface of degree 5. It follows from the work of Mumford [11] that $X=\overline{\mathscr{M}}_{0,5}$ arises as the moduli space of stable sets of 5 ordered points $[x_i,y_i]\in\mathbb{P}^1$, for $1\leqslant i\leqslant 5$. Here stability (and semistability) means that no 3 of the points coincide. The 10 lines on X are given by

$$z_{i,j}=x_iy_j-x_jy_i=0,\quad (1\leqslant i<j\leqslant 5),$$

and we write U for the complement of the lines on X.

There is a torsor $\pi:\mathscr{T}\to X$ under the torus $\mathbb{G}_m^5$. It is given as a quasi-projective variety $\mathscr{T}\subset\mathbb{P}^9$ with coordinates $z_{i,j}$ for $1\leqslant i<j\leqslant 5$ such that there are no three indices i,j,k for which

$$z_{i,j}=z_{i,k}=0, \tag{2.1}$$

and furthermore, the Grassmannian equations

$$\begin{aligned} z_{1,2}z_{3,4} - z_{1,3}z_{2,4} + z_{1,4}z_{2,3} &= 0,\\ z_{1,2}z_{3,5} - z_{1,3}z_{2,5} + z_{1,5}z_{2,3} &= 0,\\ z_{1,2}z_{4,5} - z_{1,4}z_{2,5} + z_{1,5}z_{2,4} &= 0,\\ z_{1,3}z_{4,5} - z_{1,4}z_{3,5} + z_{1,5}z_{3,4} &= 0,\\ z_{2,3}z_{4,5} - z_{2,4}z_{3,5} + z_{2,5}z_{3,4} &= 0, \end{aligned} \tag{2.2}$$

are satisfied. One finds that $\pi^{-1}(U)$ is given by these equations and the condition that $z_{i,j} \neq 0$ for $1 \leqslant i < j \leqslant 5$. The torsor $\pi : \mathscr{T} \to X$ extends in an obvious way to a torsor $\widetilde{\pi} : \widetilde{\mathscr{T}} \to \widetilde{X}$ over $\mathbb{Z}$. By properness $X(\mathbb{Q}) = \widetilde{X}(\mathbb{Z})$ and Grothendieck's version of Hilbert's Theorem 90 ensures that the map $\widetilde{\mathscr{T}}(\mathbb{Z}) \to \widetilde{X}(\mathbb{Z})$ is onto. Thus, in order to count $\mathbb{Q}$-rational points of bounded height on U, it will be enough to count $\mathbb{Z}$-integral points on $\widetilde{\mathscr{T}} \subset \mathbb{P}^9$ with all $z_{i,j} \neq 0$. The subset $\widetilde{\mathscr{T}}(\mathbb{Z}) \subset \mathbb{P}^9(\mathbb{Z})$ satisfies coprimality conditions

$$\gcd(z_{i,j}, z_{i,k}) = 1, \quad \text{for distinct } i, j, k \in \{1, 2, 3, 4, 5\}, \tag{2.3}$$

which correspond to the conditions (2.1) after reduction modulo primes. The most natural choice of anticanonical height on $U(\mathbb{Q})$ is the one corresponding to the function $|z_{13}z_{14}z_{24}z_{25}z_{35}|$ for the subset $\widetilde{\mathscr{T}}(\mathbb{Z}) \subset \mathbb{P}^9(\mathbb{Z})$. This universal torsor point of view underpins the approach taken by de la Bretèche and ultimately leads to the result [1, Lemma 3], which we record here.

Lemma 2.1. *We have $N_{U,H}(P) = 12M(P)$, where*

$$M(P) = \#\left\{\mathbf{z} = (z_{i,j})_{1\leqslant i<j\leqslant 5} \in \mathbb{N}^{10} : \begin{array}{l} (2.2), (2.3) \text{ hold} \\ z_{13}z_{14}z_{24}z_{25}z_{35} \leqslant P \end{array}\right\}.$$

We are now placed to translate the counting problem into one that takes advantage of the conic bundle structure of our split del Pezzo surface X. In this capacity it will be useful to record the map $\vartheta : \mathscr{T} \to \mathscr{T}$, acting component-wise via

$$\vartheta(z_{i,j}) = z_{i+1,j+1}, \tag{2.4}$$

for $1 \leqslant i < j \leqslant 5$. (Here, as throughout this paper, we follow the convention that $z_{i,j} = z_{j,i} = z_{i+5,j} = z_{i,j+5}$, for any indices i, j.) One checks that ϑ leaves the cardinality of $S(P)$ invariant, where $M(P) = \#S(P)$, say.

For any $\mathbf{z} \in S(P)$, we write

$$C_1 = z_{1,3}z_{2,4},\ C_2 = z_{1,3}z_{2,5},\ C_3 = z_{1,4}z_{2,5},\ C_4 = z_{1,4}z_{3,5},\ C_5 = z_{2,4}z_{3,5}.$$

It is clear that

$$\prod_{i=1}^{5} C_i = z_{1,3}^2 z_{1,4}^2 z_{2,4}^2 z_{2,5}^2 z_{3,5}^2 \leqslant P^2, \tag{2.5}$$

and furthermore, $\vartheta^i(C_1) = C_{6-i}$, for $i \pmod 5$, where θ is given by (2.4). Since $S(P)$ is left invariant under ϑ, it follows that

$$5\#\big\{\mathbf{z} \in S(P) : C_1 < \min_{i\neq 1}\{C_i\}\big\} \leqslant M(P) \leqslant 5\#\big\{\mathbf{z} \in S(P) : C_1 \leqslant \min_{i\neq 1}\{C_i\}\big\}.$$

Once combined with (2.5), it follows that $C_1 < P^{\frac{2}{5}}$ (resp. $C_1 \leqslant P^{\frac{2}{5}}$) for $\mathbf{z} \in S(P)$ belonging to the set on the left (resp. right). This therefore leads to the following conclusion.

Lemma 2.2. *We have $60\underline{M}(P) \leqslant N_{U,H}(P) \leqslant 60\overline{M}(P)$, where*

$$\overline{M}(P) = \#\left\{\mathbf{z} \in \mathbb{N}^{10} : \begin{array}{l} (2.2), (2.3) \text{ hold} \\ z_{13}z_{14}z_{24}z_{25}z_{35} \leqslant P \\ z_{1,3}z_{2,4} \leqslant \min\{z_{1,3}z_{2,5}, z_{1,4}z_{2,5}, z_{1,4}z_{3,5}, z_{2,4}z_{3,5}\} \\ z_{1,3}z_{2,4} \leqslant P^{\frac{2}{5}} \end{array}\right\},$$

and $\underline{M}(P)$ is defined similarly, but with the symbol $\leqslant$ replaced by $<$ in the final two lines.

This result is analogous to [1, Lemma 4] in the work by de la Bretèche. We henceforth focus our attention on estimating $\overline{M}(P)$, it being understood that the final estimate we obtain remains equally valid for $\underline{M}(P)$. Everything is now in place to translate our counting problem to one involving the conic bundle structures associated to X.

Let us write

$$\mathsf{L} = \{(X, Y, Z) \in \mathbb{Q}^3 : X \pm Y, X \pm Z, Y \pm Z \in \mathbb{Z}\}. \tag{2.6}$$

This defines a lattice of rank 3 and it is easy to see that $\mathsf{L} = \mathbb{Z}^3 \cup (\mathbb{Z} + \frac{1}{2})^3$. Let L_{prim} denote the set of primitive vectors in L, where we say $(X, Y, Z) \in \mathbb{Z}^3$ is primitive if and only if

$$\min\{v_p(X), v_p(Y), v_p(Z)\} = 0$$

for all primes p, and $(X, Y, Z) \in (\mathbb{Z} + \frac{1}{2})^3$ is said to be primitive if and only if

$$\min\{v_p(X), v_p(Y), v_p(Z)\} = 0$$

for all primes $p > 2$. (Note that $v_2(X) = v_2(Y) = v_2(Z) = -1$ if $(X, Y, Z) \in (\mathbb{Z} + \frac{1}{2})^3$.) We let

$$\mathsf{L}_* = \{(X, Y, Z) \in \mathsf{L}_{\text{prim}} : X, Y, Z \text{ positive and not all odd}\}.$$

Thus, any $(X, Y, Z) \in \mathsf{L}_*$ either belongs to $(\mathbb{Z} + \frac{1}{2})^3$ or it belongs to $\mathbb{Z}^3$ with $2 \mid XYZ$. Finally, we let $\mathbb{N}^3_{\text{prim}}$ denote the set of primitive vectors in $\mathbb{N}^3$.

Define the set

$$\mathscr{A} = \{(A, B, C) \in \mathbb{N}^3_{\text{prim}} : C \leqslant P^{\frac{2}{5}} \text{ and } A + B = C\}. \tag{2.7}$$

Our main aim in this section is to achieve a bijection between the vectors counted by $\overline{M}(P)$ and elements $(A, B, C; X, Y, Z) \in \mathscr{A} \times \mathsf{L}_*$ such that

$$AX^2 + BY^2 - CZ^2 = 0, \tag{2.8}$$

subject to suitable height conditions, with

$$X - Z > 0 \tag{2.9}$$

and

$$\begin{aligned} \gcd(2^{\kappa}C, X \mp Y) &\leqslant \gcd(X \pm Y, X+Z),\\ 2^{2\kappa}C &\leqslant \gcd(2^{\kappa}B, X+Z)\gcd(X \pm Y, X+Z), \end{aligned} \tag{2.10}$$

for $\kappa \in \{0,1\}$. We will henceforth refer to the latter set of relations as $(2.10)_{\kappa}$ in order to stress the dependence on κ. Once achieved, this transition will lead us to tackle a counting problem involving a family of conics. The passage to the conic bundle structure is achieved in the following result.

Lemma 2.3. *We have $\overline{M}(P) = 2L(P)$, where*

$$L(P) = \#\left\{(A,B,C;X,Y,Z) \in \mathscr{A} \times \mathsf{L}_* : \begin{array}{l} (2.8)\text{–}(2.10)_0 \text{ hold} \\ C(X+Z) \leqslant P \end{array}\right\}.$$

Proof. Let $\mathbf{z}$ be a vector counted by $\overline{M}(P)$. We proceed to construct a point that will be counted by $L(P)$. Define

$$A = z_{1,2}z_{3,4}, \quad B = z_{1,4}z_{2,3}, \quad C = z_{1,3}z_{2,4}. \tag{2.11}$$

Then it easily follows that $(A,B,C) \in \mathscr{A}$, in the notation of (2.7). One now defines

$$\begin{aligned} X &= \frac{z_{1,3}z_{2,5}z_{4,5} + z_{2,4}z_{1,5}z_{3,5}}{2} = \frac{z_{1,4}z_{2,5}z_{3,5} + z_{2,3}z_{1,5}z_{4,5}}{2},\\ Y &= \frac{z_{1,2}z_{3,5}z_{4,5} - z_{3,4}z_{1,5}z_{2,5}}{2} = \frac{z_{1,3}z_{2,5}z_{4,5} - z_{2,4}z_{1,5}z_{3,5}}{2},\\ Z &= \frac{z_{1,2}z_{3,5}z_{4,5} + z_{3,4}z_{1,5}z_{2,5}}{2} = \frac{z_{1,4}z_{2,5}z_{3,5} - z_{2,3}z_{1,5}z_{4,5}}{2}. \end{aligned}$$

The equalities in these definitions follow from the Plücker relations in (2.2). It is clear that $(X,Y,Z) \in \mathsf{L}_{\mathrm{prim}}$, where L is defined in (2.6). To see that $2 \mid XYZ$ when $(X,Y,Z) \in \mathbb{Z}^3$, we suppose that X, say, is odd. Then it follows that

$$z_{1,3}z_{2,5}z_{4,5} + z_{2,4}z_{1,5}z_{3,5} \equiv z_{1,4}z_{2,5}z_{3,5} + z_{2,3}z_{1,5}z_{4,5} \equiv 2 \pmod 4.$$

The conditions (2.3) ensure that either all four monomials are odd, or the first two monomials are even and the remaining two are odd, or the second two are even and the first two are odd. If $z_{1,3}z_{2,5}z_{4,5}$ and $z_{2,4}z_{1,5}z_{3,5}$ are both odd, then they are both congruent to 1 or -1 modulo 4, which implies that Y must be even. Alternatively, if $z_{1,3}z_{2,5}z_{4,5}$ and $z_{2,4}z_{1,5}z_{3,5}$ are both even then $z_{1,4}z_{2,5}z_{3,5}$ and $z_{2,3}z_{1,5}z_{4,5}$ must be congruent to 1 or -1 modulo 4 and one concludes that Z is even. In this way one deduces that $(X,Y,Z) \in \mathsf{L}_{\mathrm{prim}}$, with $2 \mid XYZ$ when $(X,Y,Z) \in \mathbb{Z}^3$.

To deduce the equation (2.8), we note that

$$\begin{aligned} A(X^2 - Z^2) &= z_{1,2}z_{3,4} \cdot z_{1,4}z_{2,5}z_{3,5} \cdot z_{2,3}z_{1,5}z_{4,5}\\ &= z_{1,4}z_{2,3} \cdot z_{1,2}z_{3,5}z_{4,5} \cdot z_{3,4}z_{1,5}z_{2,5} = B(Z^2 - Y^2), \end{aligned}$$

which once combined with the relation $A+B=C$ gives (2.8). Moreover, the height condition $z_{13}z_{14}z_{24}z_{25}z_{35} \leqslant P$ transforms to $C(X+Z) \leqslant P$. Next, on making repeated reference to the coprimality conditions (2.3), it is straightforward to check that the conditions in $(2.10)_0$ are satisfied, which according to our convention means that (2.10) holds with $\varkappa = 0$.

Since $z_{i,j} > 0$ for all $1 \leqslant i < j \leqslant 5$, it follows that $X, Z > 0$ and $X > Z$, as required for (2.9). We claim that there is no contribution arising from those $\mathbf{z}$ counted by $\overline{M}(P)$ for which $Y = 0$. Indeed, in view of the coprimality conditions (2.3), this can only happen when

$$z_{1,5} = z_{2,5} = z_{3,5} = z_{4,5} = 1, \quad z_{1,2} = z_{3,4} = s, \quad z_{1,3} = z_{2,4} = t,$$

say. Now the inequality $z_{1,3}z_{2,4} \leqslant z_{1,3}z_{2,5}$ implies that $t = 1$. Thus, the first equation in (2.2) yields $s^2 - 1 + z_{1,4}z_{2,3} = 0$, which clearly has no solutions in positive integers. It follows that $(X, |Y|, Z) \in \mathsf{L}_*$. We complete the proof of the upper bound $\overline{M}(P) \leqslant 2L(P)$ by observing a symmetry between solutions $(A, B, C; X, Y, Z)$ for which $Y < 0$ and those for which $Y > 0$.

Turning to the lower bound $\overline{M}(P) \geqslant 2L(P)$, we let $(A, B, C; X, |Y|, Z)$ be a vector counted by $L(P)$, for non-zero $Y \in \mathbb{Z}$, and proceed to construct a point $\mathbf{z}$ that is counted by $\overline{M}(P)$. This will be enough to complete the proof of the lemma. We define

$$\begin{aligned} z_{1,2} &= \gcd(A, Z+Y), \quad & z_{1,4} &= \gcd(B, X+Z), \quad & z_{1,3} &= \gcd(C, X+Y), \\ z_{3,4} &= \frac{A}{\gcd(A, Z+Y)}, \quad & z_{2,3} &= \frac{B}{\gcd(B, X+Z)}, \quad & z_{2,4} &= \frac{C}{\gcd(C, X+Y)}, \end{aligned}$$

and

$$\begin{aligned} z_{1,5} &= \gcd(X-Y, X-Z), \quad & z_{2,5} &= \gcd(X+Y, X+Z), \\ z_{3,5} &= \gcd(X-Y, X+Z), \quad & z_{4,5} &= \gcd(X+Y, X-Z). \end{aligned}$$

We note that $\mathbf{z} = (z_{i,j})_{1 \leqslant i < j \leqslant 5}$ defines a vector in $\mathbb{N}^{10}$. We proceed by proving that the components of $\mathbf{z}$ satisfy the coprimality conditions (2.3). Let us show, for example, that $\gcd(z_{1,2}, z_{1,i}z_{2,j}) = 1$ for $3 \leqslant i, j \leqslant 5$. The cases $i, j \in \{3, 4\}$ follow from the observation that $\gcd(A, B) = \gcd(A, C) = \gcd(B, C) = 1$ in (2.7). Suppose next that $i = j = 5$. If $p^k \mid \gcd(z_{1,2}, z_{1,5})$ then p^k divides A and $Z+Y, X-Y, X-Z$. It therefore follows that p^k divides $\gcd(A, 2X, 2Y, 2Z)$, whence $k = 1$ and $p = 2$. But this is impossible since the definition of L_* means that at least one of $X+Y, X-Y$ or $X+Z$ must be odd. Similarly, $\gcd(z_{1,2}, z_{2,5}) = 1$.

Now it follows immediately from the definitions that (2.11) holds. Moreover,

$$z_{2,4} = \frac{\gcd(C, X^2 - Y^2)}{\gcd(C, X+Y)} = \gcd(C, X-Y),$$

and so the first two lines in $(2.10)_0$ translate into the inequalities involving $z_{1,3}z_{2,4}$ in the definition of $\overline{M}(P)$. Next we must check that

$$\begin{aligned} X+Y &= z_{1,3}z_{2,5}z_{4,5}, & X+Z &= z_{1,4}z_{2,5}z_{3,5}, & Z+Y &= z_{1,2}z_{3,5}z_{4,5},\\ X-Y &= z_{2,4}z_{1,5}z_{3,5}, & X-Z &= z_{2,3}z_{1,5}z_{4,5}, & Z-Y &= z_{3,4}z_{1,5}z_{2,5}. \end{aligned} \tag{2.12}$$

For this, it is useful to observe that $X > |Y|$, $X > Z$ and $Z > |Y|$, as follows from (2.8) and (2.9). We establish the first of these identities, the remaining ones being dealt with similarly. It follows from the definitions above that

$$\begin{aligned} z_{1,3}z_{2,5}z_{4,5} &= \gcd(C, X+Y)\gcd(X+Y, X+Z)\gcd(X+Y, X-Z)\\ &= \gcd(C, X+Y)\gcd(X+Y, X^2-Z^2), \end{aligned}$$

since we clearly have $\gcd(X+Y, X+Z, X-Z) = 1$. To see that the latter expression is equal to $X+Y$, it is enough to check equality at prime powers. Let $p^\alpha \| C$, $p^\beta \| X^2 - Z^2$ and $p^\gamma \| X+Y$ for a prime p and integers $\alpha, \beta, \gamma \geqslant 0$. Our task is to show that $\min\{\alpha,\gamma\} + \min\{\beta,\gamma\} = \gamma$. If $\alpha\beta\gamma = 0$ then this is clearly trivial since $\gamma \leqslant \alpha + \beta$. Hence we may assume that $\alpha, \beta, \gamma \geqslant 1$. In particular $v_p(B) = v_p(Y-X) = 0$ and we conclude that $\gamma = \alpha + \beta$, from which the desired equality flows.

Armed with (2.11) and (2.12), it is now easy to see that $z_{13}z_{14}z_{24}z_{25}z_{35} \leqslant P$. It remains to deduce the 5 equations in (2.2). The first follows from (2.11) and the fact that $A + B = C$. We derive the remaining 4 using (2.12). For example, the second equation in (2.2) arises via

$$z_{1,2}z_{3,5} = \frac{Y+Z}{z_{4,5}} = \frac{X+Y}{z_{4,5}} - \frac{X-Z}{z_{4,5}} = z_{1,3}z_{2,5} - z_{1,5}z_{2,3}.$$

This completes the proof of the lemma. □

The set $\mathsf{L}_* \cap (\mathbb{Z} + \frac{1}{2})^3$ is slightly awkward to work with, and it is more natural to use the homogeneity of (2.8) to convert the problem to one involving $\mathbb{N}^3$. This is easily done, as the following result demonstrates.

Lemma 2.4. *We have*

$$L(P) = \sum_{(A,B,C)\in\mathscr{A}} \sum_{\kappa\in\{0,1\}} \#\mathsf{M}_\kappa(P),$$

where

$$\mathsf{M}_\kappa(P) = \left\{ (X,Y,Z) \in \mathbb{N}^3_{\mathrm{prim}} : \begin{array}{l} (2.8)\text{–}(2.10)_\kappa \textit{ hold, } C(X+Z) \leqslant 2^\kappa P \\ \gcd(XYZ, 2) = 2^{1-\kappa} \end{array} \right\}.$$

Proof. Fix a choice of $(A,B,C) \in \mathscr{A}$. Then the set of $(X,Y,Z) \in \mathsf{L}_* \cap \mathbb{Z}^3$ considered in Lemma 2.3 is equal to $\mathsf{M}_0(P)$. Thus, it remains to consider the set of $(X,Y,Z) \in \mathsf{L}_* \cap (\mathbb{Z} + \frac{1}{2})^3$. For these, we simply make the change of variables

$$(X', Y', Z') = 2(X,Y,Z).$$

This clearly produces a primitive vector in $\mathbb{N}^3$ with $2 \nmid X'Y'Z'$. By homogeneity it is clear that (2.8) and (2.9) are left invariant, whereas $(2.10)_0$ is transformed into $(2.10)_1$. Likewise, the height condition $C(X+Z) \leqslant P$ becomes $C(X'+Z') \leqslant 2P$, and it follows that the set under consideration coincides with $\mathsf{M}_1(P)$. This completes the proof of the lemma. □

At this stage, it is convenient to show that the summation over A, B, C in Lemma 2.4 can be restricted to A, B, C for which $\min\{A, B\}$ is of comparable size to $\max\{A, B\}$. This is achieved in the following result.

Lemma 2.5. *For $J \geqslant 2$ define*

$$\mathscr{A}_J = \{(A,B,C) \in \mathbb{N}^3_{\text{prim}} \ : \ A+B = C \leqslant P^{\frac{2}{5}} \text{ and } \min\{A,B\} \geqslant J^{-1}C\}.$$

Then for any $\varepsilon > 0$ we have

$$L(P) = \sum_{(A,B,C)\in\mathscr{A}_J} \sum_{\kappa\in\{0,1\}} \#\mathsf{M}_\kappa(P) + O_\varepsilon\Big(P^{\frac{4}{5}+\varepsilon} + \frac{P(\log P)^4}{J^{\frac{2}{3}}}\Big),$$

where $\mathsf{M}_\kappa(P)$ is as in the statement of Lemma 2.4.

Proof. The main tool in our proof of this result is [6, Cor. 2], which provides a uniform bound for the number of rational points of bounded height on non-singular plane conics. For the conic defined in (2.8), the underlying quadratic form has discriminant $-ABC$. Moreover, the greatest common divisor of the 2×2 minors of the associated matrix is 1, since $\gcd(A, B) = 1$. Finally, we note that any $(X, Y, Z) \in \mathsf{M}_\kappa(P)$ satisfies $X, Z \leqslant 2PC^{-1}$ and $Y < Z \leqslant 2PC^{-1}$. The latter inequalities are a trivial consequence of (2.8) and (2.9). It therefore follows from [6, Cor. 2] that

$$\sum_{\kappa\in\{0,1\}} \#\mathsf{M}_\kappa(P) \ll \tau(ABC)\Big(1 + \frac{P}{(AB)^{\frac{1}{3}}C^{\frac{4}{3}}}\Big).$$

Our task is now to sum this over all $(A, B, C) \in \mathscr{A} \setminus \mathscr{A}_J$. Thus, we must consider the contribution from (A, B, C) such that $J\min\{A, B\} \leqslant C \leqslant P^{\frac{2}{5}}$. This gives

$$\ll_\varepsilon P^{\frac{4}{5}+\varepsilon} + P\sum_{A,B} \frac{\tau(AB(A+B))}{\min\{A,B\}^{\frac{1}{3}}\max\{A,B\}^{\frac{5}{3}}},$$

on noting that $(AB)^{\frac{1}{3}}C^{\frac{4}{3}} \geqslant \min\{A,B\}^{\frac{1}{3}}\max\{A,B\}^{\frac{5}{3}}$ and using the trivial estimate $\tau(n) = O_\varepsilon(n^\varepsilon)$ for the divisor function. The first term here is satisfactory for the lemma. Since $\tau(mn) \leqslant \tau(m)\tau(n)$, we see that the second term gives the contribution

$$\ll P \sum_{JA\leqslant B\leqslant P^{\frac{2}{5}}} \frac{\tau(AB(A+B))}{A^{\frac{1}{3}}B^{\frac{5}{3}}} \ll P \sum_{B\leqslant P^{\frac{2}{5}}} \frac{\tau(B)}{B^{\frac{5}{3}}} \sum_{A\leqslant J^{-1}2B} \frac{\tau(A(A+B))}{A^{\frac{1}{3}}}.$$

For any $a \in \mathbb{N}$ and $\delta \in [0, 1)$, we claim that

$$\sum_{n\leqslant x} \frac{\tau_k(n(n+a))}{n^\delta} \ll_\delta \varphi^\ddagger(a)x^{1-\delta}(\log x)^{2(k-1)}, \tag{2.13}$$

where $\tau_k(n) = \sum_{d_1\cdots d_k=n} 1$ is the generalised divisor function and we set

$$\varphi^{\ddagger}(n) = \prod_{p\mid n}\Big(1+\frac{c}{p}\Big),$$

for a suitable constant $c>0$, which is allowed to depend on k. Applying this with $k=2$ and $\delta=\frac{1}{3}$, it therefore follows that the second term makes the contribution

$$\ll \frac{P(\log P)^2}{J^{\frac{2}{3}}}\sum_{B\leqslant P^{\frac{2}{5}}}\frac{\varphi^{\ddagger}(B)\tau(B)}{B} \ll \frac{P(\log P)^4}{J^{\frac{2}{3}}},$$

by a simple convolution argument.

It remains to establish the claimed inequality (2.13). Let $S_\delta(x)$ denote the sum that is to be estimated. Now for any $a\in\mathbb{N}$ it is clear that the polynomial $f(t)=t(t+a)$ has degree 2 and discriminant a^2. Moreover, $f(t)$ has no fixed prime divisor if a is even. If a is odd then 2 is a fixed prime divisor of f, but we may then break the sum into two sums according to whether n is even or odd and make a corresponding change of variables, absorbing the additional factor $\tau_k(2)$ into an implied constant. An application of [2, Thm. 2] now reveals that

$$S_0(x) \ll x\prod_{p\leqslant x}\Big(1-\frac{\varrho_f(p)}{p}\Big)\sum_{m\leqslant x}\frac{\tau_k(m)\varrho_f(m)}{m},$$

for $x \gg a^{\varepsilon}$, where $\varrho_f(m)$ denotes the number of roots modulo m of the congruence $f(n)\equiv 0 \,(\mathrm{mod}\, m)$. Let p^ν be a prime power and suppose that $p^\alpha\|a$ for $\alpha\geqslant 0$. When $\nu=1$ we clearly have $\varrho_f(p)\leqslant 2$. For $\nu\geqslant 2$ we claim that

$$\varrho_f(p^\nu) \leqslant 2p^{\lfloor\frac{\min\{\alpha,\nu\}}{2}\rfloor}. \tag{2.14}$$

If $\alpha=0$ then this is a direct consequence of Hensel's lemma. If $\alpha\geqslant\nu$ then $\varrho_f(p^\nu) = p^{\nu-\lceil\frac{\nu}{2}\rceil} = p^{\lfloor\frac{\nu}{2}\rfloor}$. Suppose now that $\alpha>\nu$. Then we are left with counting the number of $s \,(\mathrm{mod}\, p^{\nu-\lceil\frac{\alpha}{2}\rceil})$ such that

$$s(p^{\lceil\frac{\alpha}{2}\rceil}s + p^{\alpha-\lceil\frac{\alpha}{2}\rceil}a') \equiv 0 \,(\mathrm{mod}\, p^{\nu-\lceil\frac{\alpha}{2}\rceil}),$$

where $a'=a/p^\alpha\in\mathbb{Z}$ is coprime to p. It easily follows from Hensel's lemma that there are at most 2 values of s modulo $p^{\nu-\alpha}$, from which the claim (2.14) easily follows.

We may now deduce that

$$\begin{aligned}\sum_{m\leqslant x}\frac{\tau_k(m)\varrho_f(m)}{m} &\leqslant \prod_{p\leqslant x}\Big(1+\frac{\tau_k(p)\varrho_f(p)}{p}+\sum_{\nu\geqslant 2}\frac{\tau_k(p^\nu)\varrho_f(p^\nu)}{p^\nu}\Big)\\ &\leqslant \exp\Big(\sum_{p\leqslant x}\frac{k\varrho_f(p)}{p}+\sum_{p\leqslant x}\sum_{\nu\geqslant 2}\frac{\binom{\nu+k-1}{\nu}\varrho_f(p^\nu)}{p^\nu}\Big).\end{aligned}$$

Now

$$\begin{aligned}\prod_{p\leqslant x}\Big(1-\frac{\varrho_f(p)}{p}\Big)\exp\Big(\sum_{p\leqslant x}\frac{k\varrho_f(p)}{p}\Big) &\leqslant \exp\Big(\sum_{p\leqslant x}\frac{(k-1)\varrho_f(p)}{p}\Big)\\ &\leqslant \exp\Big(\sum_{p\leqslant x}\frac{2(k-1)}{p}\Big)\\ &\ll (\log x)^{2(k-1)},\end{aligned}$$

by Mertens' theorem. Using (2.14), the remaining contribution is seen to be

$$\begin{aligned}&\sum_{p\leqslant x}\sum_{\nu\geqslant 2}\frac{\binom{\nu+k-1}{\nu}\varrho_f(p^\nu)}{p^\nu}\\ &\quad\leqslant O(1)+\sum_{p^\alpha\|a}\Big(\sum_{2\leqslant\nu\leqslant\alpha}\frac{2\binom{\nu+k-1}{\nu}}{p^{\lceil\frac{\nu}{2}\rceil}}+\sum_{\nu>\alpha}\frac{2\binom{\nu+k-1}{\nu}p^{\lfloor\frac{\alpha}{2}\rfloor}}{p^\nu}\Big)\\ &\quad\ll 1+\sum_{p\mid a}\frac{1}{p},\end{aligned}$$

since $\binom{\nu+k-1}{\nu}\leqslant k^\nu$. We therefore obtain

$$\sum_{m\leqslant x}\frac{\tau_k(m)\varrho_f(m)}{m}\ll\varphi^\ddagger(a)\exp\Big(\sum_{p\leqslant x}\frac{k\varrho_f(p)}{p}\Big),$$

whence

$$S_0(x)\ll\varphi^\ddagger(a)x\exp\Big(\sum_{p\leqslant x}\frac{(k-1)\varrho_f(p)}{p}\Big)\ll\varphi^\ddagger(a)x(\log x)^{2(k-1)}.$$

This is satisfactory for (2.13) when $\delta=0$. When $\delta>0$ the claimed bound follows from partial summation. This therefore concludes the proof. □

Throughout the remainder of this paper, we will take

$$J=(\log P)^5.$$

In particular, it is clear that the error term in Lemma 2.5 is satisfactory from the point of view of our theorem.

3. Parametrisation of conics

Fixing a choice of $(A,B,C)\in\mathscr{A}_J$, our aim in this section is to carry out a parametrisation of the conics appearing in (2.8). For given $(A,B,C)\in\mathscr{A}_J$, it will be convenient to denote by $\mathscr{C}\subset\mathbb{P}^2$ the conic in (2.8). The crux of our treatment is the observation that the conic bundle morphism $X\to\mathbb{P}^1$ has a section. Concretely this is given by any of the points $[\pm1,\pm1,\pm1]\in\mathscr{C}$, which we can use to parametrise the points on the conic. By considering the lines passing through $\eta = [1, 1, -1]$ with multiplicity 1, one establishes a bijection between such lines defined over $\mathbb{Q}$ and the rational points on $\mathscr{C} \setminus \{\eta\}$.

Let $(A, B, C) \in \mathscr{A}_J$ and define the binary quadratic forms

$$\begin{aligned} Q_1(s,t) &= At^2 - Bs^2 + 2Bst, \\ Q_2(s,t) &= -At^2 + Bs^2 + 2Ast, \\ Q_3(s,t) &= At^2 + Bs^2. \end{aligned}$$

For $s, t \in \mathbb{Z}$, we let $\lambda = \gcd(Q_1(s,t), Q_2(s,t), Q_3(s,t))$, A straightforward calculation reveals that in terms of the set $\mathsf{M}_0(P) \cup \mathsf{M}_1(P)$, the bijection is given by

$$(X, Y, Z) = \left(\frac{Q_1(s,t)}{\lambda}, \frac{Q_2(s,t)}{\lambda}, \frac{Q_3(s,t)}{\lambda} \right),$$

for coprime $(s,t) \in \mathbb{Z}^2$ such that $t > 0$. Moreover, we will have $2 \mid \lambda^{-1} Q_j(s,t)$ for at least one index $j \in \{1, 2, 3\}$ when $(X, Y, Z) \in \mathsf{M}_0(P)$, and $2 \nmid \lambda^{-1} Q_j(s,t)$ for every $j \in \{1, 2, 3\}$ when $(X, Y, Z) \in \mathsf{M}_1(P)$. We will henceforth say that a primitive integer vector (s,t) is "0-good" (resp. "1-good") if it satisfies the first (resp. second) of these conditions. Finally, we observe that $st \neq 0$ in this parametrisation since we are only interested in positive X, Y, Z.

We must now transform the counting functions $\#\mathsf{M}_\kappa(P)$ in Lemma 2.5 into ones that involve the parameters s, t. We will show that

$$\#\mathsf{M}_\kappa(P) = \#\mathsf{K}_\kappa(P) + O(1), \tag{3.1}$$

for $\kappa \in \{0, 1\}$, where $\mathsf{K}_\kappa(P)$ is basically the set $\mathsf{M}_\kappa(P)$ written in terms of s, t. The error term $O(1)$ arises from the (s,t) that correspond to the forbidden tangent line in the bijection outlined above. We will defer writing down precise expressions for $\mathsf{K}_0(P)$ and $\mathsf{K}_1(P)$ until after we have analysed the quantity λ defined above, in addition to the relations $(2.10)_\kappa$.

Lemma 3.1. *Let (s,t) be κ-good. Then we have $\lambda = 2^{1-\kappa}\lambda_1\lambda_2\lambda_3$, where*

$$\lambda_1 = \gcd(s, A), \quad \lambda_2 = \gcd(t, B), \quad \lambda_3 = \gcd(s + t, A + B).$$

Proof. We begin with the observation that

$$\lambda = \gcd(Q_1 + Q_2, Q_2 - Q_3, Q_3) = \gcd(2(A + B)st, 2At(s + t), At^2 + Bs^2).$$

Let us write $\lambda = 2^\nu \lambda^\flat$ for $\nu \geqslant 0$ and $\lambda^\flat \in \mathbb{N}$ odd. It will be convenient to write

$$(a, b)_\flat = 2^{-v_2(\gcd(a,b))} \gcd(a, b)$$

for the odd part of the greatest common divisor. Using the primitivity of (A, B, C) and (s,t) it is now straightforward to deduce that $\lambda^\flat = \lambda_1^\flat \lambda_2^\flat \lambda_3^\flat$, where

$$\lambda_1^\flat = (s, A)_\flat, \quad \lambda_2^\flat = (t, B)_\flat, \quad \lambda_3^\flat = (s + t, A + B)_\flat.$$

Thus, it remains to calculate ν, for which purpose we set

$$\begin{aligned} &\alpha = v_2(A), \quad \beta = v_2(B), \quad \gamma = v_2(A + B) = v_2(C), \\ &\sigma = v_2(s), \quad \tau = v_2(t), \quad \xi = v_2(s + t). \end{aligned}$$

Our expression for λ therefore leads to the conclusion that

$$\nu = \min\{1+\tau+\min\{\gamma+\sigma, \alpha+\xi\}, v_2(At^2+Bs^2)\}. \tag{3.2}$$

There are three basic cases to consider depending on which of α, β, γ is positive. We will deal here only with the case $\alpha \geqslant 1$, the other cases being treated similarly. In particular $\beta = \gamma = 0$ and so $\lambda_2 = \lambda_2^\flat, \lambda_3 = \lambda_3^\flat$. It therefore remains to show that

$$2^\nu \lambda_1^\flat = 2^{1-\kappa}\lambda_1, \tag{3.3}$$

if (s,t) is κ-good. If $\sigma = 0$ then (3.2) implies that $\nu = 0$, and moreover, $2^{-\nu}Q_j(s,t)$ is odd for $1 \leqslant j \leqslant 3$. Hence, (s,t) is 1-good and clearly (3.3) holds in this case. Suppose now that $\sigma \geqslant 1$. Then a little thought reveals that

$$\nu = \min\{1+\sigma, \alpha\}$$

in (3.2). We have two cases to consider according to whether $\alpha < 1+\sigma$ or $\alpha \geqslant 1+\sigma$. A straightforward calculation reveals that for $\sigma \geqslant 1$ the first inequality holds if and only if (s,t) is 1-good and the second if and only if (s,t) is 0-good. When (s,t) is 1-good, so that $\alpha < 1+\sigma$, it immediately follows that $\nu = \alpha$, whence $2^\nu\lambda_1^\flat = 2^\alpha\lambda_1^\flat = \lambda_1$. Alternatively, when (s,t) is 0-good, so that $\alpha \geqslant 1+\sigma \geqslant 2$, we conclude that $2^\nu\lambda_1^\flat = 2^{1+\sigma}\lambda_1^\flat = 2\lambda_1$. The establishes (3.3) and so completes the proof of the lemma. □

It is now time to investigate how the conditions in $(2.10)_\kappa$ are incorporated into the argument, for $\kappa \in \{0,1\}$. It will be convenient to first collect expressions for the various terms $X \pm Y, X \pm Z, Z \pm Y$. The bijection recorded at the start of the section yields

$$\begin{aligned}
X+Y &= \tfrac{-2(A+B)st}{\lambda}, & X-Y &= \tfrac{2(s+t)(At-Bs)}{\lambda},\\
X+Z &= \tfrac{2t(At-Bs)}{\lambda}, & X-Z &= \tfrac{-2Bs(s+t)}{\lambda},\\
Z+Y &= \tfrac{-2s(At-Bs)}{\lambda}, & Z-Y &= \tfrac{2At(s+t)}{\lambda}.
\end{aligned}$$

We may now establish the following result, in which we recall the notation for $\lambda_1, \lambda_2, \lambda_3$ introduced in Lemma 3.1.

Lemma 3.2. *For $(X,Y,Z) \in \mathsf{M}_\kappa(P)$, we have*

$$\begin{aligned}
\gcd(2^\kappa B, X+Z) &= 2^\kappa\lambda_2,\\
\gcd(2^\kappa C, X-Y) &= 2^\kappa\lambda_3,\\
\gcd(2^\kappa C, X+Y) &= \frac{2^\kappa C}{\lambda_3},
\end{aligned}$$

and

$$\begin{aligned}
\gcd(X+Y,X+Z)&=\gcd(Z-Y,X+Z)=\tfrac{2^\kappa t}{\lambda_2},\\
\gcd(X-Y,X-Z)&=\gcd(Z-Y,X-Z)=\tfrac{2^\kappa(s+t)}{\lambda_3},\\
\gcd(X-Y,X+Z)&=\gcd(Z+Y,X+Z)=\tfrac{2^\kappa(At-Bs)}{\lambda_1\lambda_2\lambda_3},\\
\gcd(X+Y,X-Z)&=\gcd(Z+Y,X-Z)=\tfrac{2^\kappa s}{\lambda_1}.
\end{aligned}$$

Proof. This all follows from direct calculation based on Lemma 3.1. In doing so it is critical to observe that $\lambda_1,\lambda_2,\lambda_3$ are coprime to each other. We will illustrate the sort of calculations involved by evaluating $\gcd(2^\kappa B,X+Z)$ and $\gcd(Z-Y,X-Z)$.

Beginning with the former, we note that $\lambda_1\lambda_2\lambda_3 \mid At-Bs$. But then

$$\begin{aligned}
\gcd(2^\kappa B,X+Z)&=\gcd\Big(2^\kappa B,\frac{2t(At-Bs)}{2^{1-\kappa}\lambda_1\lambda_2\lambda_3}\Big)\\
&=2^\kappa\lambda_2\gcd\Big(\frac{B}{\lambda_2},\frac{t}{\lambda_2}\cdot\frac{At-Bs}{\lambda_1\lambda_2\lambda_3}\Big)\\
&=2^\kappa\lambda_2\gcd\Big(\frac{B}{\lambda_2},\frac{At-Bs}{\lambda_1\lambda_2\lambda_3}\Big)\\
&=2^\kappa\lambda_2.
\end{aligned}$$

Next we observe that

$$\begin{aligned}
\gcd(Z-Y,X-Z)&=\gcd\Big(\frac{2^\kappa At(s+t)}{\lambda_1\lambda_2\lambda_3},\frac{2^\kappa Bs(s+t)}{\lambda_1\lambda_2\lambda_3}\Big)\\
&=\frac{2^\kappa(s+t)}{\lambda_3}\gcd\Big(\frac{A}{\lambda_1}\cdot\frac{t}{\lambda_2},\frac{B}{\lambda_2}\cdot\frac{s}{\lambda_1}\Big)\\
&=\frac{2^\kappa(s+t)}{\lambda_3}.
\end{aligned}$$

The remaining cases are dispatched in much the same spirit, which readily leads us to the statement of the lemma. $\square$

We are now ready to return to (3.1) and to consider the effect of our arguments on the expressions for $\mathsf{K}_0(P)$ and $\mathsf{K}_1(P)$. This is the subject of the following result.

Lemma 3.3. *We have*

$$\sum_{\kappa\in\{0,1\}}\#\mathsf{K}_\kappa(P)=\#\left\{(u,v)\in\mathbb{N}^2_{\mathrm{prim}}:\begin{array}{l}0<Q_1(u,v),Q_2(u,v),v>u\\ v(Av+Bu)\leqslant\frac{\lambda_1\lambda_2\lambda_3P}{C},\\ \max\{C,\lambda_2\lambda_3\}\leqslant v,\\ C\lambda_1\max\{\lambda_2,\lambda_3\}\leqslant Av+Bu\end{array}\right\}+O(1),$$

where

$$\lambda_1=\gcd(u,A),\quad \lambda_2=\gcd(v,B),\quad \lambda_3=\gcd(u-v,A+B).$$

Proof. Drawing together Lemma 3.1 and Lemma 3.2, we see in $(2.10)_\kappa$ that

$$\gcd(2^\kappa C, X-Y) \leqslant \gcd(X+Y, X+Z)$$

if and only if $\lambda_2\lambda_3 \leqslant t$. Similarly,

$$\gcd(2^\kappa C, X+Y) \leqslant \gcd(X-Y, X+Z)$$

if and only if $\lambda_1\lambda_2 \leqslant \frac{At-Bs}{C}$. Next we note that the pair of inequalities

$$2^{2\kappa}C \leqslant \gcd(2^\kappa B, X+Z)\gcd(X \pm Y, X+Z)$$

are equivalent to $C \leqslant t$ and $\lambda_1\lambda_3 \leqslant \frac{At-Bs}{C}$. The remaining expressions in $(2.10)_\kappa$ give no new information. Invoking (2.9) we deduce that

$$\mathsf{K}_\kappa(P) = \left\{ (s,t) \in \mathbb{Z}^2_{\text{prim}} : \begin{array}{l} (s,t) \text{ is } \kappa\text{-good and } |s|, t > 0, \\ 0 < Q_1(s,t), Q_2(s,t), Q_3(s,t), \\ Q_1(s,t) > Q_3(s,t), \\ Q_1(s,t) + Q_3(s,t) \leqslant \frac{2^\kappa \lambda P}{C}, \\ \max\{C, \lambda_2\lambda_3\} \leqslant t, \\ \lambda_1 \max\{\lambda_2, \lambda_3\} \leqslant \frac{At-Bs}{C} \end{array} \right\}.$$

for $\kappa \in \{0,1\}$. Furthermore, Lemma 3.1 implies that $2^\kappa\lambda = 2\lambda_1\lambda_2\lambda_3$ and we observe that any $(s,t) \in \mathbb{Z}^2$ is either 0-good or 1-good. Thus, if we sum $\mathsf{K}_\kappa(P)$ over $\kappa \in \{0,1\}$ then the right hand side remains the same, but with $2^\kappa\lambda$ replaced by $2\lambda_1\lambda_2\lambda_3$ and the first line replaced by the condition $|s|, t > 0$.

Now it is clear that $Q_3(s,t)$ is always positive. Furthermore, if $Q_1(s,t)$ and $Q_2(s,t)$ are both positive then certainly $Q_1(s,t) + Q_2(s,t) > 0$, whence $s > 0$. We write $(s,t) = (u,v)$ and observe that $Q_1(s,t) > Q_3(s,t)$ if and only if $v > u$, and $Q_1(s,t) + Q_3(s,t) = 2v(Av + Bu)$. This completes the proof. □

4. Counting lattice points

In this section, we transform our problem into a lattice point counting problem for fixed $(A,B,C) \in \mathscr{A}_J$ in the notation of Lemma 2.5. One sees that the inequalities $v > u$ and $Q_1(u,v), Q_2(u,v) > 0$ in Lemma 3.3 are equivalent to

$$u < v < u\Big(1 + \sqrt{1 + \frac{B}{A}}\Big). \tag{4.1}$$

We define the region

$$\mathscr{R}(A,B;\boldsymbol{\lambda}) = \left\{ (u,v) \in \mathbb{R}^2_{>0} : \begin{array}{l} (4.1) \text{ holds and } v(Av+Bu) \leqslant X, \\ \alpha \leqslant v \text{ and } \beta \leqslant Av + Bu \end{array} \right\}, \tag{4.2}$$

with

$$X = \frac{\lambda_1\lambda_2\lambda_3 P}{C}, \quad \alpha = \max\{C, \lambda_2\lambda_3\}, \quad \beta = C\lambda_1 \max\{\lambda_2, \lambda_3\} \tag{4.3}$$

and $C = A + B$. The inequality $v(Av + Bu) \leqslant X$ obviously implies $Av^2 < X$ and $Bu^2 < X$ since $v > u$. We conclude that

$$\mathscr{R}(A, B; \boldsymbol{\lambda}) \subseteq \Big(0, \sqrt{\frac{X}{B}}\Big) \times \Big(0, \sqrt{\frac{X}{A}}\Big), \tag{4.4}$$

with X as in (4.3).

For any $\ell \in \mathbb{N}$, $\mathbf{k} = (k_1, k_2, k_3) \in \mathbb{N}^3$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{N}^3$, we will need to work with the integer sublattice

$$(\mathbf{k}, \boldsymbol{\lambda}, \ell) = \{(u, v) \in \mathbb{Z}^2 : \ [k_1\lambda_1, \ell] \mid u, [k_2\lambda_2, \ell] \mid v, k_3\lambda_3 \mid u - v\}, \tag{4.5}$$

where $[a, b]$ denotes the least common multiple of $a, b \in \mathbb{N}$. In view of the fact that each $(A, B, C) \in \mathscr{A}_J$ has pairwise coprime coordinates, it is clear that for distinct choices of $i, j \in \{1, 2, 3\}$, we have $\gcd(k_i\lambda_i, k_j\lambda_j) = 1$, whenever $k_1\lambda_1 \mid A$, $k_2\lambda_2 \mid B$ and $k_3\lambda_3 \mid C$. We will make frequent use of this observation in the remainder of this paper. Note that $(\mathbf{k}, \boldsymbol{\lambda}, \ell)$ has rank 2 and determinant

$$\det(\mathbf{k}, \boldsymbol{\lambda}, \ell) = \frac{k_1k_2k_3\lambda_1\lambda_2\lambda_3\ell^2}{\gcd(k_1k_2k_3\lambda_1\lambda_2\lambda_3, \ell)}, \tag{4.6}$$

a calculation that is implicit in the proof of Lemma 4.2 below.

Ultimately we will be led to estimate asymptotically the number of lattice points that are constrained to lie in a suitable planar region. A useful companion in this endeavour will be the upper bound

$$\#(\mathbb{Z}^2_{\mathrm{prim}} \cap \cap \mathscr{R}) \leqslant 4\Big(\frac{\operatorname{meas}\mathscr{R}}{\det} + 1\Big), \tag{4.7}$$

which is due to Heath-Brown [9, Lemma 2] and valid for any rank 2 lattice $\subseteq \mathbb{Z}^2$ and any ellipse $\mathscr{R} \subset \mathbb{R}^2$.

Bringing together Lemma 2.5, Lemma 3.3 and (3.1), we obtain

$$L(P) = \sum_{(A,B,C)\in\mathscr{A}_J} \sum_{\lambda_1\mid A} \sum_{\lambda_2\mid B} \sum_{\lambda_3\mid C} L_1(P) + O\big(P(\log P)^{\frac{3}{2}}\big),$$

where $L_1(P)$ denotes the number of $(u, v) \in \mathbb{Z}^2_{\mathrm{prim}} \cap \mathscr{R}(A, B; \boldsymbol{\lambda})$, in the notation of (4.2) and (4.3), for which

$$\begin{cases} \lambda_1 \mid u, \quad \lambda_2 \mid v, \quad \lambda_3 \mid u - v, \\ \gcd(\frac{u}{\lambda_1}, \frac{A}{\lambda_1}) = \gcd(\frac{v}{\lambda_2}, \frac{B}{\lambda_2}) = \gcd(\frac{u-v}{\lambda_3}, \frac{A+B}{\lambda_3}) = 1. \end{cases}$$

Applying Möbius inversion to remove the latter coprimality conditions, we easily arrive at the expression

$$L(P) = \sum_{(A,B,C)\in\mathscr{A}_J} \sum_{k_1\lambda_1\mid A} \sum_{k_2\lambda_2\mid B} \sum_{k_3\lambda_3\mid C} \mu(\mathbf{k})L_2(P) + O\big(P(\log P)^{\frac{3}{2}}\big),$$

where $\mu(\mathbf{k}) = \mu(k_1)\mu(k_2)\mu(k_3)$ and

$$L_2(P) = \#\big(\mathbb{Z}^2_{\mathrm{prim}} \cap (\mathbf{k}, \boldsymbol{\lambda}, 1) \cap \mathscr{R}(A, B; \boldsymbol{\lambda})\big). \tag{4.8}$$

We now have three tasks remaining. Firstly, we must replace $\mathbb{Z}^2_{\text{prim}}$ by $\mathbb{Z}^2$ in $L_2(P)$. Secondly, we must reduce the range of summation for $\mathbf{k}$. The final task is more subtle and arises from the observation that the quantity to be estimated is zero unless $\boldsymbol{\lambda}$ is constrained to lie in a certain region. For any $R > 0$ and $(A, B, C) \in \mathscr{A}_J$, let

$$V_C(R) = \left\{ \mathbf{t} \in \mathbb{R}^3_{>0} : \begin{array}{l} C^4 \leqslant t_1 t_2 t_3 R, \\ C^2 t_i t_j \leqslant t_k R \text{ for } \{i, j, k\} = \{1, 2, 3\} \end{array} \right\}. \tag{4.9}$$

We have the following result.

Lemma 4.1. *Let $K, L \geqslant 2$ and let $\varepsilon > 0$. Then $L(P)$ is equal to*

$$\sum_{(A,B,C)\in\mathscr{A}_J} \sum_{\substack{k_1\lambda_1|A \\ k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B \\ k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C \\ k_3\leqslant K}} \mu(\mathbf{k}) \chi_{(k_1\lambda_1,k_2\lambda_2,k_3\lambda_3)}\Big(\frac{P}{L}\Big) \sum_{\ell\leqslant P^{\frac{7}{10}}} \mu(\ell) L_3(P)$$
$$+ O_\varepsilon\Big(P^{\frac{4}{5}+\varepsilon} L + P(\log P)^3 (\log JKL) + \frac{P(\log P)^6}{K}\Big),$$

where $\chi_{\mathbf{t}}(R) = \chi_{\mathbf{t}}(R; C)$ is the characteristic function of (4.9) and

$$L_3(P) = \#\big((\mathbf{k}, \boldsymbol{\lambda}, \ell) \cap \mathscr{R}(A, B; \boldsymbol{\lambda})\big).$$

Proof. We begin by recording an upper bound for $L_2(P)$ in (4.8). Combining (4.4) with (4.3), (4.6) and (4.7), it easily follows that

$$L_2(P) \ll 1 + \frac{X}{\sqrt{AB}} \cdot \frac{1}{k_1 k_2 k_3 \lambda_1 \lambda_2 \lambda_3} = 1 + \frac{P}{\sqrt{AB} C k_1 k_2 k_3}. \tag{4.10}$$

Armed with this it is now straightforward to show that the range of summation for $\mathbf{k}$ can be reduced to $\max k_i \leqslant K$ with negligible error. Indeed, if we let $E(P)$ denote the overall contribution to the main term arising from $\mathbf{k}$ satisfying $\max k_i > K$, we deduce that

$$E(P) \ll \sum_{(A,B,C)\in\mathscr{A}_J} \sum_{k_1\lambda_1|A} \sum_{k_2\lambda_2|B} \sum_{k_3\lambda_3|C} \Big(1 + \frac{P}{\sqrt{AB} C K}\Big).$$

The number of available k_i, λ_i in the sum is clearly at most $\tau_3(AB(A+B))$. Applying (2.13) with $k = 3$ and $\delta = \frac{1}{2}$, together with the trivial estimate $\tau_3(n) = O_\varepsilon(n^\varepsilon)$, we therefore obtain

$$\begin{aligned} E(P) &\ll_\varepsilon P^{\frac{4}{5}+\varepsilon} + \frac{P}{K} \sum_{A,B\leqslant P^{\frac{2}{5}}} \frac{\tau_3(AB(A+B))}{\sqrt{AB}(A+B)} \\ &\ll_\varepsilon P^{\frac{4}{5}+\varepsilon} + \frac{P(\log P)^4}{K} \sum_{B\leqslant P^{\frac{2}{5}}} \frac{\varphi^\dagger(B)\tau_3(B)}{B^{\frac{1}{2}}} \\ &\ll_\varepsilon P^{\frac{4}{5}+\varepsilon} + \frac{P(\log P)^6}{K}, \end{aligned}$$

by a simple convolution argument.

Next we deduce from (4.1) and (4.4) that any (u,v) under consideration in $L_2(P)$ is contained in the region

$$1 \leqslant u \leqslant \sqrt{\frac{X}{B}}, \quad 1 \leqslant v \leqslant \sqrt{\frac{X}{A}}, \quad v-u \leqslant \sqrt{\frac{CX}{AB}}, \tag{4.11}$$

with X given in (4.3). In fact, we have a better lower bound for v available through the fact that $\max\{C,\lambda_2\lambda_3\} \leqslant v$ in (4.2). This leads to the inequalities $AC^3 \leqslant \lambda_1\lambda_2\lambda_3 P$ and $AC\lambda_2\lambda_3 \leqslant \lambda_1 P$. On recalling the definition of $\mathscr{A}_J$ from Lemma 2.5, these imply that

$$C^4 \leqslant \lambda_1\lambda_2\lambda_3 PJ, \quad C^2\lambda_2\lambda_3 \leqslant \lambda_1 PJ.$$

Next we deduce from the the inequality $C\lambda_1\max\{\lambda_2,\lambda_3\} \leqslant Av+Bu$ in (4.2) that

$$\max\{C\lambda_1\lambda_2, C\lambda_1\lambda_3\} \leqslant \sqrt{2}\sqrt{X}(\sqrt{A}+\sqrt{B}) \ll \sqrt{\lambda_1\lambda_2\lambda_3 P},$$

whence

$$C^2\lambda_1\lambda_2 \leqslant \lambda_3 PJ, \quad C^2\lambda_1\lambda_3 \leqslant \lambda_2 PJ.$$

Using the fact that $\max k_i \leqslant K$, our argument so far shows that $L(P)$ can be approximated by

$$\sum_{(A,B,C)\in\mathscr{A}_J} \sum_{\substack{k_1\lambda_1|A\\ k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B\\ k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C\\ k_3\leqslant K}} \mu(\mathbf{k})\chi_{(k_1\lambda_1,k_2\lambda_2,k_3\lambda_3)}(PJK^2)L_2(P),$$

with an error of $O\big(P(\log P)^{\frac{3}{2}} + PK^{-1}(\log B)^6\big)$, where $\chi_{\mathbf{t}}(R)$ is the characteristic function of (4.9).

We would now like to show that this estimate is valid with PL^{-1} in place of PJK^2, with an acceptable error. To do so, it suffices to estimate the overall contribution to the main term from values of $\boldsymbol{\lambda}$ for which

$$(k_1\lambda_1, k_2\lambda_2, k_3\lambda_3) \in V_C(PJK^2) \setminus V_C(PL^{-1}),$$

in the notation of (4.9). This forces $\boldsymbol{\lambda}$ to satisfy one of four further inequalities. Let us show how to handle the contribution $E'(P)$, say, corresponding to $\boldsymbol{\lambda}$ satisfying

$$\frac{C^4}{k_1k_2k_3PJK^2} \leqslant \lambda_1\lambda_2\lambda_3 < \frac{C^4L}{k_1k_2k_3P}, \tag{4.12}$$

the remaining cases being handled in an identical manner. Applying (4.10), we see as before that

$$E'(P) \ll_\varepsilon P^{\frac{4}{5}+\varepsilon} + \sum_{(A,B,C)\in\mathscr{A}_J} \sum_{\substack{k_1\lambda_1|A\\ k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B\\ k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C\\ k_3\leqslant K\\ (4.12)\text{ holds}}} \frac{P}{\sqrt{AB}Ck_1k_2k_3}.$$

Suppose without loss of generality that $B \geqslant A$ in the outer summation. Then $E'(P)$ is

$$\ll_\varepsilon P^{\frac{4}{5}+\varepsilon} + P \sum_{\max k_i \leqslant K} \frac{1}{k_1k_2k_3} \sum_{\substack{J^{-1}B \leqslant A \leqslant B \leqslant P^{\frac{2}{5}} \\ \gcd(A,B)=1}} \frac{1}{A^{\frac{1}{2}}B^{\frac{3}{2}}} \sum_{k_1\lambda_1 | A} \sum_{k_2\lambda_2|B} \sum_{\substack{k_3\lambda_3|A+B \\ (4.12) \text{ holds}}} 1.$$

We write $E'_{\mathbf{k}}(P)$ for the inner summation over A, B and $\boldsymbol{\lambda}$.

To estimate $E'_{\mathbf{k}}(P)$, we break the sum over A, B into dyadic intervals and interchange the order of summation with $\boldsymbol{\lambda}$. This shows that $E'_{\mathbf{k}}(P)$ is

$$\leqslant \sum_{\substack{A_0 \leqslant B_0 \leqslant P^{\frac{2}{5}} \\ A_0 \gg J^{-1}B_0}} \frac{1}{A_0^{\frac{1}{2}} B_0^{\frac{3}{2}}} \sum_{\boldsymbol{\lambda}} \#\left\{(A,B) \in \mathbb{Z}^2_{\mathrm{prim}} \cap (\mathbf{k}, \boldsymbol{\lambda}, 1) : \begin{array}{l} A_0 < A \leqslant 2A_0 \\ B_0 < B \leqslant 2B_0 \end{array}\right\}$$

in the notation of (4.5), where the sum over $\boldsymbol{\lambda} \in \mathbb{N}^3$ is restricted by the inequalities

$$\lambda_1 \leqslant 2A_0, \quad \lambda_2 \leqslant 2B_0, \quad \lambda_3 \leqslant 4B_0, \quad \frac{B_0^4}{k_1k_2k_3PJK^2} \ll \lambda_1\lambda_2\lambda_3 \ll \frac{B_0^4 L}{k_1k_2k_3P}.$$

A repeat application of (4.6) and (4.7) now shows that $E'_{\mathbf{k}}(P)$ is

$$\ll \sum_{\substack{A_0 \leqslant B_0 \leqslant P^{\frac{2}{5}} \\ A_0 \gg J^{-1}B_0}} \frac{1}{A_0^{\frac{1}{2}} B_0^{\frac{3}{2}}} \sum_{\boldsymbol{\lambda}} \left(\frac{A_0B_0}{k_1k_2k_3\lambda_1\lambda_2\lambda_3} + 1 \right)$$

$$\ll \left(\frac{\log JKL}{k_1k_2k_3} \sum_{A_0 \leqslant B_0 \leqslant P^{\frac{2}{5}}} \frac{A_0^{\frac{1}{2}}}{B_0^{\frac{1}{2}}} + \frac{L}{Pk_1k_2k_3} \sum_{\substack{A_0 \leqslant B_0 \leqslant P^{\frac{2}{5}} \\ A_0 \gg J^{-1}B_0}} \frac{B_0^{\frac{5}{2}}}{A_0^{\frac{1}{2}}} \right) \sum_{\lambda_1, \lambda_2 \leqslant P^{\frac{2}{5}}} \frac{1}{\lambda_1\lambda_2}$$

$$\ll \frac{(\log P)^3 (\log JKL)}{k_1k_2k_3} + \frac{J^{\frac{1}{2}} L (\log P)^2}{P^{\frac{1}{5}} k_1k_2k_3}.$$

Inserting this into the above gives

$$E'(P) \ll_\varepsilon P^{\frac{4}{5}+\varepsilon} L + P (\log P)^3 (\log JKL),$$

thereby concluding the proof that

$$L(P) = \sum_{(A,B,C) \in \mathscr{A}_J} \sum_{\substack{k_1\lambda_1|A \\ k_1 \leqslant K}} \sum_{\substack{k_2\lambda_2|B \\ k_2 \leqslant K}} \sum_{\substack{k_3\lambda_3|C \\ k_3 \leqslant K}} \mu(\mathbf{k}) \chi_{(k_1\lambda_1, k_2\lambda_2, k_3\lambda_3)}\left(\frac{P}{L}\right) L_2(P)$$
$$+ O_\varepsilon\left(P^{\frac{4}{5}+\varepsilon} L + P(\log P)^3 (\log JKL) + \frac{P(\log P)^6}{K} \right).$$

In order to complete the proof of the lemma, it remains to replace $\mathbb{Z}^2_{\mathrm{prim}}$ by $\mathbb{Z}^2$ in the definition (4.8) of $L_2(P)$. But this follows from a direct application of Möbius inversion and the deduction from (4.4) that $L_2(P) = 0$ unless $\ell \leqslant \sqrt{\frac{X}{A}} \leqslant P^{\frac{7}{10}}$ for any ℓ dividing $(u,v) \in (\mathbf{k}, \boldsymbol{\lambda}, 1) \cap \mathscr{R}(A,B;\boldsymbol{\lambda})$. $\square$

In our work, we will take

$$J = K = (\log P)^5, \quad L = (\log P)^{100}.$$

With this choice, the error term in Lemma 4.1 is satisfactory from the point of view of our main theorem. Our task is now to estimate $L_3(P)$, which is the object of the following result.

Lemma 4.2. *We have*

$$L_3(P) = \frac{\operatorname{meas}(\mathscr{R}(A,B;\boldsymbol{\lambda}))}{\det(\mathbf{k},\boldsymbol{\lambda},\ell)} + O\left(\frac{\gcd(\ell,\lambda_1\lambda_2\lambda_3)\sqrt{\lambda_1\lambda_2\lambda_3 PJ}}{\ell C} F\right),$$

where $\det(\mathbf{k},\boldsymbol{\lambda},\ell)$ *is given by* (4.6) *and*

$$F = \frac{\max\{\lambda_1,\lambda_2,\lambda_3\}}{\lambda_1\lambda_2\lambda_3} + \frac{1}{\max\{\lambda_1,\lambda_2,\lambda_3\}}.$$

Proof. Let $= (\mathbf{k},\boldsymbol{\lambda},\ell) \subseteq \mathbb{Z}^2$ be the lattice defined in (4.5) and write $\mathscr{R} = \mathscr{R}(A,B;\boldsymbol{\lambda})$ for the region (4.2). For any $(u,v) \in$, we have $D_1 \mid u$ and $D_2 \mid v$ and $D_3 \mid u - v$, with

$$\mathbf{D} = ([k_1\lambda_1,\ell],[k_2\lambda_2,\ell],[k_3\lambda_3,\ell]).$$

The condition that $(u,v) \in \mathscr{R}$ implies that (4.11) holds with X as in (4.3). Moreover, it is easily seen that $\gcd(D_1,D_2,D_3) = \ell$.

We make the change of variables $u = D_1\sigma$ and $v = D_2\tau$. Then we are interested in counting the number of vectors $(\sigma,\tau) \in \mathbb{N}^2$ for which

$$D_1'\sigma - D_2'\tau \equiv 0 \pmod{D_3'}$$

and $(D_1\sigma, D_2\tau) \in \mathscr{R}$, where $D_i' = \ell^{-1}D_i$. Note that $\gcd(D_i',D_j') = 1$ for $i \neq j$. We will estimate this quantity in two ways: first by fixing σ and estimating the number of available τ, and then by arguing in the reverse direction.

Let us fix a choice of σ and estimate the number N_σ of available τ. It easily follows that

$$N_\sigma = \frac{V_\sigma}{D_3'} + O(1),$$

where $V_\sigma = \operatorname{meas}\{\tau \in \mathbb{R} : (D_1\sigma, D_2\tau) \in \mathscr{R}\}$. We now wish to sum this over the relevant σ, which by (4.11) gives

$$\#(\cap \mathscr{R}) = \sum_\sigma \left(\frac{V_\sigma}{D_3'} + O(1)\right) = \frac{1}{D_3'}\sum_\sigma V_\sigma + O\left(\frac{\sqrt{X}}{D_1\sqrt{B}}\right).$$

Here the summation is over all $\sigma \in \mathbb{N}$ for which $(D_1\sigma, D_2\tau) \in \mathscr{R}$, for some $\tau \in \mathbb{R}$. On noting that $V_\sigma \leqslant \frac{\sqrt{X}}{D_2\sqrt{A}}$ for any $\sigma \in \mathbb{R}$, as follows from (4.4), we deduce from an application of partial summation that the σ-summation is

$$\operatorname{meas}\left\{(\sigma,\tau)\in\mathbb{R}^2 : (D_1\sigma, D_2\tau)\in\mathscr{R}\right\} + O\left(\frac{\sqrt{X}}{D_2\sqrt{A}}\right).$$

Putting everything together, we therefore conclude that

$$\#(\cap\mathscr{R}) = \frac{\operatorname{meas}(\mathscr{R})}{D_1D_2D_3'} + O\left(\frac{\sqrt{X}}{D_2D_3'\sqrt{A}}\right) + O\left(\frac{\sqrt{X}}{D_1\sqrt{B}}\right).$$

If we begin instead by fixing τ and estimating the number of available σ, we are led to the alternative estimate

$$\#(\cap\mathscr{R}) = \frac{\operatorname{meas}(\mathscr{R})}{D_1D_2D_3'} + O\left(\frac{\sqrt{X}}{D_1D_3'\sqrt{B}}\right) + O\left(\frac{\sqrt{X}}{D_2\sqrt{A}}\right).$$

So far, we made the change of variables $(u,v)=(D_1\sigma, D_2\tau)$ and considered the congruence $D_1'\sigma - D_2'\tau \equiv 0 \pmod{D_3'}$. We are obviously free to make the change of variables $(u, u-v) = (D_1\sigma, D_3\tau)$ instead and to consider the resulting congruence $D_1'\sigma - D_3'\tau \equiv 0 \pmod{D_2'}$, where now $|D_3\tau| \leqslant \sqrt{\frac{CX}{AB}}$ by (4.11). One may now repeat the above argument more or less verbatim. Carrying this out and combining the result with our previous estimates, we are led to the expression

$$\#(\cap\mathscr{R}) = \frac{\operatorname{meas}(\mathscr{R})}{D_1D_2D_3'} + O(\sqrt{X}E),$$

where

$$E = \max\left\{\frac{1}{D_2D_3'\sqrt{A}}, \frac{1}{D_1D_3'\sqrt{B}}, \frac{1}{D_1D_2'\sqrt{B}}\right\} + \min\left\{\frac{1}{D_2\sqrt{A}}, \frac{1}{D_1\sqrt{B}}, \frac{\sqrt{C}}{D_3\sqrt{AB}}\right\}.$$

It remains to insert the definitions of D_i, D_i' that were recorded above. This reveals that the main term is

$$\frac{\operatorname{meas}(\mathscr{R})}{D_1D_2D_3'} = \frac{\operatorname{meas}(\mathscr{R})\ell}{[k_1\lambda_1,\ell][k_2\lambda_2,\ell][k_3\lambda_3,\ell]} = \frac{\operatorname{meas}(\mathscr{R})\gcd(\ell, k_1k_2k_3\lambda_1\lambda_2\lambda_3)}{k_1k_2k_3\lambda_1\lambda_2\lambda_3\ell^2} = \frac{\operatorname{meas}(\mathscr{R})}{\det},$$

by (4.6), as claimed in the lemma. Moreover, the error term is $O(\sqrt{X}E)$, with

$$E \leqslant \frac{\gcd(\ell,\lambda_1\lambda_2\lambda_3)}{\ell\sqrt{AB}}\left(\frac{\sqrt{A}}{\lambda_1\lambda_3} + \frac{\sqrt{B}}{\lambda_2\lambda_3} + \frac{\sqrt{B}}{\lambda_1\lambda_2} + \min\left\{\frac{\sqrt{A}}{\lambda_1}, \frac{\sqrt{B}}{\lambda_2}, \frac{\sqrt{C}}{\lambda_3}\right\}\right).$$

Here we have dropped the dependence on k_1, k_2, k_3, it not being useful in the final analysis to retain their influence on E. Finally, we take $\max\{A,B\} \leqslant C$

and $\min\{A, B\} \geqslant J^{-1}C$, as guaranteed by the definition of $\mathscr{A}_J$. This therefore concludes the proof of the lemma. □

The remainder of this section will be concerned with showing that once inserted into the main term in Lemma 4.1, the error term in Lemma 4.2 makes a satisfactory overall contribution. Let us write $E(P)$ for this contribution. Note that

$$\sum_{\ell \leqslant x} |\mu(\ell)| \frac{\gcd(n, \ell)}{\ell} \ll \tau(n) \log x,$$

for any $x \geqslant 2$. Carrying out the summations over ℓ and $\mathbf{k}$ we readily arrive at the estimate

$$\ll \sqrt{PJ}K^3 \log P \sum_{(A,B,C)\in\mathscr{A}_J} \sum_{\lambda_1|A} \sum_{\lambda_2|B} \sum_{\lambda_3|C} \chi_{\boldsymbol{\lambda}}\Big(\frac{PK^3}{L}\Big) \frac{\tau(\lambda_1\lambda_2\lambda_3)\sqrt{\lambda_1\lambda_2\lambda_3}}{C} F$$

for $E(P)$, where $\chi_{\mathbf{t}}(R)$ is the characteristic function of (4.9) and F is as in the statement of Lemma 4.2. Note that $\sqrt{J}K^3 \log P \leqslant K^4$ and $PK^3L^{-1} \leqslant PL^{-\frac{2}{3}}$. We open up the divisibility conditions by writing

$$A = \lambda_1\mu_1, \quad B = \lambda_2\mu_2, \quad C = \lambda_3\mu_3.$$

It follows that

$$E(P) \ll \sqrt{P}K^4 \sum_{(\boldsymbol{\lambda},\boldsymbol{\mu})\in\mathscr{C}} \frac{\tau(\lambda_1\lambda_2\lambda_3)\sqrt{\lambda_1\lambda_2}}{\sqrt{\lambda_3}\mu_3} F,$$

where reference to (4.9) yields

$$\mathscr{C} = \left\{ (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{N}^3 : \begin{array}{l} \gcd(\lambda_i\mu_i, \lambda_j\mu_j) = 1 \text{ for } i \neq j, \\ \min\{\lambda_1\mu_1, \lambda_2\mu_2\} \geqslant J^{-1}\lambda_3\mu_3, \\ \lambda_1\mu_1 + \lambda_2\mu_2 = \lambda_3\mu_3 \leqslant P^{\frac{2}{5}}, \\ \lambda_3^3\mu_3^4 \leqslant \lambda_1\lambda_2 PL^{-\frac{2}{3}},\ \lambda_1\lambda_2\lambda_3\mu_3^2 \leqslant PL^{-\frac{2}{3}}, \\ \lambda_3^3\mu_3^2\lambda_1 \leqslant \lambda_2 PL^{-\frac{2}{3}},\ \lambda_3^3\mu_3^2\lambda_2 \leqslant \lambda_1 PL^{-\frac{2}{3}} \end{array} \right\}. \tag{4.13}$$

Moreover, F is composed of two terms and can be written $F = F_1 + F_2$, say. We let $E_i(P)$ denote the contribution to $E(P)$ from F_i for $i = 1, 2$.

For given $\mathbf{L} = (L_1, L_2, L_3), \mathbf{M} = (M_1, M_2, M_3) \in \mathbb{R}^3_{\geqslant 2}$, we will need to understand the behaviour of the sum

$$S(\mathbf{L}, \mathbf{M}) = \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} \tau(\lambda_1\lambda_2\lambda_3),$$

where the sum is over all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{N}^3$ such that $\lambda_1\mu_1 + \lambda_2\mu_2 = \lambda_3\mu_3$ and

$$L_i < \lambda_i \leqslant 2L_i, \quad M_i < \mu_i \leqslant 2M_i, \quad \gcd(\lambda_i\mu_i, \lambda_j\mu_j) = 1, \tag{4.14}$$

for $1 \leqslant i < j \leqslant 3$.

Lemma 4.3. *We have* $S(\mathbf{L}, \mathbf{M}) \ll (L_1L_2L_3M_1M_2M_3)^{\frac{2}{3}}(\log L_1L_2L_3)^3$.

Proof. Taking $\tau(\lambda_1\lambda_2\lambda_3) \leqslant \tau(\lambda_1)\tau(\lambda_2)\tau(\lambda_3)$ and opening up the divisor function, it easy to see that

$$S(\mathbf{L},\mathbf{M}) \ll \sum_{d_i \leqslant \sqrt{2L_i}} \sum_{\boldsymbol{\lambda}',\boldsymbol{\mu}} 1 = \sum_{d_i \leqslant \sqrt{2L_i}} S_{\mathbf{d}}(\mathbf{L},\mathbf{M}),$$

say, where $S_{\mathbf{d}}(\mathbf{L},\mathbf{M})$ is the number of $\boldsymbol{\lambda}',\boldsymbol{\mu} \in \mathbb{N}^3$ such that

$$\frac{L_i}{d_i} < \lambda_i' \leqslant \frac{2L_i}{d_i}, \quad M_i < \mu_i \leqslant 2M_i, \quad \gcd(d_i\lambda_i'\mu_i, d_j\lambda_j'\mu_j) = 1$$

and $d_1\lambda_1'\mu_1 + d_2\lambda_2'\mu_2 = d_3\lambda_3'\mu_3$.

Now for fixed $\mathbf{d}, \boldsymbol{\lambda}'$, the latter equation forces $\boldsymbol{\mu}$ to lie on an integer sublattice of rank 2. It follows from a result due to Heath-Brown [9, Lemma 3] that the number of available $\boldsymbol{\mu}$ is

$$\ll \frac{M_1M_2M_3}{\max d_i\lambda_i'M_i} + 1 \ll \frac{(M_1M_2M_3)^{\frac{2}{3}}}{(L_1L_2L_3)^{\frac{1}{3}}} + 1,$$

on taking $\max\{a,b,c\} \geqslant (abc)^{\frac{1}{3}}$ in the denominator. Summing over the $\boldsymbol{\lambda}'$ gives

$$S_{\mathbf{d}}(\mathbf{L},\mathbf{M}) \ll \frac{(L_1L_2L_3M_1M_2M_3)^{\frac{2}{3}}}{d_1d_2d_3} + \frac{L_1L_2L_3}{d_1d_2d_3}.$$

Repeating the process, but reversing the order of summation of $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}'$, one obtains

$$S_{\mathbf{d}}(\mathbf{L},\mathbf{M}) \ll \frac{(L_1L_2L_3M_1M_2M_3)^{\frac{2}{3}}}{d_1d_2d_3} + M_1M_2M_3,$$

as a companion estimate.

Combining our two estimates, we may now conclude that

$$\begin{aligned} S(\mathbf{L},\mathbf{M}) \ll (L_1L_2L_3M_1M_2M_3)^{\frac{2}{3}}&(\log L_1L_2L_3)^3 \\ &+ \sum_{d_i \leqslant \sqrt{2L_i}} \min\Big\{\frac{L_1L_2L_3}{d_1d_2d_3}, M_1M_2M_3\Big\}. \end{aligned}$$

Taking $\min\{a,b\} \leqslant a^{\frac{1}{3}}b^{\frac{2}{3}}$ in the second term, we thereby complete the proof of Lemma 4.3. □

We are now ready to estimate $E_i(P)$ for $i = 1,2$. To do so, we will fix dyadic intervals for the $\boldsymbol{\lambda},\boldsymbol{\mu}$, writing $E_i(P;\mathbf{L},\mathbf{M})$ for the overall contribution to $E_i(P)$ for $\boldsymbol{\lambda},\boldsymbol{\mu}$ such that (4.14) holds. Writing $P_0 = PL^{-\frac{2}{3}}$, it is clear from (4.13) that $E_i(P;\mathbf{L},\mathbf{M}) = 0$ unless

$$\begin{array}{ll} L_3^3M_3^4 \ll L_1L_2P_0, & L_1L_2L_3M_3^2 \ll P_0, \\ L_3^3M_3^2L_1 \ll L_2P_0, & L_3^3M_3^2L_2 \ll L_1P_0, \\ \max\{L_1M_1, L_2M_2\} \ll L_3M_3, & L_i, M_i \gg 1. \end{array} \tag{4.15}$$

It will be convenient to write $\widetilde{L} = L_1L_2L_3$ and $\widetilde{M} = M_1M_2M_3$ in what follows.

Beginning with the contribution from F_2, we deduce that

$$E_2(P;\mathbf{L},\mathbf{M}) \ll \sqrt{P}K^4 \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} \frac{\tau(\lambda_1\lambda_2\lambda_3)\sqrt{L_1L_2}}{\sqrt{L_3}M_3 \max\{L_1,L_2,L_3\}}.$$

Now it follows from the fifth inequality in (4.15) that $L_3M_3 \geqslant \widetilde{L}^{\frac{1}{3}}\widetilde{M}^{\frac{1}{3}}$. Once combined with the fact that $\max\{L_1,L_2,L_3\} \geqslant \sqrt{L_1L_2}$, we obtain

$$\frac{\sqrt{L_1L_2}}{\sqrt{L_3}M_3 \max\{L_1,L_2,L_3\}} \ll \frac{1}{\sqrt{L_3M_3}\sqrt{M_3}} \ll \frac{1}{\widetilde{L}^{\frac{1}{6}}(M_1M_2)^{\frac{1}{6}}M_3^{\frac{2}{3}}}.$$

It now follows from Lemma 4.3 that

$$E_2(P;\mathbf{L},\mathbf{M}) \ll \frac{\sqrt{P}K^4(\log P)^3\widetilde{L}^{\frac{2}{3}}\widetilde{M}^{\frac{2}{3}}}{\widetilde{L}^{\frac{1}{6}}(M_1M_2)^{\frac{1}{6}}M_3^{\frac{2}{3}}} = \frac{\sqrt{P}K^4(\log P)^3\widetilde{L}^{\frac{1}{2}}\widetilde{M}^{\frac{1}{2}}}{M_3^{\frac{1}{2}}}.$$

The middle pair of inequalities in (4.15) combine to give $\widetilde{L}\widetilde{M} \ll L_3^3M_3^3 \ll M_3P_0$, whence

$$E_2(P;\mathbf{L},\mathbf{M}) \ll \sqrt{P}K^4(\log P)^3P_0^{\frac{1}{2}} = PK^4(\log P)^3L^{-\frac{1}{3}}.$$

On summing over the $O((\log P)^6)$ possible dyadic intervals for $\boldsymbol{\lambda},\boldsymbol{\mu}$, we conclude that $E_2(P) \ll PK^6L^{-\frac{1}{3}} \ll P$, which is satisfactory for Theorem 1.1

Turning to the estimation of $E_1(P)$, which constitutes the contribution from F_1, it will again be fruitful to analyse the contribution $E_1(P;\mathbf{L},\mathbf{M})$ from $\boldsymbol{\lambda},\boldsymbol{\mu}$ restricted by (4.14). Furthermore, we may continue to assume that $\mathbf{L},\mathbf{M}$ are constrained by (4.15). Applying Lemma 4.3, we deduce that

$$\begin{aligned} E_1(P;\mathbf{L},\mathbf{M}) &\ll \sqrt{P}K^4 \sum_{\boldsymbol{\lambda},\boldsymbol{\mu}} \frac{\tau(\lambda_1\lambda_2\lambda_3)\max\{L_1,L_2,L_3\}}{L_3M_3\widetilde{L}^{\frac{1}{2}}} \\ &\ll \sqrt{P}K^4(\log P)^3 \frac{\widetilde{L}^{\frac{1}{6}}\widetilde{M}^{\frac{2}{3}}\max\{L_1,L_2,L_3\}}{L_3M_3}. \end{aligned}$$

Now it follows from (4.15) that

$$\frac{\widetilde{L}^{\frac{1}{6}}\widetilde{M}^{\frac{2}{3}}\max\{L_1,L_2,L_3\}}{L_3M_3} \ll L_3M_3\max\{L_1,L_2,L_3\}\widetilde{L}^{-\frac{1}{2}} \ll \sqrt{P_0}.$$

Summing over the $O((\log P)^6)$ possible dyadic intervals for $\boldsymbol{\lambda},\boldsymbol{\mu}$, we therefore conclude that $E_1(P) \ll PK^6L^{-\frac{1}{3}} \ll P$. This completes our proof that the error term in Lemma 4.2 makes the overall contribution $E(P) = O(P)$ to the main term in Lemma 4.1, which is satisfactory for Theorem 1.1.

5. A divisor problem for binary cubic forms

In this section, we complete the proof of Theorem 1.1, by studying the contribution from the main term in Lemma 4.2 in Lemma 4.1. Recalling the definition (4.2) of the region $\mathscr{R}(A, B; \boldsymbol{\lambda})$, with X, α, β given by (4.3), the following result follows from making the change of variables $u = \sqrt{\frac{X}{B}}s$ and $v = \sqrt{\frac{X}{A}}t$.

Lemma 5.1. *We have*

$$\operatorname{meas} \mathscr{R}(A, B; \boldsymbol{\lambda}) = \frac{\lambda_1\lambda_2\lambda_3 P}{\sqrt{ABC}} \operatorname{meas} \mathscr{S}_{\sqrt{\frac{B}{A}}}(\alpha', \beta'),$$

where

$$\alpha' = \max\Big\{\frac{AC^3}{\lambda_1\lambda_2\lambda_3 P}, \frac{\lambda_2\lambda_3 AC}{\lambda_1 P}\Big\}^{\frac{1}{2}}, \quad \beta' = \max\Big\{\frac{C^3\lambda_1\lambda_2}{\lambda_3 AP}, \frac{C^3\lambda_1\lambda_3}{\lambda_2 AP}\Big\}^{\frac{1}{2}},$$

and for any $\delta > 0$ we set

$$\mathscr{S}_\delta(\alpha', \beta') = \left\{ (s,t) \in (0,1)^2 : \begin{array}{l} s < \delta t < s(1 + \sqrt{1+\delta^2}), \\ t(t+\delta s) \leqslant 1, \\ \alpha' \leqslant t \text{ and } \beta' \leqslant t + \delta s \end{array} \right\}.$$

We now insert the main term described in Lemma 4.2 into Lemma 4.1. Letting Σ_1 denote this contribution, the remainder of this paper will be devoted to proving that

$$\Sigma_1 = c_1 P(\log P)^4 + O\left(P(\log P)^3(\log\log P)^3\right), \tag{5.1}$$

where

$$c_1 = \frac{\pi^2}{8640}\prod_p \Big(1 - \frac{1}{p}\Big)^5\Big(1 + \frac{5}{p} + \frac{1}{p^2}\Big).$$

Inserting this asymptotic formula for Σ_1 into Lemmas 2.2 and 2.3, we obtain

$$N_{U,H}(P) \leqslant c_U P(\log P)^4 + O\big(P(\log P)^3(\log\log P)^3\big),$$

with $c_U = 60 \cdot 2 \cdot c_1$, as in (1.2). As remarked after Lemma 2.2, the corresponding lower bound is achieved with trivial changes to the argument, which thereby concludes the proof of Theorem 1.1, subject to the verification of (5.1).

To begin with, we carry out the summation over ℓ in Lemma 4.1. For this, we note that

$$\sum_{\ell \leqslant x} \mu(\ell)\frac{\gcd(n,\ell)}{\ell^2} = \frac{1}{\zeta(2)\varphi^\dagger(n)} + O\Big(\frac{\tau(n)}{x}\Big), \tag{5.2}$$

for any $n \in \mathbb{N}$ and $x \geqslant 1$, where $\varphi^\dagger(n) = \prod_{p|n}(1+\frac{1}{p})$. Thus it follows from (4.6) and Lemma 5.1 that for fixed $A, B, C, \mathbf{k}, \boldsymbol{\lambda}$ we have

$$\sum_{\ell \leqslant P^{\frac{7}{10}}} \mu(\ell) \frac{\operatorname{meas}(\mathscr{R}(A,B;\boldsymbol{\lambda}))}{\det(\mathbf{k},\boldsymbol{\lambda},\ell)} = \frac{P}{\zeta(2)\sqrt{ABC}} \cdot \frac{\operatorname{meas} \mathscr{S}_{\sqrt{\frac{B}{A}}}(\alpha',\beta')}{k_1k_2k_3\varphi^\dagger(k_1k_2k_3\lambda_1\lambda_2\lambda_3)} + O(1).$$

The overall contribution from the error term here makes the satisfactory contribution $O_\varepsilon(P^{\frac{4}{5}+\varepsilon})$ to (5.1), by the trivial estimate for the divisor function. It follows that

$$\Sigma_1 = \frac{P}{\zeta(2)} \sum_{(A,B,C)\in\mathscr{A}_J} \frac{1}{\sqrt{ABC}} \Sigma_2 + O_\varepsilon(P^{\frac{4}{5}+\varepsilon}), \tag{5.3}$$

where

$$\Sigma_2 = \sum_{\substack{k_1\lambda_1|A\\ k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B\\ k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C\\ k_3\leqslant K}} \frac{\mu(\mathbf{k})\chi_{(k_1\lambda_1,k_2\lambda_2,k_3\lambda_3)}(\frac{P}{L}) \operatorname{meas} \mathscr{S}_{\sqrt{\frac{B}{A}}}(\alpha',\beta')}{k_1k_2k_3\varphi^\dagger(k_1k_2k_3\lambda_1\lambda_2\lambda_3)}.$$

We proceed by simplifying the dependence on $\mathscr{S}_{\sqrt{\frac{B}{A}}}(\alpha',\beta')$, as in the following result.

Lemma 5.2. *We have*

$$\Sigma_2 = f\Big(\sqrt{\frac{B}{A}}\Big) \sum_{\substack{k_1\lambda_1|A\\ k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B\\ k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C\\ k_3\leqslant K}} \frac{\mu(\mathbf{k})\chi_{(k_1\lambda_1,k_2\lambda_2,k_3\lambda_3)}(\frac{P}{L})}{k_1k_2k_3\varphi^\dagger(k_1k_2k_3\lambda_1\lambda_2\lambda_3)} + O\left(\frac{\tau(A)\tau(B)\tau(C)(\log K)^3}{\log P}\right),$$

where, for any $\delta > 0$ we set

$$f(\delta) = \frac{\log(1+\delta^2)}{4\delta}.$$

Proof. We seek to remove the conditions $\alpha' \leqslant t$ and $\beta' \leqslant t + \sqrt{\frac{B}{A}}s$ from $\mathscr{S}_{\sqrt{\frac{B}{A}}}(\alpha',\beta')$. First, we show that the volume of this region can be taken over $(s,t) \in (\frac{1}{\log P}, 1)^2$ with an acceptable error. For this we note that the contribution to Σ_2 from $s \in (0, \frac{1}{\log P})$ is

$$\ll \frac{1}{\log P} \sum_{\substack{k_1\lambda_1|A\\ k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B\\ k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C\\ k_3\leqslant K}} \frac{1}{k_1k_2k_3} \ll \frac{\tau(A)\tau(B)\tau(C)(\log K)^3}{\log P},$$

But, we may now note that for $(s,t) \in (\frac{1}{\log P}, 1)^2$ we automatically have $\alpha' \leqslant t$ and $\beta' \leqslant t + \sqrt{\frac{B}{A}}s$ for any $\mathbf{k}, \lambda, A, B, C$ featuring in Σ_2, since they are subsumed by (4.9). Indeed, we have

$$\Big(\frac{AC^3}{\lambda_1\lambda_2\lambda_3 P}\Big)^{\frac{1}{2}} \leqslant K^{\frac{3}{2}}\Big(\frac{C^4}{k_1\lambda_1k_2\lambda_2k_3\lambda_3 P}\Big)^{\frac{1}{2}} \leqslant \frac{K^{\frac{3}{2}}}{L^{\frac{1}{2}}} \leqslant t,$$

and the remaining inequalities are checked similarly.

Repeating our argument, we restore the range of integration to $(s,t) \in (0,1)^2$, with an acceptable error. This leads to the statement of the lemma with

$$f(\delta) = \mathrm{meas}\{(s,t) \in (0,1)^2 : s < \delta t < s(1+\sqrt{1+\delta^2}),\ t(t+\delta s) \leqslant 1\}.$$

It remains to prove that $f(\delta) = \frac{\log(1+\delta^2)}{4\delta}$. To see this, we note that for fixed t, the variable s is restricted to lie in an interval I, say, with

$$\begin{aligned}
\mathrm{meas}(I) &= \max\Big\{0, \min\Big\{\frac{1-t^2}{\delta t}, \delta t\Big\} - \frac{\delta t}{1+\sqrt{1+\delta^2}}\Big\} \\
&= \begin{cases} \big(\frac{\delta\sqrt{1+\delta^2}}{1+\sqrt{1+\delta^2}}\big)t, & \text{if } 0 < t < (1+\delta^2)^{-\frac{1}{2}}, \\ \frac{1}{\delta t} - \big(\frac{1}{\delta} + \frac{\delta}{1+\sqrt{1+\delta^2}}\big)t, & \text{if } (1+\delta^2)^{-\frac{1}{2}} \leqslant t < (1+\delta^2)^{-\frac{1}{4}}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

A tedious but routine calculation now completes the proof. □

Returning to (5.3), it now follows from Lemma 5.2 that

$$\Sigma_1 = \frac{P}{\zeta(2)} \sum_{(A,B,C)\in\mathscr{A}_J} \frac{f(\sqrt{\frac{B}{A}})}{\sqrt{ABC}} \sum_{\substack{k_1\lambda_1|A\\k_1\leqslant K}} \sum_{\substack{k_2\lambda_2|B\\k_2\leqslant K}} \sum_{\substack{k_3\lambda_3|C\\k_3\leqslant K}} \frac{\mu(\mathbf{k})\chi_{(k_1\lambda_1,k_2\lambda_2,k_3\lambda_3)}(\frac{P}{L})}{k_1k_2k_3\varphi^\dagger(k_1k_2k_3\lambda_1\lambda_2\lambda_3)}$$
$$+ O\left(P(\log P)^3(\log\log P)^3\right),$$

since the error term makes the overall contribution

$$\ll \frac{P(\log K)^3}{\log P} \sum_{A,B\leqslant P^{\frac{2}{5}}} \frac{\tau(A)\tau(B)\tau(A+B)}{\sqrt{AB}(A+B)} \ll P(\log P)^3(\log\log P)^3,$$

by an application of (2.13) with $k=2$ and $\delta = \frac{1}{2}$. Next, we can extend the summation over $\mathbf{k}$ to infinity with error $\ll K^{-1}P(\log P)^6 \ll P\log P$.

For any arithmetic function h and any $N \in \mathbb{N}$, we have

$$\sum_{k\lambda|N} \frac{\mu(k)h(k\lambda)}{k} = \sum_{n|N} h(n) \sum_{k|n} \frac{\mu(k)}{k} = \sum_{n|N} \varphi^*(n)h(n),$$

where $\varphi^*(n)=\prod_{p|n}(1-\frac{1}{p})$. It follows that we may write

$$\Sigma_1=\frac{P}{\zeta(2)}\sum_{(A,B,C)\in\mathscr{A}_J}\frac{f(\sqrt{\frac{B}{A}})}{\sqrt{ABC}}\sum_{n_1|A}\sum_{n_2|B}\sum_{n_3|C}\chi_{\mathbf{n}}\Big(\frac{P}{L}\Big)\prod_{i=1}^{3}\frac{\varphi^*(n_i)}{\varphi^\dagger(n_i)}\\ +O\left((\log P)^3(\log\log P)^3\right).$$

Now the only possible choice of $(A,B,C)\in\mathscr{A}_J$ with $A=B$ is the vector $(1,1,2)$. This term contributes $O(P)$ to Σ_1. For the remaining contribution, we break the summation over $(A,B,C)\in\mathscr{A}_J$ into those vectors for which $B>A$ and those for which $B<A$. This allows us to take

$$\Sigma_1=\frac{P}{\zeta(2)}\Sigma_3+O\left(P(\log P)^3(\log\log P)^3\right),$$

with

$$\Sigma_3=\sum_{\substack{(A,B,C)\in\mathscr{A}_J\\ B>A}}\frac{F(\frac{B}{A})}{C^2}\sum_{\substack{\mathbf{n}\in\mathbb{N}^3\\ n_1|A,\ n_2|B,\ n_3|C}}\chi_{\mathbf{n}}\Big(\frac{P}{L}\Big)\prod_{i=1}^{3}\frac{\varphi^*(n_i)}{\varphi^\dagger(n_i)},$$

and where

$$\begin{aligned}F(u)&=\Big(f(\sqrt{u})+f\Big(\frac{1}{\sqrt{u}}\Big)\Big)\Big(\sqrt{u}+\frac{1}{\sqrt{u}}\Big)\\ &=\frac{u+1}{4u}\big(\log(u+1)+u\log(u^{-1}+1)\big),\end{aligned}\tag{5.4}$$

for any $u>1$. In particular it is clear that $F(u)\ll\log u$.

In terms of Dirichlet convolution, we clearly have $\frac{\varphi^*(n)}{\varphi^\dagger(n)}=(1*h)(n)$, with h given multiplicatively by

$$h(p^e)=\begin{cases}1, & \text{if } e=0,\\ \frac{-2}{p+1}, & \text{if } e=1,\\ 0, & \text{otherwise.}\end{cases}\tag{5.5}$$

For any $t\geqslant 1$, let

$$\mathscr{B}_t=\{\mathbf{x}\in\mathbb{R}^2_{>0}: tx_1<x_2<x_1+x_2\leqslant 1,\ x_1\geqslant J^{-1}(x_1+x_2)\}$$

and define the linear forms

$$L_1(\mathbf{x})=x_1,\quad L_2(\mathbf{x})=x_2,\quad L_3(\mathbf{x})=x_1+x_2.$$

Adopting the notation $X\mathscr{R}=\{X\mathbf{x}:\mathbf{x}\in\mathscr{R}\}$ for a region $\mathscr{R}\subset\mathbb{R}^2$, and reserving i for a generic index from the set $\{1,2,3\}$, we may now write

$$\Sigma_1=\frac{P}{\zeta(2)}\sum_{\substack{\mathbf{l}\in\mathbb{N}^3\\ \gcd(m_i,m_j)=1}}h(m_1)h(m_2)h(m_3)\Sigma_3(\mathbf{l})+O\left(P(\log P)^3(\log\log P)^3\right),\tag{5.6}$$

where

$$\Sigma_3(\mathbf{l}) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\cap P^{\frac{2}{5}}\mathscr{B}_1\\ \gcd(x_1,x_2)=1\\ m_i \mid L_i(\mathbf{x})}} \frac{F(\frac{x_2}{x_1})}{(x_1+x_2)^2} \sum_{\substack{\mathbf{d}\in\mathbb{N}^3\\ d_i \mid m_i^{-1}L_i(\mathbf{x})}} \chi_{(d_1m_1,d_2m_2,d_3m_3)}\Big(\frac{P}{L}\Big).$$

Before evaluating $\Sigma_3(\mathbf{l})$ asymptotically, it will be useful to have an upper bound which is uniform in all of the relevant parameters. Recall the definition of $\varphi^\dagger$ from (5.2). The main result in [2] implies that for $X \geqslant 2$ there exists an absolute constant $c > 0$ such that

$$\sum_{\substack{|x_1|,|x_2|\leqslant X\\ m_i \mid L_i(\mathbf{x})}} \tau\Big(\frac{L_1(\mathbf{x})}{m_1}\Big)\tau\Big(\frac{L_2(\mathbf{x})}{m_2}\Big)\tau\Big(\frac{L_3(\mathbf{x})}{m_3}\Big) \ll_\varepsilon \varphi^\dagger(Q)^c\Big(\frac{X^2(\log X)^3}{Q} + X^{1+\varepsilon}\Big), \quad (5.7)$$

for any $\varepsilon > 0$, where $Q = m_1m_2m_3$. Here we have used the fact that the conditions $m_i \mid L_i(\mathbf{x})$ define a lattice $\;= (\mathbf{1},\mathbf{l},1)$, in the notation of (4.5), which (4.6) confirms has determinant Q. We now observe that

$$\sum_{\max m_i > T} \frac{|h(m_1)h(m_2)h(m_3)|}{(m_1m_2m_3)^\delta} \ll_\varepsilon T^{-\delta+\varepsilon}$$

and

$$\sum_{\max m_i \leqslant T} |h(m_1)h(m_2)h(m_3)| = \left(\sum_{\max m \leqslant T} |h(m)|\right)^3 \ll (\log T)^6,$$

for any $\delta, T > 0$. Furthermore, we have $F(\frac{x_2}{x_1}) \ll \log(\frac{x_2}{x_1}) \ll \log\log P$ in our expression for $\Sigma_3(\mathbf{l})$. With these estimates to hand, we deduce from (5.7) that the overall contribution to Σ_1 from $\max m_i > \log P$ is

$$\begin{aligned} &\ll P\log\log P \sum_{\log P < \max m_i} |h(m_1)h(m_2)h(m_3)| \\ &\quad \times \sum_{\substack{|x_1|,|x_2|\leqslant P^{\frac{2}{5}}\\ m_i \mid L_i(\mathbf{x})}} \frac{\tau(\frac{L_1(\mathbf{x})}{m_1})\tau(\frac{L_2(\mathbf{x})}{m_2})\tau(\frac{L_3(\mathbf{x})}{m_3})}{(x_1+x_2)^2} \\ &\ll P(\log P)^3(\log\log P). \end{aligned}$$

This therefore makes an acceptable contribution to the error in (5.6).

Let $P_0 = PL^{-1}$, $t \geqslant 1$, $X \geqslant 2$ and let $\mathbf{l} \in \mathbb{N}^3$ such that $\gcd(m_i, m_j) = 1$. Our asymptotic formula for $\Sigma_3(\mathbf{l})$ hinges upon an investigation of the sums

$$\begin{aligned} S_t(X;\mathbf{l}) &= \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\cap X\mathscr{B}_t \\ \gcd(x_1,x_2)=1 \\ m_i \mid L_i(\mathbf{x})}} \sum_{\substack{\mathbf{d}\in\mathbb{N}^3 \\ d_i \mid m_i^{-1}L_i(\mathbf{x})}} \chi_{(d_1m_1,d_2m_2,d_3m_3)}(P_0) \\ &= \sum_{\substack{\mathbf{x}\in(\mathbf{m})\cap X\mathscr{B}_t \\ \gcd(x_1,x_2)=1}} \#\left\{ \mathbf{d}\in\mathbb{N}^3 : \begin{array}{l} d_i \mid m_i^{-1}L_i(\mathbf{x}), \\ (\boldsymbol{\delta}+\boldsymbol{\mu},\boldsymbol{\xi})\in V, \\ \gcd(d_im_i, d_jm_j) = 1 \end{array} \right\}, \end{aligned}$$

where $(\mathbf{m}) = \{\mathbf{x} \in \mathbb{Z}^2 : m_i \mid L_i(\mathbf{x})\}$ and

$$\delta_i = \frac{\log d_i}{\log P_0}, \quad \mu_i = \frac{\log m_i}{\log P_0}, \quad \xi_i = \frac{\log L_i(\mathbf{x})}{\log P_0},$$

and finally,

$$V = \left\{ \mathbf{v} \in [0,1]^6 : \begin{array}{l} 4v_4 \leqslant 1 + v_1 + v_2 + v_3, \\ 2v_4 + v_i + v_j \leqslant 1 + v_k \text{ for } \{i,j,k\} = \{1,2,3\}, \\ \max\{v_1, v_2, v_3\} \leqslant v_4 \end{array} \right\}.$$

Note that $V = V^* \times [0,1]^2$ for an appropriate subset $V^* \subseteq [0,1]^4$. Estimating this sum is the focus of the following result.

Lemma 5.3. *We have*

$$S_t(X;\mathbf{l}) = g(\mathbf{m})(\log P_0)^3 \int_{\mathbf{x}\in X\mathscr{B}_t} \int_{\{\boldsymbol{\tau}\in\mathbb{R}^3 : (\boldsymbol{\tau},\boldsymbol{\xi})\in V\}} \mathrm{d}\boldsymbol{\tau}\mathrm{d}\mathbf{x} + O_\varepsilon(E_{\mathbf{m}}(X,P)),$$

where $E_{\mathbf{m}}(X,P) = (m_1m_2m_3)^{-1}X^2(\log P)^2 + X^{\frac{7}{4}+\varepsilon}$ *and*

$$g(\mathbf{m}) = \frac{1}{m_1m_2m_3} \prod_{p \mid m_1m_2m_3} \Big(1 + \frac{3}{p} - \frac{1}{p^2}\Big)^{-1} \prod_p \Big(1 - \frac{1}{p}\Big)^3 \Big(1 + \frac{3}{p} - \frac{1}{p^2}\Big).$$

Proof. To estimate $S_t(X;\mathbf{l})$ we appeal to work of Browning [5], which is concerned with sums of the shape

$$\sum \tau(L_1(\mathbf{x}))\tau(L_2(\mathbf{x}))\tau(L_3(\mathbf{x})),$$

for arbitrary non-proportional linear forms $L_i \in \mathbb{Z}[x_1, x_2]$ and $\mathbf{x} = (x_1, x_2)$ running over integers restricted to a suitable expanding region.

Let us write $S(X) = S_t(X;\mathbf{l})$ for short. Removing the coprimality condition on $\mathbf{x}$ using Möbius inversion, we obtain

$$S(X) = \sum_{\ell=1}^{\infty} \mu(\ell) S_\ell(X),$$

where $S_\ell(X)$ is defined as for $S(X)$ but with $\gcd(x_1, x_2) = 1$ replaced by $\ell \mid \mathbf{x}$. We have $S_\ell(X) \ll_\varepsilon \ell^{-2}X^{2+\varepsilon}$ for any $\varepsilon > 0$, using the trivial estimate for the divisor function. Hence, we can restrict to $\ell \leqslant X^{\frac{1}{4}}$ in the above sum with error $O_\varepsilon(X^{\frac{7}{4}+\varepsilon})$.

In our analysis of $S_\ell(X)$ for fixed ℓ we will need to arrange things so that we are only considering small divisors in the summand. It is easy to see that the overall contribution to the sum from $\mathbf{d}$ such that $d_j^2 = m_j^{-1}L_j(\mathbf{x})$ for some $j \in \{1,2,3\}$ is

$$\ll_\varepsilon X^\varepsilon \sum_{d_j \leqslant \sqrt{X}} \#\{\mathbf{x} \in \mathbb{Z}^2 \cap X\mathscr{B}_t \,:\, \ell \mid \mathbf{x},\; L_j(\mathbf{x}) = m_j d_j^2\} \ll_\varepsilon \ell^{-1} X^{\frac{3}{2}+\varepsilon}.$$

It follows that we may write

$$S(X) = \sum_{\ell \leqslant X^{\frac{1}{4}}} \sum_{\sigma \in \{\pm 1\}^3} S_\ell^{(\sigma)}(X) + O_\varepsilon(X^{\frac{7}{4}+\varepsilon}), \tag{5.8}$$

where $S_\ell^{(\sigma)}(X)$ is the contribution from $\sigma_i d_i \leqslant \sigma_i \sqrt{m_i^{-1}L_i(\mathbf{x})}$.

We indicate how to estimate $S_\ell^{(1,1,-1)}(X) = S_\ell^{+,+,-}(X)$, say, which is typical. Putting $m_3^{-1}L_3(\mathbf{x}) = d_3 f_3$, we see that $f_3 \leqslant \sqrt{m_3^{-1}L_3(\mathbf{x})}$ and

$$\delta_3 = \frac{\log(m_3^{-1} f_3^{-1} L_3(\mathbf{x}))}{\log P_0} = \xi_3 - \frac{\log f_3}{\log P_0} - \mu_3.$$

On relabelling the variables, we may therefore write $S_\ell^{+,+,-}(X)$ as a sum

$$\sum_{\substack{\mathbf{x} \in (\mathbf{m}) \cap X\mathscr{B}_t \\ \ell \mid \mathbf{x}}} \#\left\{\mathbf{d} \in \mathbb{N}^3 \,:\, \begin{array}{l} d_i \mid m_i^{-1}L_i(\mathbf{x}),\; d_i \leqslant \sqrt{m_i^{-1}L_i(\mathbf{x})}, \\ (\boldsymbol{\delta}, \boldsymbol{\xi}) \in V_{+,+,-}(\mathbf{m}), \\ \gcd(m_i d_i, m_j d_j) = 1 \end{array}\right\},$$

where

$$V_{+,+,-}(\mathbf{m}) = \{(\boldsymbol{\delta}, \boldsymbol{\xi}) \in \mathbb{R}^6 \,:\, (\delta_1 + \mu_1, \delta_2 + \mu_2, \xi_3 - \delta_3, \boldsymbol{\xi}) \in V\}.$$

Let $D_i = [m_i d_i, \ell]$ and $D_i' = \ell^{-1} D_i$. Since $\gcd(m_i d_i, m_j d_j) = 1$ each pair D_i', D_j' is coprime. Interchanging the order of summation, we obtain

$$S_\ell^{+,+,-}(X) = \sum_{\substack{\mathbf{d} \in \mathbb{N}^3 \\ \gcd(m_i d_i, m_j d_j) = 1}} \#\left\{\mathbf{x} \in (\mathbf{D}) \cap X\mathscr{B}_t \,:\, \boldsymbol{\xi} \in V_{+,+,-}(\mathbf{m}; \mathbf{d})\right\},$$

where $\boldsymbol{\xi} \in V_{+,+,-}(\mathbf{m}; \mathbf{d})$ if and only if $(\boldsymbol{\delta}, \boldsymbol{\xi}) \in V_{+,+,-}(\mathbf{m})$ and $2\delta_i + \mu_i \leqslant \xi_i$. The underlying region that appears here is a compact subset of $\mathbb{R}^2$ whose boundary is a piecewise continuously differentiable closed curve with absolutely bounded length. Making the change of variables $\mathbf{x} = \ell \mathbf{y}$ and applying [5, Lemma 1], we deduce that

$$S_\ell^{+,+,-}(X) = \sum_{\substack{\mathbf{d}\in\mathbb{N}^3\\ \gcd(m_id_i,m_jd_j)=1}} \frac{\operatorname{meas}\{\mathbf{x}\in X\mathscr{B}_t : \boldsymbol{\xi}\in V_{+,+,-}(\mathbf{m};\mathbf{d})\}\varrho(\mathbf{D}')}{(\ell D_1'D_2'D_3')^2} + O_\varepsilon\Big(\frac{X^{\frac{7}{4}+\varepsilon}}{\ell}+X^{\frac{3}{2}+\varepsilon}\Big).$$

The error term makes the overall contribution $O_\varepsilon(X^{\frac{7}{4}+\varepsilon})$ once summed over $\ell\leqslant X^{\frac{1}{4}}$. Moreover, it is easy to see that

$$\frac{\varrho(\mathbf{D}')}{(\ell D_1'D_2'D_3')^2} = \frac{1}{\ell^2 D_1'D_2'D_3'} = \frac{\gcd(\ell, m_1m_2m_3d_1d_2d_2)}{\ell^2}\cdot\frac{1}{m_1m_2m_3d_1d_2d_2},$$

by [5, Lemma 3].

Bringing in the summation over ℓ, we set

$$S^{+,+,-}(X) = \sum_{\ell\leqslant X^{\frac{1}{4}}} \mu(\ell) S_\ell^{+,+,-}(X).$$

Let

$$c_{\mathbf{m}} = \frac{1}{\zeta(2)m_1m_2m_3\varphi^\dagger(m_1m_2m_3)},$$

in the notation of (5.2). It therefore follows from this that

$$\begin{aligned}
S^{+,+,-}(X) &= c_{\mathbf{m}}\sum_{\mathbf{d}\in\mathbb{N}^3}\frac{\operatorname{meas}\{\mathbf{x}\in X\mathscr{B}_t : \boldsymbol{\xi}\in V_{+,+,-}(\mathbf{m};\mathbf{d})\}f_{\mathbf{m}}(\mathbf{d})}{d_1d_2d_3}\\
&\quad + O_\varepsilon(X^{\frac{7}{4}+\varepsilon})\\
&= c_{\mathbf{m}}\int_{\mathbf{x}\in X\mathscr{B}_t}\sum_{\substack{\mathbf{d}\in\mathbb{N}^3\\ 2\delta_i+\mu_i\leqslant\xi_i}}\frac{\chi_V(\delta_1+\mu_1,\delta_2+\mu_2,\xi_3-\mu_3,\boldsymbol{\xi})f_{\mathbf{m}}(\mathbf{d})}{d_1d_2d_3}\mathrm{d}\mathbf{x}\\
&\quad + O_\varepsilon(X^{\frac{7}{4}+\varepsilon}),
\end{aligned}$$

where χ_V is the characteristic function of the set V and

$$f_{\mathbf{m}}(\mathbf{d}) = \begin{cases}\prod_{i=1}^3\frac{\varphi^\dagger(\gcd(d_i,m_i))}{\varphi^\dagger(d_i)}, & \text{if } \gcd(m_id_i,m_jd_j)=1,\\ 0, & \text{otherwise.}\end{cases}$$

We now write $f_{\mathbf{m}} = h_{\mathbf{m}} * 1$ as a convolution, for a multiplicative arithmetic function $h_{\mathbf{m}}$. The condition $\gcd(m_i,m_j)=1$ is automatic. Let us calculate the function $f_{\mathbf{m}}$ at prime powers. Writing $m_i = p^{\alpha_i}$ and $d_i = p^{\gamma_i}$, and assuming without loss of generality that $\alpha_1 \geqslant 0$ and $\alpha_2 = \alpha_3 = 0$, we see that

$$f_{p^{\alpha_1},p^{\alpha_2},p^{\alpha_3}}(p^{\gamma_1}, p^{\gamma_2}, p^{\gamma_3}) = \begin{cases} 1, & \text{if } \gamma_1 = \gamma_2 = \gamma_3 = 0, \\ 1, & \text{if } \alpha_1, \gamma_1 \geqslant 1 \text{ and } \gamma_2 = \gamma_3 = 0, \\ (1 + \frac{1}{p})^{-1}, & \text{if } \gamma_1 \geqslant 1 \text{ and } \alpha_1 = \gamma_2 = \gamma_3 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let us write $h(p^{\beta_1}, p^{\beta_2}, p^{\beta_3}) = h_{p^{\alpha_1},p^{\alpha_2},p^{\alpha_3}}(p^{\beta_1}, p^{\beta_2}, p^{\beta_3})$ for short. Continuing under the hypothesis that $\alpha_1 \geqslant 0$ and $\alpha_2 = \alpha_3 = 0$, we find that

$$h(1,p,1) = h(1,1,p) = \begin{cases} \frac{-1}{p+1}, & \text{if } \alpha_1 = 0, \\ -1, & \text{if } \alpha_1 \geqslant 1, \end{cases} \quad h(p,1,1) = \begin{cases} \frac{-1}{p+1}, & \text{if } \alpha_1 = 0, \\ 0, & \text{if } \alpha_1 \geqslant 1, \end{cases}$$

$$h(p,p,1) = h(p,1,p) = \begin{cases} \frac{1-p}{p+1}, & \text{if } \alpha_1 = 0, \\ 0, & \text{if } \alpha_1 \geqslant 1, \end{cases} \quad h(1,p,p) = \begin{cases} \frac{1-p}{p+1}, & \text{if } \alpha_1 = 0, \\ 1, & \text{if } \alpha_1 \geqslant 1, \end{cases}$$

and

$$h(1,1,1) = 1, \quad h(p,p,p) = \begin{cases} \frac{2p-1}{p+1}, & \text{if } \alpha_1 = 0, \\ 0, & \text{if } \alpha_1 \geqslant 1, \end{cases}$$

with $h(p^{\beta_1}, p^{\beta_2}, p^{\beta_3}) = 0$ in all other cases.

Opening up the convolution, we obtain

$$S^{+,+,-}(X) = c_{\mathbf{m}} \sum_{\mathbf{k} \in \mathbb{N}^3} \frac{h_{\mathbf{m}}(\mathbf{k})}{k_1 k_2 k_3} \int_{\mathbf{x} \in X\mathscr{B}_t} M(X) \mathrm{d}\mathbf{x} + O_\varepsilon(X^{\frac{7}{4}+\varepsilon}),$$

where for $\kappa_i = \frac{\log k_i}{\log P_0}$ we set

$$M(X) = \sum_{\substack{\mathbf{d} \in \mathbb{N}^3 \\ 2\delta_i + 2\kappa_i + \mu_i \leqslant \xi_i}} \frac{\chi_V(\delta_1 + \mu_1 + \kappa_1, \delta_2 + \mu_2 + \kappa_2, \xi_3 - \delta_3 - \kappa_3, \boldsymbol{\xi})}{d_1 d_2 d_3}.$$

This sum depends on $\mathbf{m}, \mathbf{k}, \mathbf{x}$ in addition to X. It is now clear from repeated applications of Euler–Maclaurin summation that $M(X)$ can be approximated by

$$\int_{2\frac{\log t_i}{\log P_0} + 2\kappa_i + \mu_i \leqslant \xi_i} \frac{\chi_V(\frac{\log t_1}{\log P_0} + \mu_1 + \kappa_1, \frac{\log t_2}{\log P_0} + \mu_2 + \kappa_2, \xi_3 - \frac{\log t_3}{\log P_0} - \kappa_3, \boldsymbol{\xi})}{t_1 t_2 t_3} \mathrm{d}\mathbf{t} + O((\log P)^2),$$

uniformly in $\mathbf{m}, \mathbf{k}, \mathbf{x}$. Making the change of variables

$$\tau_1 = \frac{\log t_1}{\log P_0} + \mu_1 + \kappa_1, \quad \tau_2 = \frac{\log t_2}{\log P_0} + \mu_2 + \kappa_2, \quad \tau_3 = \xi_3 - \frac{\log t_3}{\log P_0} - \kappa_3,$$

we obtain

$$M(X) = (\log P_0)^3 \int_{\substack{2\tau_1 \leqslant \xi_1+\mu_1 \\ 2\tau_2 \leqslant \xi_2+\mu_2 \\ 2\tau_3 > \xi_3+\mu_3}} \chi_V(\boldsymbol{\tau}, \boldsymbol{\xi}) \mathrm{d}\boldsymbol{\tau} + O((\log P)^2).$$

We now reintroduce the summation over $\mathbf{k}$ and the integration over $\mathbf{x}$. This gives the asymptotic formula

$$c_{\mathbf{m}} (\log P_0)^3 \sum_{\mathbf{k} \in \mathbb{N}^3} \frac{h_{\mathbf{m}}(\mathbf{k})}{k_1 k_2 k_3} \int_{\mathbf{x} \in X\mathscr{B}_t} \int_{\substack{2\tau_1 \leqslant \xi_1+\mu_1 \\ 2\tau_2 \leqslant \xi_2+\mu_2 \\ 2\tau_3 > \xi_3+\mu_3}} \chi_V(\boldsymbol{\tau}, \boldsymbol{\xi}) \mathrm{d}\boldsymbol{\tau} \mathrm{d}\mathbf{x} + O(E_{\mathbf{m}}(X, P)),$$

for $S^{+,+,-}(X)$, where $E_{\mathbf{m}}(X, P)$ is as in the statement of Lemma 5.3.

Our final task involves summing over the various $\mathbf{m}$ in (5.8), assuming analogous formulae for all the sums $S^{\pm,\pm,\pm}(X)$. This shows that $S(X)$ is

$$c_{\mathbf{m}} (\log P_0)^3 \sum_{\mathbf{k} \in \mathbb{N}^3} \frac{h_{\mathbf{m}}(\mathbf{k})}{k_1 k_2 k_3} \int_{\mathbf{x} \in X\mathscr{B}_t} \int_{\boldsymbol{\tau} \in \mathbb{R}^3} \chi_V(\boldsymbol{\tau}, \boldsymbol{\xi}) \mathrm{d}\boldsymbol{\tau} \mathrm{d}\mathbf{x} + O(E_{\mathbf{m}}(X, P)).$$

We are now free to carry out the summation over $\mathbf{k}$, finding that

$$\begin{aligned} \sum_{\mathbf{k} \in \mathbb{N}^3} \frac{h_{\mathbf{m}}(\mathbf{k})}{k_1 k_2 k_3} = &\prod_{p \nmid m_1 m_2 m_3} \Big(1 - \frac{1}{p}\Big)^2 \Big(1 + \frac{1}{p}\Big)^{-1} \Big(1 + \frac{3}{p} - \frac{1}{p^2}\Big) \\ &\times \prod_{p \mid m_1 m_2 m_3} \Big(1 - \frac{1}{p}\Big)^2. \end{aligned}$$

Multiplying this by $c_{\mathbf{m}}$ readily leads to the function $\mathrm{g}(\mathbf{m})$ defined in the statement of the lemma. □

It is clear that $\mathbf{x}$ restricted to $\frac{X}{\log P}\mathscr{B}_t$ make an overall contribution

$$\ll (m_1 m_2 m_3)^{-1} X^2 \log P \ll E_{\mathbf{m}}(X, P)$$

to the main term in Lemma 5.3's estimate for $S_t(X; \mathbf{m})$. Recall that $V = V^* \times [0,1]^2$ for an appropriate subset $V^* \subseteq [0,1]^4$. Hence, we have that $S_t(X; \mathbf{m})$ is

$$\mathrm{g}(\mathbf{m}) (\log P_0)^3 \int_{\substack{\mathbf{x} \in X\mathscr{B}_t \\ x_1 + x_2 > \frac{X}{\log P}}} \int_{\{\boldsymbol{\tau} \in \mathbb{R}^3 : (\boldsymbol{\tau}, \xi) \in V^*\}} \mathrm{d}\boldsymbol{\tau} \mathrm{d}\mathbf{x} + O_\varepsilon(E_{\mathbf{m}}(X, P)),$$

where now $\xi = \frac{\log x_1 + x_2}{\log P_0}$. But a simple change of variables shows that this is equal to

$$\begin{aligned} &\mathrm{g}(\mathbf{m}) X^2 (\log P_0)^3 \operatorname{meas}(\mathscr{B}_{t,P}) \int_{\{\boldsymbol{\tau} \in \mathbb{R}^3 : (\boldsymbol{\tau}, \frac{\log X}{\log P_0}) \in V^*\}} \mathrm{d}\boldsymbol{\tau} \\ &\qquad + O(E_{\mathbf{m}}(X, P) \log\log P), \end{aligned}$$

where $\mathscr{B}_{t,P} = \{\mathbf{x} \in \mathscr{B}_t : x_1 + x_2 > \frac{1}{\log P}\}$. Now it is clear that

$$\operatorname{meas}(\mathscr{B}_{t,P}) = \operatorname{meas}(\mathscr{B}_t) + O\Big(\frac{1}{\log P}\Big) = \frac{1}{2(t+1)} + O\Big(\frac{1}{\log P} + \frac{1}{J}\Big).$$

We therefore conclude that

$$S_t(X;\mathbf{l}) = \frac{\mathrm{g}(\mathbf{m})X^2(\log P_0)^3}{2(t+1)} \int_{\{\tau\in\mathbb{R}^3 : (\tau, \frac{\log X}{\log P_0})\in V^*\}} \mathrm{d}\tau + O(E_{\mathbf{m}}(X,P)\log\log P).$$

Note that $F(u) = F(1) + \int_1^u F'(v)\mathrm{d}v$ for any $u > 1$ in (5.4). Furthermore, we may write $(x_1+x_2)^{-2} = P^{-\frac{4}{5}} + 2\int_{x_1+x_2}^{P^{\frac{2}{5}}} w^{-3}\mathrm{d}w$. Hence, we deduce that

$$\Sigma_3(\mathbf{l}) = 2\int_1^{P^{\frac{2}{5}}} \Big(F(1)S_1(w;\mathbf{m}) + \int_1^J F'(v)S_v(w;\mathbf{m})\mathrm{d}v\Big)\frac{\mathrm{d}w}{w^3} + O(\mathscr{E}),$$

where

$$\mathscr{E} = P^{-\frac{4}{5}} \sum_{\substack{\mathbf{x}\in\mathbb{N}^2 \\ J^{-1}x_2<x_1<x_2\leqslant P^{\frac{2}{5}} \\ m_i \mid L_i(\mathbf{x})}} \tau\Big(\frac{L_1(\mathbf{x})}{m_1}\Big)\tau\Big(\frac{L_2(\mathbf{x})}{m_2}\Big)\tau\Big(\frac{L_3(\mathbf{x})}{m_3}\Big)F\Big(\frac{x_2}{x_1}\Big).$$

We have $F(\frac{x_2}{x_1}) \ll \log\frac{x_2}{x_1} \ll \log\log P$ here. Applying (5.7), we easily deduce that

$$\mathscr{E} \ll_\varepsilon \frac{(\log P)^3\log\log P}{m_1m_2m_3} + P^{-\frac{2}{5}+\varepsilon},$$

which makes a satisfactory overall contribution to Σ_1 in (5.6).

We now call upon our work above to estimate the sums $S_1(w;\mathbf{m})$ and $S_v(w;\mathbf{m})$ in the integral. One finds that $F'(v) > 0$ for each $v > 1$, whence

$$|F(1)| + \int_1^J |F'(v)|\mathrm{d}v = F(J) \ll \log\log P.$$

The error term thus makes a satisfactory contribution to Σ_1. Once substituted into (5.6), we deduce that

$$\Sigma_1 = \omega_\infty\omega_{\mathrm{local}}P(\log P)^3 \int_1^{P^{\frac{2}{5}}} \int_{\{\tau\in\mathbb{R}^3 : (\tau, \frac{\log w}{\log P_0})\in V^*\}} \frac{\mathrm{d}\tau\mathrm{d}w}{w} + O\big(P(\log P)^3(\log\log P)^3\big),$$

on extending the summation over $\mathbf{m}$ to infinity, where

$$\omega_{\mathrm{local}} = \frac{1}{\zeta(2)} \sum_{\substack{\mathbf{l}\in\mathbb{N}^3 \\ \gcd(m_i,m_j)=1}} h(m_1)h(m_2)h(m_3)\mathrm{g}(\mathbf{m})$$

and $\omega_\infty = \frac{F(1)}{2} + \int_1^J \frac{F'(v)}{v+1}\mathrm{d}v$. Now it follows from (5.4) that

$$\omega_\infty = \int_1^\infty \frac{F(v)}{(v+1)^2}\mathrm{d}v + O\Big(\frac{\log J}{J}\Big) = \frac{\pi^2}{24} + O((\log P)^{-5}\log\log P).$$

Moreover, on recalling the definition (5.5), an easy calculation reveals that

$$\omega_{\mathrm{local}} = \prod_p \Big(1-\frac{1}{p}\Big)^5\Big(1+\frac{5}{p}+\frac{1}{p^2}\Big).$$

Finally, we find that

$$\begin{aligned}\int_1^{P^{\frac{2}{5}}} \int_{\{\tau\in\mathbb{R}^3\,:\,(\tau,\frac{\log w}{\log P_0})\in V^*\}} \frac{\mathrm{d}\tau\mathrm{d}w}{w} &= \mathrm{meas}\Big\{(\tau,u)\in V^*\ :\ u\leqslant \frac{2}{5}\Big\}\log P_0\\ &= \frac{\log P_0}{360}.\end{aligned}$$

This completes the proof of (5.1), and so the proof of Theorem 1.1.

References

[1] de la Bretèche, Régis. Nombre de points de hauteur bornée sur les surfaces de del Pezzo de degré 5. *Duke Math. J.* **113** (2002), no. 3, 421–464. MR1909606, Zbl 1054.14025, doi: 10.1215/S0012-7094-02-11332-5. 1194, 1196, 1197

[2] de la Bretèche, Régis; Browning, Timothy D. Sums of arithmetic functions over values of binary forms. *Acta Arith.* **125** (2006), no. 3, 291–304. MR2276196, Zbl 1159.11035, arXiv:math/0604119, doi: 10.4064/aa125-3-6. 1202, 1221

[3] de la Bretèche, Régis; Browning, Timothy D. Manin's conjecture for quartic del Pezzo surfaces with a conic fibration. *Duke Math. J.* **160** (2011), no. 1, 1–69. MR2838351, Zbl 1193.14028, arXiv:0710.1560, doi: 10.5802/aif.2462. 1194

[4] Browning, Timothy D. Quantitative arithmetic of projective varieties. Progress in Mathematics, 277. *Birkhäuser Verlag, Basel,* 2009. xiv+160 pp. ISBN: 978-3-0346-0128-3. MR2559866, Zbl 1188.14001, doi: 10.1007/978-3-0346-0129-0. 1194

[5] Browning, Timothy D. The divisor problem for binary cubic forms *J. Théorie Nombres Bordeaux* **23** (2011), no. 3, 579–602. MR2861076, Zbl 1271.11091, arXiv:1006.3476, doi: 10.5802/jtnb.778. 1195, 1222, 1223, 1224

[6] Browning, Timothy D.; Heath–Brown, D. R. Counting rational points on hypersurfaces. *J. Reine Angew. Math.* **584** (2005), 83–115. MR2155086, Zbl 1079.11033, arXiv:math/0404456, doi: 10.1515/crll.2005.2005.584.83. 1201

[7] Franke, Jens; Manin, Yuri; Tschinkel, Yuri. Rational points of bounded height on Fano varieties. *Invent. Math.* **95** (1989), no. 2, 421–435. MR0974910, Zbl 0674.14012, doi: doi.org/10.1007/BF01393904. 1193

[8] Greaves, George. On the divisor-sum problem for binary cubic forms. *Acta Arith.* **17** (1970), 1–28. MR0263761, Zbl 0198.37903, doi: 10.4064/aa-17-1-1-28. 1195

[9] Heath–Brown, D. R. Diophantine approximation with square-free numbers. *Math. Z.* **187** (1984), no. 3, 335–344. MR0757475, Zbl 0539.10026, doi: 10.1007/BF01161951. 1208, 1215

[10] Manin, Yu. I.; Tschinkel, Yu. Points of bounded height on del Pezzo surfaces. *Compositio Math.* **85** (1993), no. 3, 315–332. MR1214451, Zbl 0782.14033. 1193

[11] Mumford, David; Suominen, Kalevi. Introduction to the theory of moduli. *Algebraic geometry*, Oslo 1970 *(Proc. Fifth Nordic Summer-School in Math.)*, 171–222. *Wolters–Noordhoff, Groningen*, 1972. MR0437531, Zbl 0242.14004. 1195

[12] Peyre Emmanuel. Hauteurs et mesures de Tamagawa sur les variétés de Fano. *Duke Math. J.* **79** (1995), no. 1, 101–218. MR1340296, Zbl 0901.14025, doi: 10.1215/S0012-7094-95-07904-6. 1193, 1194

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