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A new look at local maps on algebraic structures of matrices and operators

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Abstract. In a very general setting, we introduce a new type of local maps, a new sort of reflexive closure of a given class of transformations relative to a given operation that we call operational reflexive closure, and a corresponding concept of reflexivity. We calculate the operational reflexive closures of some important classes of transformations and significantly strengthen former 2-reflexivity results concerning the automorphism groups of various operator structures. A typical new result is this: if ϕ is a map from the unitary group over a separable infinite dimensional Hilbert space into itself with the property that for any pair V, W of unitaries there is a group automorphism $\alpha_{V,W}$ of the unitary group such that $\phi(V)\phi(W) = \alpha_{V,W}(VW)$, then either ϕ itself or $-\phi$ is a group automorphism. This result substantially generalizes a former one on the 2-reflexivity of the automorphism group of the unitary group. We also present open problems and questions for further study.

Contents

1. Introduction

The study of linear maps on linear spaces of matrices or, more generally, on linear spaces of Hilbert (or Banach) space operators which locally belong to a given class of transformations has a long history started with the seminal works of Kadison [14], and Larson and Sourour [16]. The former paper concerns linear maps on von Neumann algebras which are so-called local derivations, the latter one concerns linear maps on full operator algebras over Banach spaces

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which are local derivations or local automorphisms. Important generalizations of the results in [16] were obtained by Brešar and Šemrl in [5] which is the third most influential paper on that research area.

In [35], Šemrl introduced the concept of 2-local maps (specifically, 2-local automorphisms and 2-local derivations) in order to drop the linearity assumptions in the previously mentioned results. Roughly speaking, his definition of 2-local automorphisms is as follows. For a given algebra $\mathcal{A}$, a map ϕ (importantly, the linearity of ϕ is not assumed) on $\mathcal{A}$ is called a 2-local automorphism if for any two elements in $\mathcal{A}$, ϕ can be interpolated at those two elements by an algebra automorphism of $\mathcal{A}$. More precisely, this means that for any pair $A, B \in \mathcal{A}$ of points we have an automorphism $\phi_{A,B}$ of $\mathcal{A}$ (depending on A, B) such that

$$\phi(A) = \phi_{A,B}(A), \quad \text{and} \quad \phi(B) = \phi_{A,B}(B). \tag{1}$$

It was proved in [35] that, quite surprisingly, every 2-local automorphism of the full operator algebra $\mathbb{B}(H)$ over a separable infinite dimensional complex Hilbert space H is necessarily an algebra automorphism and a similar result was presented for derivations, too. In short, one can describe this phenomenon by using the expression of 2-reflexivity, in particular, saying that the automorphism group of the algebra $\mathbb{B}(H)$ is 2-reflexive. These remarkable results attracted serious attention and motivated a number of further investigations, for a highly incomplete list of references, see, e.g., the introduction in [26].

In the paper [26], we observed that the group of all algebra *-automorphisms of $\mathbb{B}(H)$ has a much stronger property. Namely, we proved the following. If $\phi : \mathbb{B}(H) \to \mathbb{B}(H)$ is a map which either has the property that for any $A, B \in \mathbb{B}(H)$ there is an algebra *-automorphism of $\mathbb{B}(H)$ such that

$$\phi(A) + \phi(B) = \alpha_{A,B}(A + B),$$

or has the property that for any $A, B \in \mathbb{B}(H)$ there is an algebra *-automorphism of $\mathbb{B}(H)$ such that

$$\phi(A)\phi(B) = \alpha_{A,B}(AB),$$

then ϕ is "more or less" an algebra *-automorphism. Indeed, in the former case, we proved that ϕ is an algebra *-automorphism if $\dim H \geq 3$ and it is an algebra *-automorphism or an algebra *-antiautomorphism if $\dim H = 2$, see Theorem 1 in [26]. In the latter case, we proved that ϕ or $-\phi$ is an algebra *-automorphism, see Theorem 2 in [26]. Consequently, the two equations in (1) defining 2-local maps can be squeezed into one equation still producing essentially the same conclusion as before. It is worth mentioning that this observation turned to be quite useful since that way we could verify the 2-reflexivity of the group of all isometries of $\mathbb{B}(H)$ (not only the subgroup of all linear isometries of $\mathbb{B}(H)$), see Theorem 3 in [26]. (The result for the group of the linear isometries was obtained quite some time ago in [19]). This application itself has already motivated further research, see [29] and Section 4 in [30].

Let us point out also the following. In [35], Šemrl referred to a famous result by Kowalski and Slodkowski [15] as the main motivation for the introduction of his concept. He writes the following. “In particular, their result shows that at the cost of requiring the local behaviour like a character at every two points, the condition of linearity can be dropped [1, Corollary 3.7]. More precisely, if $\theta : \mathcal{A} \to \mathbb{C}$ is a function having the property that for every a and b in $\mathcal{A}$ there exists a multiplicative linear functional $\theta_{a,b}$ on $\mathcal{A}$ such that $\theta(a) = \theta_{a,b}(a)$ and $\theta(b) = \theta_{a,b}(b)$, then θ itself is linear and multiplicative.″ But the truly authentic reformulation of the famous result by Kowalski and Slodkowski, Theorem 1.2 in [15], for a commutative Banach algebra $\mathcal{A}$ should, in fact, be the following. If $\theta : \mathcal{A} \to \mathbb{C}$ is a function having the property that $\theta(0) = 0$ and for every a and b in $\mathcal{A}$ there exists a multiplicative linear functional $\theta_{a,b}$ on $\mathcal{A}$ such that $\theta(a)-\theta(b) = \theta_{a,b}(a-b)$, then θ itself is linear and multiplicative. Probably one can agree that the kind of locality that we are proposing in the present paper is the one which really fits in spirit to that celebrated result by Kowalski and Slodkowski.

After having discussed this, we can put the above phenomenon in a much wider and more general perspective as follows. Let $\mathcal{A}, \mathcal{B}$ be algebraic structures, $\mathcal{G}, \mathcal{F}$ be given collections of transformations from $\mathcal{A}$ into $\mathcal{B}$ such that $\mathcal{G} \subset \mathcal{F}$. Assume that $\circ$, $\diamond$ are (binary) operations on $\mathcal{A}$ and $\mathcal{B}$, respectively, and that the elements of $\mathcal{G}$ are homomorphisms with respect to that pair of operations. Denote by $\mathcal{R}(\mathcal{F}, \mathcal{G}, \circ, \diamond)$ the set of all maps $\phi \in \mathcal{F}$ with the property that for any $A, B \in \mathcal{A}$ there is an $\alpha_{A,B} \in \mathcal{G}$ such that

$$\phi(A) \diamond \phi(B) = \alpha_{A,B}(A \circ B) = \alpha_{A,B}(A) \diamond \alpha_{A,B}(B).$$

Having set this, the basic question we are interested in (and we investigated in [26] in some particular cases) concerns the description of $\mathcal{R}(\mathcal{F}, \mathcal{G}, \circ, \diamond)$ and the study of how the collection $\mathcal{R}(\mathcal{F}, \mathcal{G}, \circ, \diamond)$ (which evidently includes $\mathcal{G}$) relates to $\mathcal{G}$. Is it true that $\mathcal{R}(\mathcal{F}, \mathcal{G}, \circ, \diamond)$ is “more or less″ equal to $\mathcal{G}$? In the case where $\mathcal{A} = \mathcal{B}$ and $\circ = \diamond$, we call $\mathcal{R}(\mathcal{F}, \mathcal{G}, \circ, \circ)$ the $\circ$-reflexive closure or, if the operation $\circ$ what we consider is unambiguous, simply the operational reflexive closure of $\mathcal{G}$ in $\mathcal{F}$. Assuming further that $\mathcal{F}$ is the collection of all maps from $\mathcal{A}$ into itself, we call $\mathcal{R}(\mathcal{A}^{\mathcal{A}}, \mathcal{G}, \circ, \circ)$ the $\circ$-reflexive closure or simply the operational reflexive closure of $\mathcal{G}$. In the case where $\mathcal{R}(\mathcal{A}^{\mathcal{A}}, \mathcal{G}, \circ, \circ) = \mathcal{G}$ holds, we say that the collection $\mathcal{G}$ is $\circ$-reflexive or simply operationally reflexive. After this, we can place the results in [26] into this general frame. Let H be a separable Hilbert space, $\mathcal{A} = \mathcal{B} = \mathbb{B}(H)$, let $\mathcal{G}$ be the group of all algebra *-automorphisms of $\mathbb{B}(H)$, i.e., the group of all unitary similarity transformations, and let $\mathcal{F}$ be the set of all functions from $\mathbb{B}(H)$ into itself. Theorem 1 in [26] tells that, in the case where $\dim H \geq 3$, the $+$-reflexive closure of the *-automorphism group of $\mathbb{B}(H)$ equals itself, while in the case where $\dim H = 2$, it coincides with the larger group consisting of all algebra *-automorphisms together with all algebra *-antiautomorphisms of $\mathbb{B}(H)$. Hence, for $\dim H \geq 3$, we obtain that the group of all algebra *-automorphisms (the group of all unitary similarity transformations) is $+$-reflexive, while for $\dim H = 2$ (in which case each operator is unitarily similar to its transpose), the larger group consisting of all algebra *-automorphisms and all algebra *-antiautomorphisms is $+$-reflexive. If we change the operation of addition to the operator multiplication but keeping $\mathcal{G}$ unchanged (being the group of all algebra *-automorphisms), by Theorem 2 in [26] we have that then the operational reflexive closure of $\mathcal{G}$ is $\mathcal{G} \cup (-\mathcal{G})$.

As mentioned before, the above results significantly strengthen the 2-reflexivity of the group of all *-automorphisms of $\mathbb{B}(H)$ that follows from the statements in [35]. In the recent paper [11], we presented several results of similar spirit concerning the automorphism groups of certain quantum structures formed by Hilbert space operators which also substantially extend existing 2-reflexivity results for those automorphism groups. Namely, in [11], $\mathcal{A} = \mathcal{B}$ was either the set of all bounded observables, i.e., the collection of all bounded self-adjoint operators on a separable complex Hilbert space H, or the set of all projections on H, or the set of all density operators (positive semidefinite operators with unit trace) on H, or the collection of all Hilbert space effects (positive semidefinite operators majorized by the identity I) on H, or the set of all positive definite operators on H. Each of those collections is equipped with some binary operations, and we considered $\mathcal{G}$s to be the corresponding groups of automorphims. We computed their operational reflexive closures and, in several cases, we obtained the operational reflexivity of the automorphism groups in questions.

In the present paper we continue and broaden our investigations along this line, we present several results in this new spirit that concern the automorphism groups of various important algebraic structures of operators and matrices and propose open problems and questions for further study.

2. Results

We begin with mentioning the not surprising fact that operational reflexivity is a much stronger property than 2-reflexivity. Indeed, consider the following example. Take any C^*-algebra $\mathcal{A}$, its positive definite cone $\mathcal{A}_{++}$ (i.e., the set of all positive invertible elements of $\mathcal{A}$) equipped with the usual addition making it a commutative semigroup. Then an arbitrary map $\phi : \mathcal{A}_{++} \to \mathcal{A}_{++}$ has the property that for any $A, B \in \mathcal{A}_{++}$, there exists an additive bijection (i.e., semigroup automorphism) $\alpha_{A,B}$ of $\mathcal{A}_{++}$ such that $\phi(A) + \phi(B) = \alpha_{A,B}(A + B)$ holds. Indeed, this follows from the apparent observation that for any $C, D \in \mathcal{A}_{++}$ we have an invertible $T \in \mathcal{A}$ such that $D = TCT^*$ (take, e.g., $T = D^{1/2}C^{-1/2}$). Clearly, the congruence transformation $A \mapsto TAT^*$ is an additive bijection of $\mathcal{A}_{++}$. (Let us mention that the precise structure of all additive semigroup automorphisms of $\mathcal{A}_{++}$ is known: those maps are exactly the above congruence transformations composed by the (linear) Jordan *-automorphisms of the underlying algebra $\mathcal{A}$, see Lemma 8 in [3]. In the case of the algebra $\mathbb{B}(H)$, the corresponding automorphism group of the positive definite cone is generated by the congruence transformations together with the transposition.) Therefore, the operational reflexive closure of the automorphism group of $\mathcal{A}_{++}$ equipped with addition is not small, just the opposite, it is the collection of all functions on $\mathcal{A}_{++}$, the largest possible collection of maps. On the other hand, as for the property of 2-reflexivity, we have that for any separable Hilbert space H, that automorphism group of the positive definite cone of $\mathbb{B}(H)$ is 2-reflexive. To verify this, we mention the following. The group of all Thompson isometries of the positive definite cone of $\mathbb{B}(H)$ is known to be 2-reflexive. (For the definition of the Thompson metric and for the structure of the corresponding surjective isometries, we refer to the papers [23, 13, 24]. What is important for us is that the surjective Thompson isometries of the positive definite cone of $\mathbb{B}(H)$ are exactly the transformations $A \mapsto TAT^*$ and $A \mapsto TA^{-1}T^*$, where T is an invertible bounded either linear or conjugate linear operator on H). In the infinite dimensional separable case this 2-reflexivity property was proved in Theorem 3.1 in [4], while in the finite dimensional case it immediately follows from Theorem 1 in [25] which, in particular, implies that, in finite dimension, any map preserving the Thompson distance is automatically surjective. Now, one can see that the group of the bijective additive maps of the positive definite cone of $\mathbb{B}(H)$ is a subgroup (of index 2) of the Thompson isometry group. Indeed, as mentioned above, the Thompson isometry group in the considered case is generated by the group of all additive bijections and the operation of the (multiplicative) inverse. It follows that any 2-local semigroup automorphism ϕ of the positive definite cone of $\mathbb{B}(H)$ is a surjective Thompson isometry, from which it is not difficult to conclude that ϕ is necessarily an additive bijection. (Actually, what we need to check in order to prove this is that the map $A \mapsto A^{-1}$ is not a 2-local additive automorphism, i.e., there is a pair A, B of positive definite operators on H for which there is no invertible bounded linear or conjugate linear operator T on H such that $A^{-1} = TAT^*$ and $B^{-1} = TBT^*$ hold. This can easily be done by finding A, B such that $A^{-1}B$ is not similar to AB^{-1}.)

To sum up, above we have given an example for such a situation where we have that the set what one could call the 2-reflexive closure of an automorphism group is the smallest possible (in fact, we have the 2-reflexivity property of that automorphism group) but the corresponding operational reflexive closure is the largest possible set, the collection of all maps. Therefore, from the reflexivity point of view, operational reflexivity seems to be a very special, rather extreme type of reflexivity meaning a particularly strong rigidity property of the considered automorphism group (or, more generally, of a collection of transformations).

In what follows, we present our new results on operational reflexive closures and operational reflexivity. Recalling again the results in [26], one can immediately and very naturally ask what the operational closures of the group of all algebra automorphisms, i.e., all similarity transformations on $\mathbb{B}(H)$ are. The sad fact is that we have an answer only in the finite dimensional case, for the matrix algebra $\mathbb{M}_n(\mathbb{C})$. Namely, by Theorem 1 in [20] (or see Theorem 3.4.1 in [22]), which describes the nonlinear maps on matrix algebras preserving the eigenvalues of products, we immediately have Proposition 2.1 below. (Let us mention here that the next three results are also valid for matrix algebras over fields more general than $\mathbb{C}$ but in this paper we are not interested in extensions of our investigations into that direction.) Recall that the algebra automorphisms of $\mathbb{M}_n(\mathbb{C})$ (and also of $\mathbb{B}(H)$) are exactly the similarity transformations and the algebra antiautomorphisms are exactly the similarity transformations composed by the transposition.

Proposition 2.1. *Let $\phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ be a map with the property that for any $A, B \in \mathbb{M}_n(\mathbb{C})$ there is an algebra automorphism $\alpha_{A,B}$ of $\mathbb{M}_n(\mathbb{C})$ such that*

$$\phi(A)\phi(B) = \alpha_{A,B}(AB).$$

Then ϕ or $-\phi$ is an algebra automorphism of $\mathbb{M}_n(\mathbb{C})$.

Proof. By the above mentioned result in [20] on maps preserving the eigenvalues of products, it follows that $\phi(A) = \lambda TAT^{-1}$ for all $A \in \mathbb{M}_n(\mathbb{C})$ or $\phi(A) = \lambda TA^tT^{-1}$ for all $A \in \mathbb{M}_n(\mathbb{C})$, where $\lambda = \pm 1$ and $T \in \mathbb{M}_n(\mathbb{C})$ is invertible. Here ^{t} denotes the transposition. Assuming that we have the second case, it follows that A^tB^t is similar to AB for any $A, B \in \mathbb{M}_n(\mathbb{C})$. But this is not true as we can easily see already in the 2×2 case. Indeed, just take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \tag{2}$$

and observe that the set of the eigenvalues of A^tB^t is different from the set of the eigenvalues of AB. This proves the statement. □

Observe that the converse statement in Proposition 2.1 is trivially true and hence we have that the operational reflexive closure of the group of all similarity transformations on $\mathbb{M}_n(\mathbb{C})$ with respect to matrix multiplication consists exactly of all similarity transformations and their negatives.

If, in Proposition 2.1, we replace the operation of matrix product by addition, using a result on adjacency preserving maps, we obtain the following statement.

Proposition 2.2. *Let $n \geq 3$. Assume $\phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is a map with the property that for any $A, B \in \mathbb{M}_n(\mathbb{C})$ there is an algebra automorphism $\alpha_{A,B}$ of $\mathbb{M}_n(\mathbb{C})$ such that*

$$\phi(A) + \phi(B) = \alpha_{A,B}(A + B).$$

Then ϕ is an algebra automorphism or an algebra antiautomorphism of $\mathbb{M}_n(\mathbb{C})$. The converse statement also holds.

Proof. First observe that $\phi(0) = 0$ and for any $A \in \mathbb{M}_n(\mathbb{C})$ we have $\phi(A) + \phi(-A) = 0$ implying that $\phi(-A) = -\phi(A)$. Therefore, $\phi(A) - \phi(B) = \phi(A) + \phi(-B)$ is similar to $A - B$ for any $A, B \in \mathbb{M}_n(\mathbb{C})$. It immediately follows that $\phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is an adjacency preserving map, i.e., for any $A, B \in \mathbb{M}_n(\mathbb{C})$ such that $A - B$ has rank 1, we have that $\phi(A) - \phi(B)$ also has rank 1. We can apply the deep result Theorem 4.1 in [36] due to Šemrl on the description of such transformations. To formulate it, we need the following notation. If $\tau : \mathbb{C} \to \mathbb{C}$ is a ring endomorphism, then for any $A \in \mathbb{M}_n(\mathbb{C})$, A^τ denotes the matrix obtained from A by applying τ on every entry of it. Šemrl's theorem states that we have the following three possibilities for ϕ:

(i) either there exist invertible matrices $T, S \in \mathbb{M}_n(\mathbb{C})$, a nonzero ring endomorphism $\tau : \mathbb{C} \to \mathbb{C}$, and a matrix $L \in \mathbb{M}_n(\mathbb{C})$ with the property that $I + A^\tau L \in \mathbb{M}_n(\mathbb{C})$ is invertible for every $A \in \mathbb{M}_n(\mathbb{C})$ such that

$$\phi(A) = T(I + A^\tau L)^{-1} A^\tau S, \quad A \in \mathbb{M}_n(\mathbb{C}), \tag{3}$$

(ii) or there exist invertible matrices $T, S \in \mathbb{M}_n(\mathbb{C})$, a nonzero ring endomorphism $\tau : \mathbb{C} \to \mathbb{C}$, and a matrix $L \in \mathbb{M}_n(\mathbb{C})$ with the property that $I + (A^\tau)^t L \in \mathbb{M}_n(\mathbb{C})$ is invertible for every $A \in \mathbb{M}_n(\mathbb{C})$ such that

$$\phi(A) = T(I + (A^\tau)^t\, L)^{-1} (A^\tau)^t S, \quad A \in \mathbb{M}_n(\mathbb{C}),$$

(iii) or ϕ is degenerate, it is of some special form described in Examples 3.4, 3.5 and Definition 3.6 in [36].

First, we can exclude the degenerate case. Indeed, inspecting the corresponding parts of [36], in that case we would obtain that for all rank-one idempotents $P, Q \in \mathbb{M}_n(\mathbb{C})$, the linear combinations of $\phi(P), \phi(Q)$ were all rank at most 1 elements which is clearly not true for our transformation ϕ.

Assume now that (i) holds for ϕ. Since, as it can be easily seen, our transformation ϕ maps scalar multiples of the identity to themselves, we obtain from (3) that

$$\lambda(I + \tau(\lambda)L)T^{-1} = \tau(\lambda)S, \quad \lambda \in \mathbb{C}.$$

Linearizing this identity in λ, i.e., writing $\lambda + \mu$ in the place of λ, we can easily conclude that

$$(\lambda\tau(\mu) + \mu\tau(\lambda))LT^{-1} = 0, \quad \lambda, \mu \in \mathbb{C}.$$

Since τ is nonzero, we deduce that $L = 0$. It follows that $\phi(A) = TA^\tau S$, $A \in \mathbb{M}_n(\mathbb{C})$. Again, using the property that ϕ fixes the scalar multiples of the identity, it follows easily that τ is the identity and $TS = I$. Consequently, ϕ is a similarity transformation, hence an algebra automorphism of $\mathbb{M}_n(\mathbb{C})$.

In a similar manner, in the case where (ii) holds for ϕ, we obtain that $\phi(A) = TA^tT^{-1}$, $A \in \mathbb{M}_n(\mathbb{C})$, i.e., ϕ is an algebra antiautomorphism of $\mathbb{M}_n(\mathbb{C})$.

The converse statement follows from the fact that every element of $\mathbb{M}_n(\mathbb{C})$ is similar to its transpose. □

As far as we know, although it is suspected that Šemrl's result which we have used in the above proof holds also in the case $n = 2$, it has not yet been proved. Therefore, it would be interesting to know if our statement above holds in the 2-dimensional case, too.

The previous proposition tells us that, for $\dim H \geq 3$, the +-reflexive closure of the group of all algebra automorphisms (similarity transformations) of $\mathbb{M}_n(\mathbb{C})$ consists exactly of all algebra automorphisms together with all algebra antiautomorphisms. Therefore, we can also obtain that the group of all similarity transformations on $\mathbb{M}_n(\mathbb{C})$ together with their compositions with the transposition is operational reflexive with respect to the operation of addition (recall again that every matrix is similar to its transpose). Moreover, one can see that both Proposition 2.1 and Proposition 2.2 significantly strengthen the 2-reflexivity property of the group of all algebra automorphims of $\mathbb{M}_n(\mathbb{C})$ (see Remark in [35] or Corollary 2 in [20]).

We also mention at this point that Proposition 2.2 can be used to give a (hopefully somewhat interesting) characterization of similarity transformations and their compositions with the transposition. In fact, one can say that (in the case $n \geq 3$) a map $\phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ (no linearity is assumed) is a similarity transformation or a similarity transformation composed with transposition if and only if for every $A, B \in \mathbb{M}_n(\mathbb{C})$, the matrix $\phi(A) + \phi(B)$ is similar to $A + B$. We point out that the other results in the present paper can also be used for similar characterizations of some specific collections of transformations.

Let us proceed with mentioning that above we have considered algebras of matrices but the general frame introduced in the paper makes it possible to consider other algebraic structures, too. In what follows we examine important (multiplicative) groups of matrices or operators, namely, the general linear group and the unitary group. The next proposition tells that for the group $\mathcal{G}$ of all inner automorphisms of the general linear group $\mathbb{GL}_n(\mathbb{C})$ (all similarity transformations on $\mathbb{GL}_n(\mathbb{C})$), its operational reflexive closure for matrix multiplication is the smallest possible, it is just $\mathcal{G} \cup (-\mathcal{G})$.

Proposition 2.3. *Let $\phi : \mathbb{GL}_n(\mathbb{C}) \to \mathbb{GL}_n(\mathbb{C})$ be a map. Assume that for every $A, B \in \mathbb{GL}_n(\mathbb{C})$ we have $T_{A,B} \in \mathbb{GL}_n(\mathbb{C})$ such that*

$$\phi(A)\phi(B) = T_{A,B}(AB)T_{A,B}^{-1}. \tag{4}$$

Then ϕ or $-\phi$ is an inner automorphism of $\mathbb{GL}_n(\mathbb{C})$. The converse statement is also true.

Proof. Assume that $\phi : \mathbb{GL}_n(\mathbb{C}) \to \mathbb{GL}_n(\mathbb{C})$ has the above property described in (4). Clearly, it follows that $\operatorname{Tr} \phi(A)\phi(B) = \operatorname{Tr} AB$, $A, B \in \mathbb{GL}_n(\mathbb{C})$. (Tr denotes the usual trace functional.) Choose a Hamel base (A_{ij}) in $\mathbb{M}_n(\mathbb{C})$ whose elements belong to $\mathbb{GL}_n(\mathbb{C})$. For example, let $A_{ij} = I + E_{ij}$, $i, j = 1, \dots, n$. The elements $\phi(A_{ij})$ are also linearly independent. Indeed, if

$$\sum_{i,j} \lambda_{ij}\phi(A_{ij}) = 0$$

holds for the scalars λ_{ij}, then

$$0 = \sum_{i,j} \lambda_{ij} \operatorname{Tr} \phi(A_{ij})\phi(A) = \sum_{i,j} \lambda_{ij} \operatorname{Tr} A_{ij}A = \operatorname{Tr}(\sum_{i,j} \lambda_{ij}A_{ij})A$$

for all $A \in \mathbb{GL}_n(\mathbb{C})$ which implies that $\sum_{ij} \lambda_{ij}A_{ij} = 0$. We conclude that all λ_{ij}'s are 0 meaning that the system $(\phi(A_{ij}))$ is linearly independent.

Define a map $\psi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ as follows. Let

$$\psi(A) = \sum_{i,j} \lambda_{ij}\phi(A_{ij}) \quad \text{for any } A = \sum_{i,j} \lambda_{ij}A_{ij}.$$

Clearly, ψ is a bijective linear transformation on $\mathbb{M}_n(\mathbb{C})$. Furthermore, it obviously satisfies $\operatorname{Tr}\psi(A)\psi(B) = \operatorname{Tr} AB$ for all $A, B \in \mathbb{M}_n(\mathbb{C})$. We assert that $\psi(A) = \phi(A)$ holds for any $A \in \mathbb{GL}_n(\mathbb{C})$. Indeed, selecting an arbitrary $B \in \mathbb{M}_n(\mathbb{C})$ with $B = \sum_{i,j} \lambda_{ij} A_{ij}$, we compute

$$\operatorname{Tr}\phi(A)\psi(B) = \sum_{i,j} \lambda_{ij} \operatorname{Tr} AA_{ij} = \sum_{i,j} \lambda_{ij} \operatorname{Tr}\psi(A)\psi(A_{ij}) = \operatorname{Tr}\psi(A)\psi(B)$$

which implies that $\psi(A) = \phi(A)$.

Consequently, $\psi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_n(\mathbb{C})$ is a bijective linear map preserving invertible elements. By the result Theorem 2.1 in [18] due to Marcus and Purves, we have that there are invertible matrices $M, N \in \mathbb{M}_n(\mathbb{C})$ such that either $\psi(A) = MAN$, $A \in \mathbb{M}_n(\mathbb{C})$ or $\psi(A) = MA^tN$, $A \in \mathbb{M}_n(\mathbb{C})$.

Since the composition of the original map ϕ with any similarity transformation on $\mathbb{M}_n(\mathbb{C})$ also has the property described in (4), we can assume that $\psi(A) = AN$, $A \in \mathbb{M}_n(\mathbb{C})$ or $\psi(A) = A^tN$, $A \in \mathbb{M}_n(\mathbb{C})$ for some $N \in \mathbb{GL}_n(\mathbb{C})$. By (4), $\phi(I)$ is an involution, hence we have that N is necessarily an involution.

In the first case where $\psi(A) = AN$, $A \in \mathbb{M}_n(\mathbb{C})$, it follows that $\operatorname{Tr} ANBN = \operatorname{Tr} AB$ holds for all $A, B \in \mathbb{GL}_n(\mathbb{C})$ implying that $NBN = B$, i.e., $NB = BN$ holds for all $B \in \mathbb{GL}_n(\mathbb{C})$. Therefore, N commutes with every invertible matrix and thus also with every element of $\mathbb{M}_n(\mathbb{C})$. This gives us that N is a scalar multiple of the identity and since it is also an involution, we deduce that N is either I or $-I$. Consequently, in the present case we obtain that ϕ or $-\phi$ is an inner automorphism of $\mathbb{GL}_n(\mathbb{C})$.

In the second case where $\psi(A) = A^tN$, $A \in \mathbb{M}_n(\mathbb{C})$, we obtain in a similar way that $N = \pm I$. This implies that $\phi(A) = A^t$, $A \in \mathbb{GL}_n(\mathbb{C})$ or $\phi(A) = -A^t$, $A \in \mathbb{GL}_n(\mathbb{C})$. Therefore, A^tB^t is similar to AB for any $A, B \in \mathbb{GL}_n(\mathbb{C})$ which is untenable, see the proof of Proposition 2.1.

The converse statement is obviously true also. □

It should be mentioned that the group of all inner automorphisms (i.e., similarity transformations) is only a proper subgroup of the automorphism group of $\mathbb{GL}_n(\mathbb{C})$. In fact, there is a classical result usually attributed to Schreier and van der Waerden [32] saying that the generators of the full automorphism group of $\mathbb{GL}_n(\mathbb{C})$ are the following maps: the inner automorphisms $A \mapsto TAT^{-1}$, the automorphisms $A \mapsto A^\tau$ induced by ring isomorphisms τ of the underlying field $\mathbb{C}$, the contragredient automorphism $A \mapsto (A^{-1})^t$, and the radial automorphisms $A \mapsto \gamma(A)A$ corresponding to the characters γ of $\mathbb{GL}_n(\mathbb{C})$. (For a more general result concerning general linear groups of matrix rings over commutative rings, see [38].)

The following natural questions arise. In the finite dimensional case, it would be interesting to see if the operational reflexive closure of the full automorphism group (or at least some of its subgroups strictly larger than the group of inner automorphisms) is as small as that of the group of all inner automorphisms (i.e., it equals the group itself together with the collection of the negatives of its elements). The second natural question concerns the infinite dimensional case.

In fact, the automorphism group of the general linear group over an infinite dimensional complex Hilbert space is much simpler than it is in the finite dimensional case. For the precise result see Theorem 3.1 in [27]. Roughly speaking, we can tell that in the infinite dimensional case the ring isomorphisms τ of $\mathbb{C}$, the abundance of which appear above in the finite dimensional case, are only the identity and the conjugation on $\mathbb{C}$, and the only possible radial automorphism is the identity. In the same paper it was proved that the full automorphism group is 2-reflexive in infinite dimension. Hence, a natural problem to investigate is the following.

Problem 2.4. Let H be a separable infinite dimensional Hilbert space. What is the operational reflexive closure of the automorphism group of the general linear group over H? And, referring back to Propositions 2.1 and 2.2, do those statements remain valid for the automorphism group of the whole algebra $\mathbb{B}(H)$?

Of course, one could be even more demanding and ask what happens in the case of, say, general von Neumann algebras but to answer such a question seems to be out of reach at the moment.

We now turn to the unitary group and we primarily deal with the infinite dimensional case. Let us recall that it was proved in [31] that any norm continuous group isomorphism between the unitary groups of two AW^*-factors extends to a linear or conjugate-linear *-isomorphism of the factors themselves. In the cases of von Neumann factors not of type I_n, $n < \infty$, combining the results Theorem 2 in [10] and Theorem 1 in [6], we obtain the same conclusion without imposing the continuity assumption, see Theorem 1.2 in [1]. As a particular case, concerning the unitary group $\mathbb{U}(H)$ in $\mathbb{B}(H)$, we have that any norm continuous group automorphism $\phi : \mathbb{U}(H) \to \mathbb{U}(H)$ is of the form

$$\phi(V) = UVU^{-1}, \qquad V \in \mathbb{U}(H) \tag{5}$$

with some either unitary or antiunitary operator U on H. Moreover, if H is infinite dimensional, the same holds even without the assumption of continuity.

The next theorem tells that, in the case of a separable infinite dimensional Hilbert space, the operational reflexive closure of the automorphism group of $\mathbb{U}(H)$ is as small as it is in Proposition 2.3, i.e., it consists exactly of the automorphisms and their negatives. (This may provide additional motivation to study Problem 2.4.)

Theorem 2.5. *Let H be a separable infinite dimensional Hilbert space and let $\phi : \mathbb{U}(H) \to \mathbb{U}(H)$ be a map with the property that for every $V, W \in \mathbb{U}(H)$ there is an automorphism $\alpha_{V,W}$ of $\mathbb{U}(H)$ such that*

$$\phi(V)\phi(W) = \alpha_{V,W}(VW). \tag{6}$$

Then ϕ or $-\phi$ is an automorphism of $\mathbb{U}(H)$. The converse statement is also true.

Proof. It follows from the assumption that for any $V, W \in \mathbb{U}(H)$, there is a unitary or antiunitary operator U on H (depending on V, W) such that $\phi(V)\phi(W) = UVWU^{-1}$.

Observe the following. For each $V \in \mathbb{U}(H)$ we have $\phi(V)\phi(V^{-1}) = I$ implying that $\phi(V)^{-1} = \phi(V^{-1})$. It follows that for any pair $V, W \in \mathbb{U}(H)$ there is a unitary or antiunitary operator U on H such that

$$\phi(V)\phi(W)^{-1} = \phi(V)\phi(W^{-1}) = U(VW^{-1})U^{-1}.$$

Easy calculation shows that

$$\phi(V) = UV(V^{-1}U^{-1}\phi(V))$$

and

$$\phi(W) = UW(V^{-1}U^{-1}\phi(V)).$$

It implies, in particular, that ϕ is a 2-local isometry of $\mathbb{U}(H)$ (relative to the metric coming from the operator norm). We know that the isometry group of $\mathbb{U}(H)$ is 2-reflexive, see Theorem 2.1 in [4], and hence we have that ϕ is a surjective isometry. By Theorem 8 in [12] describing the structure of all surjective isometries of $\mathbb{U}(H)$, we infer that there are U, U' both unitary or both antiunitary operators on H such that either

$$\phi(V) = UVU', \quad V \in \mathbb{U}(H)$$

or

$$\phi(V) = UV^{-1}U', \quad V \in \mathbb{U}(H).$$

We call a selfadjoint unitary, i.e., unitary involution, a symmetry. Clearly, by (6), ϕ maps symmetries to symmetries. Consequently, for any symmetry $S \in \mathbb{U}(H)$ we have $USU'USU' = I$ implying $SU'US = (U'U)^{-1}$. In particular, $U'U$ is a symmetry and then we obtain that $S(U'U)S = U'U$ holds for each symmetry $S \in \mathbb{U}(H)$. This implies that $U'U$ commutes with every symmetry $S \in \mathbb{U}(H)$ from which we can infer that $U'U$ is a central symmetry in $\mathbb{B}(H)$. This gives us that $U'U = I$ or $U'U = -I$. Therefore, we have the following four possibilities for ϕ:

$$\phi(V) = UVU^{-1}, \phi(V) = -UVU^{-1}, \phi(V) = UV^{-1}U^{-1}, \phi(V) = -UV^{-1}U^{-1} \tag{7}$$

in each case the equality holding for all $V \in \mathbb{U}(H)$. We assert that the last two possibilities cannot occur. Indeed, in the third case we have $UV^{-1}U^{-1} = \phi(V) = \phi(V)\phi(I)$ which, by (6) is unitarily or antiunitarily similar to V. Consequently, we would get that V, V^{-1} were unitarily or antiunitarily similar for every $V \in \mathbb{U}(H)$ which is a clear contradiction. In the remaining, fourth case, a similar argument applies and this finishes the proof of the statement.

The validity of the converse statement is just obvious. □

Let us remark that in the finite dimensional case, the same conclusion as in Theorem 2.5 holds for the group of all norm continuous automorphisms of $\mathbb{U}(H)$. In fact, we can follow the same argument as above. We mention that the 2-reflexivity of the isometry group of $\mathbb{U}(H)$ in that case is apparent, it follows from the fact that any distance preserving map of any compact metric space is automatically surjective. It is, of course, quite a natural question that what happens to the full automorphism group of the unitary group in the finite dimensional case? We recall that the structure of that group is complicated like the structure of the automorphism group of the general linear group, see [9].

Above we have considered full matrix and operator algebras and two of their multiplicative subgroups. We next turn to another type of structure, namely, to positive definite cones. In fact, if we consider the additive structure of $\mathcal{A}_{++}$ for any C^*-algebra $\mathcal{A}$ then, as we have already seen at the beginning of this section, the operational reflexive closure of the automorphism group of $\mathcal{A}_{++}$ is the largest possible, the set of all functions on $\mathcal{A}_{++}$. Now let us consider multiplicative structures on $\mathcal{A}_{++}$. We list three important multiplication-like operations on $\mathcal{A}_{++}$. The first one is the Jordan triple product $(A, B) \mapsto ABA$, which plays important role in general ring theory. The second one is what we call the inverted Jordan triple product $(A, B) \mapsto AB^{-1}A$. This is important for the following reason: for a quite large family of metrics (or even generalized distance measures or divergences) defined on the positive definite cone including the Thompson metric, the corresponding surjective isometries are necessarily automorphisms with respect to that operation. This follows from our work on generalized Mazur-Ulam theorems, see [24]. The third operation is the natural K-loop operation $(A, B) \mapsto A^{1/2}BA^{1/2}$ which one can meet, for example, in the quantum theory of measurements under the name “sequential product″ introduced by Gudder and Nagy, and which has connections also to hyperbolic geometry and other parts of physics (Einstein’s velocity addition in the special theory of relativity), see the introduction in [3].

Let us briefly examine the automorphism groups corresponding to the previous three operations which are in fact closely related. Indeed, one can easily see that for a general C^*-algebra $\mathcal{A}$, a map $\phi : \mathcal{A}_{++} \to \mathcal{A}_{++}$ is an automorphism with respect to the Jordan triple product if and only if it is an automorphism with respect to the natural K-loop operation. Moreover, it is an automorphism with respect to the inverted Jordan triple product if and only if $\phi(I)^{-1/2}\phi(.)\phi(I)^{-1/2}$ is an automorphism with respect to the Jordan triple product. This means that it is sufficient to determine only the Jordan triple automorphisms (i.e., the bijective maps respecting the operation of Jordan triple product). The continuous Jordan triple isomorphisms were completely characterized in Theorem 5 in [24] for the case of factor von Neumann algebras. Since we do not have reflexivity results for the automorphism groups in that generality, we instead refer to Theorem 1 in [21] for the case of the full operator algebra over a Hilbert space. So, if H is a separable infinite dimensional Hilbert space, then a bijective map $\phi : \mathbb{B}(H)_{++} \to \mathbb{B}(H)_{++}$ is a continuous Jordan triple automorphism (equivalently, a continuous automorphism with respect to the natural K-loop operation) if and only if there is a unitary or antiunitary operator U on H such that either

$$\phi(A) = UAU^*, \quad A \in \mathbb{B}(H)_{++}$$

or

$$\phi(A) = UA^{-1}U^*, \quad A \in \mathbb{B}(H)_{++}.$$

As a consequence, we obtain that a bijective map $\phi : \mathbb{B}(H)_{++} \to \mathbb{B}(H)_{++}$ is a continuous automorphims with respect to the inverted Jordan triple product if and only if there is an invertible bounded either linear or conjugate linear operator T on H such that either

$$\phi(A) = TAT^*, \quad A \in \mathbb{B}(H)_{++}$$

or

$$\phi(A) = TA^{-1}T^*, \quad A \in \mathbb{B}(H)_{++}.$$

Arguing in the same way as at the beginning of this section, we see that the operational reflexive closure of the automorphism group with respect to the inverted Jordan triple product is the largest possible, it consists of all maps on the positive definite cone. As the other extreme, in [11] we proved that the group of all continuous automorphisms of $\mathbb{B}(H)_{++}$ equipped with the natural K-loop operation is operationally reflexive. The key idea in the proof was the use of Thompson metric. However, that idea is not applicable concerning Jordan triple automorphisms. Therefore, we need to modify the argument significantly in order to verify the following statement.

Theorem 2.6. *Let H be a separable infinite dimensional Hilbert space. Let $\phi : \mathbb{B}(H)_{++} \to \mathbb{B}(H)_{++}$ be a map with the property that for any $A, B \in \mathbb{B}(H)_{++}$ there is a continuous Jordan triple automorphism $\alpha_{A,B}$ of $\mathbb{B}(H)_{++}$ such that*

$$\phi(A)\phi(B)\phi(A) = \alpha_{A,B}(ABA).$$

Then ϕ itself is a continuous Jordan triple automorphism of $\mathbb{B}(H)_{++}$.

Proof. For any operator $A \in \mathbb{B}(H)_{++}$, denote $A^{[-1]}$ either A or A^{-1}. We know that for any $A, B \in \mathbb{B}(H)_{++}$ there is a unitary or antiunitary operator U on H depending on A, B such that

$$\phi(A)\phi(B)\phi(A) = U(ABA)^{[-1]}U^*.$$

We observe the following. First, $\phi(I)^3 = I$ which implies that $\phi(I) = I$. It follows that for any $A \in \mathbb{B}(H)_{++}$ we have that $\phi(A) = \phi(I)\phi(A)\phi(I) = UA^{[-1]}U^*$ holds for some unitary or antiunitary operator U on H. In particular, for any positive real number t we have that $\phi(tI)$ is either tI or $(1/t)I$. Assume that for some positive numbers t, s different from 1, $\phi(tI) = tI$ and $\phi(sI) = (1/s)I$. Considering $(t^2/s)I = \phi(tI)\phi(sI)\phi(tI) = (t^2s)^{[-1]}I$ we arrive at a contradiction. Therefore, we have either $\phi(tI) = tI$ for all positive t or we have $\phi(tI) = (1/t)I$ for all positive t.

In what follows we can assume that $\phi(tI) = tI, t > 0$ (otherwise, we consider the map $A \mapsto \phi(A)^{-1}$). Select any operator $A \in \mathbb{B}(H)_{++}$ for which $I \leq A$, $A \neq I$. Take any $t > 1$. We have that $t^2\phi(A) = \phi(tI)\phi(A)\phi(tI)$. The left hand side is unitarily or antiunitarily congruent to $t^2A^{[-1]}$ while the right hand side is unitarily or antiunitarily congruent to $(t^2A)^{[-1]}$. This can happen only if $\phi(A)$ is unitarily or antiunitarily congruent to A.

Pick any rank-one projection P on H. It follows that $\phi(I+P)$ is unitarily or antiunitarily congruent to $I+P$ meaning that $\phi(I+P) = I+\psi(P)$ for some rank-one projection $\psi(P)$. This induces a map ψ on the set $\mathbb{P}_1(H)$ of all rank-one projections on H. Pick any two rank-one projections $P, Q \in \mathbb{P}_1(H)$. We have that there is a unitary or antiunitary operator U on H such that

$$\phi(I+P)\phi(I+Q)\phi(I+P) = U((I+P)(I+Q)(I+P))^{[-1]}U^*.$$

Since the triple product on the left hand side is greater than or equal to I, it follows that on the right hand side the inverse does not show up. Therefore, we have that

$$(I+\psi(P))(I+\psi(Q))(I+\psi(P)) = U(I+P)(I+Q)(I+P)U^*.$$

Performing the multiplications, subtracting I from both sides and then taking traces, we easily have $\operatorname{Tr}\psi(P)\psi(Q) = \operatorname{Tr} PQ$. Therefore, ψ is a Wigner transformation, a map of the set $\mathbb{P}_1(H)$ of rank-one projections preserving the trace of products. By the nonsurjective version of Wigner's famous theorem there is a linear or conjugate linear isometry J on H such that

$$\psi(P) = JPJ^*, \quad P \in \mathbb{P}_1(H), \tag{8}$$

see, e.g., Section 2.1 in [22]. We show that J is in fact either a unitary or an antiunitary operator. To verify this, let (P_n) be a sequence of pairwise orthogonal elements of $\mathbb{P}_1(H)$ whose ranges generate H. Pick a strictly decreasing sequence (λ_n) of positive real numbers with finite sum. Set $K = \sum_n \lambda_n P_n$. Since $\phi(I+K)$ is unitarily or antiunitarily congruent to $I+K$, we have $\phi(I+K) = I+K'$ where K' is of the form $K' = \sum_n \lambda_n P'_n$ with some sequence (P'_n) in $\mathbb{P}_1(H)$ whose elements are pairwise orthogonal and their ranges generate H. It follows that there is a unitary or antiunitary operator U on H such that

$$\begin{aligned}(I+JP_1J^*)(I+K')(I+JP_1J^*) &= \phi(I+P_1)\phi(I+K)\phi(I+P_1)\\ &= U(I+P_1)(I+K)(I+P_1)U^*.\end{aligned} \tag{9}$$

If we perform the multiplications, subtract I from both sides, take trace and simplify, then we arrive at

$$\operatorname{Tr} K'JP_1J^* = \operatorname{Tr} KP_1.$$

By the particular forms of K, K', one can easily deduce from this equality that we necessarily have $JP_1J^* = P'_1$. Now, considering the equality similar to (9) but with P_2 in the place of P_1, we have

$$\operatorname{Tr} K'JP_2J^* = \operatorname{Tr} KP_2.$$

Since JP_2J^* is orthogonal to $JP_1J^* = P'_1$, one can deduce from this equality that JP_2J^* is necessarily equal to P'_2, and so forth. Consequently, we obtain that $JP_nJ^* = P'_n$ holds for all positive integer n and this gives us that J is a surjective linear or conjugate linear isometry. Considering the transformation $J^*\phi(.)J$ in place of ϕ, for the rest of the proof we can assume that $\phi(I+P) = I+P$ holds for all $P \in \mathbb{P}_1(H)$. What remains is to show that ϕ is the identity on the full positive definite cone $\mathbb{B}(H)_{++}$.

Let $T \in \mathbb{B}(H)$ be a positive trace class operator. We know that $\phi(I+T) = I+T'$ where $T' \in \mathbb{B}(H)$ is unitarily or antiunitarily congruent to T. On the other hand, we know that $(I+P)(I+T')(I+P) = \phi(I+P)\phi(I+T)\phi(I+P)$ is unitarily or antiunitarily congruent to $(I+P)(I+T)(I+P)$. Just as above, from this fact we easily derive that

$$\operatorname{Tr} T'P = \operatorname{Tr} TP$$

holds for every $P \in \mathbb{P}_1(H)$, which clearly implies that $T' = T$, i.e., $\phi(I+T) = I+T$. In particular, we obtain that $\phi(I+tP) = I+tP$ holds for all $P \in \mathbb{P}_1(H)$ and nonnegative real number t.

Let now $T \in \mathbb{B}(H)_{++}$ be such that $T \geq I$ and set $S = \phi(T)$. In what follows we will use the following two facts. First, if $A, B \in \mathbb{B}(H)_{++}$, then AB^2A is unitarily congruent to BA^2B which follows from the polar decomposition of BA. The second easy fact is that for any nonzero vectors $x, y \in H$, we have that $I + x \otimes y$ is noninvertible if and only if $\langle x, y \rangle = -1$.

Pick any nonnegative real number t. Then we know that $(I+tP)S(I+tP) = \phi(I+tP)\phi(T)\phi(I+tP)$ is unitarily or antiunitarily congruent to $(I+tP)T(I+tP)$. From this we infer that $S^{1/2}(I+tP)^2S^{1/2}$ is unitarily or antiunitarily congruent to $T^{1/2}(I+tP)^2T^{1/2}$ implying that $S + (2t+t^2)S^{1/2}PS^{1/2}$ is unitarily or antiunitarily congruent to $T + (2t+t^2)T^{1/2}PT^{1/2}$ for each nonnegative real number t. Let λ be a positive number greater than the largest element of the spectrum of S, T. Then the operators $S - \lambda I, T - \lambda I$ are negative definite and we know that $S - \lambda I + (2t+t^2)S^{1/2}PS^{1/2}$ is unitarily or antiunitarily congruent to $T - \lambda I + (2t+t^2)T^{1/2}PT^{1/2}$ for each nonnegative real number t. Consequently, for any such t, we have that $S - \lambda I + (2t+t^2)S^{1/2}PS^{1/2}$ is invertible if and only if $T - \lambda I + (2t+t^2)T^{1/2}PT^{1/2}$ is invertible and hence $I + (2t+t^2)(S-\lambda I)^{-1}S^{1/2}PS^{1/2}$ is invertible if and only if $I + (2t+t^2)(T-\lambda I)^{-1}T^{1/2}PT^{1/2}$ is invertible. This easily gives that

$$(2t+t^2)\operatorname{Tr}(S-\lambda I)^{-1}S^{1/2}PS^{1/2} = -1$$

if and only if

$$(2t+t^2)\operatorname{Tr}(T-\lambda I)^{-1}T^{1/2}PT^{1/2} = -1.$$

It follows that

$$(2t+t^2)\operatorname{Tr}(S(S-\lambda I)^{-1})P = -1$$

if and only if

$$(2t+t^2)\operatorname{Tr}(T(T-\lambda I)^{-1})P = -1.$$

Since the operators $S(S-\lambda I)^{-1}, T(T-\lambda I)^{-1}$ are negative definite, $(2t+t^2)$ runs through all positive numbers, we obtain that $\operatorname{Tr}(S(S-\lambda I)^{-1})P = \operatorname{Tr}(T(T-\lambda I)^{-1})P$ holds for all projections $P \in \mathbb{P}_1(H)$. This gives $S(S-\lambda I)^{-1} = T(T-\lambda I)^{-1}$. Since the function $x \mapsto x/(x-\lambda)$ is injective on the spectrum of S, T, we can infer that $S = T$. Therefore, we derive that $\phi(T) = T$ holds whenever $T \geq I$.

Let now $T \in \mathbb{B}(H)_{++}$ be arbitrary. Select any $P \in \mathbb{P}_1(H)$. For any positive integer n, we have that

$$((P+nP^{\perp})\phi(T)(P+nP^{\perp}))^{-1} = (\phi(P+nP^{\perp})\phi(T)\phi(P+nP^{\perp}))^{-1}$$

is unitarily or antiunitarily congruent either to $((P + nP^{\perp})T(P + nP^{\perp}))^{-1}$ or to $(P + nP^{\perp})T(P + nP^{\perp})$. Since the sequence $((P + nP^{\perp})\phi(T)(P + nP^{\perp}))^{-1}$ is norm bounded while $(P + nP^{\perp})T(P + nP^{\perp})$ is not, we deduce that, for large enough n, the operator $((P + nP^{\perp})\phi(T)(P + nP^{\perp}))^{-1}$ is unitarily or antiunitarily congruent to $((P + nP^{\perp})T(P + nP^{\perp}))^{-1}$. Taking norms and letting n tend to infinity, we obtain that

$$\|P\phi(T)^{-1}P\| = \|PT^{-1}P\|$$

holds for all $P \in \mathbb{P}_1(H)$. From this, one can easily conclude that $\phi(T) = T$ and this finishes the proof of the theorem. □

As mentioned in the introduction, the research concerning local maps originally started with the study of local derivations and local automorphisms. The results above give new information on the rigidity of automorphism groups. We close the section with an operational reflexivity result concerning the collection of derivations.

Let $\mathcal{A}, \mathcal{B}$ be algebras, denote by $M_2(\mathcal{B})$ the algebra of all 2×2 matrices with entries in $\mathcal{B}$. Let $D : \mathcal{A} \to \mathcal{B}$ be a function and consider the map $\tilde{D} : \mathcal{A} \to M_2(\mathcal{B})$ defined by

$$\tilde{D}(A) = \begin{bmatrix} A & D(A) \\ 0 & A \end{bmatrix}. \tag{10}$$

The multiplicativity of $\tilde{D}$ means exactly that $D(AB) = D(A)B + AD(B)$, $A, B \in \mathcal{A}$, i.e., that D satisfies the so-called Leibniz rule. Maps with this property are called (multiplicative) derivations. When considering derivations on algebras, it is common to assume that they are also linear. However, by a result of Daif [8] (also see [34]), multiplicative derivations on algebras with some mild conditions concerning the existence of idempotents are automatically additive. In particular, if $\mathcal{A}$ is a so-called standard operator algebra over a normed space X (which means that $\mathcal{A}$ is subalgebra of the algebra $\mathbb{B}(X)$ of all bounded linear operators on X containing the collection $\mathbb{F}(X)$ of all finite rank elements of $\mathbb{B}(X)$), then every multiplicative derivation D from $\mathcal{A}$ into $\mathbb{B}(X)$ is additive. Moreover, if X is an infinite dimensional Banach space, then D is linear as well, see [33]. By Corollary 3.4 in [7], we know that every linear derivation D on any standard operator algebra $\mathcal{A}$ over a normed space X into $\mathbb{B}(X)$ is spatial, i.e., it is of the form $D(A) = TA - AT$, $A \in \mathcal{A}$ with some $T \in \mathbb{B}(X)$.

In the next result we consider maps $\phi : \mathcal{A} \to \mathbb{B}(H)$ on a standard operator algebra $\mathcal{A}$ over a Hilbert space H with the property that for any $A, B \in \mathcal{A}$ we have a linear derivation $D_{A,B} : \mathcal{A} \to \mathbb{B}(H)$ such that $\tilde{\phi}(A)\tilde{\phi}(B) = \tilde{D}_{A,B}(AB)$ holds and prove that then ϕ is necessarily a linear derivation. The result significantly generalizes Šemrl's result Theorem 2 in [35] on 2-local derivations.

Theorem 2.7. *Let H be a Hilbert space and $\mathcal{A}$ be a standard operator algebra over H. Consider a map $\phi : \mathcal{A} \to \mathbb{B}(H)$ with the property that for any $A, B \in \mathcal{A}$ we have a linear derivation $D_{A,B} : \mathcal{A} \to \mathbb{B}(H)$ such that*

$$\phi(A)B + A\phi(B) = D_{A,B}(AB). \tag{11}$$

Then ϕ is a linear derivation.

Proof. Let us fix $B \in \mathbb{F}(H)$ and consider an arbitrary $A \in \mathcal{A}$. Then, by (11) and the spatiality of derivations, we have

$$\operatorname{Tr}(\phi(A)B + A\phi(B)) = 0, \quad A \in \mathcal{A}. \tag{12}$$

It follows that $A \mapsto \operatorname{Tr}(\phi(A)B)$ is linear for every $B \in \mathbb{F}(H)$ which easily implies the linearity of ϕ.

We assert that for every $A \in \mathbb{F}(H)$ we have $\phi(A)(\ker A) \subset \operatorname{rng} A$. Let $x \in H$ be such that $Ax = 0$. Select $B \in \mathbb{F}(H)$ for which $\operatorname{rng} B \subset \ker A$ and $x \in \operatorname{rng} B$. There exists an $R \in \mathbb{B}(H)$ such that

$$\phi(A)B + A\phi(B) = R(AB) - (AB)R.$$

Since $AB = 0$, we easily obtain that $\phi(A)x \in \operatorname{rng} A$. Therefore, the linear map $\phi : \mathcal{A} \to \mathbb{B}(H)$ has the following property: for any $A \in \mathbb{F}(H)$, the operator $\phi(A)$ maps the kernel of A into the range of A. With this property of ϕ in mind, following the argument given in the proof of Theorem 3 in [39] to the second displayed equality on page 1369 (or see the proof of Lemma 2.2 (i) in [17]), one can verify that we have linear operators T, S on H such that

$$\phi(x \otimes y) = Tx \otimes y + x \otimes Sy, \quad x, y \in H. \tag{13}$$

We show that T, S are in fact bounded. To see this, first observe that for $A, B \in \mathcal{A}$ with $AB = 0$ we have $\phi(A)B + A\phi(B) = 0$. After that, choosing nonzero vectors $x, y, x', y' \in H$ such that $\langle x', y\rangle = 0$ and setting $A = x \otimes y, B = x' \otimes y'$, by (13) we have that

$$0 = (Tx \otimes y + x \otimes Sy)x' \otimes y' + x \otimes y(Tx' \otimes y' + x' \otimes Sy'). \tag{14}$$

This reduces to

$$0 = \langle x', Sy\rangle x \otimes y' + \langle Tx', y\rangle x \otimes y'$$

which implies that

$$\langle x', Sy\rangle = -\langle Tx', y\rangle$$

holds for any pair x', y of orthogonal vectors in H. It is now easy to check that the image of the unit ball is weakly bounded both under S and T implying that S, T are bounded linear operators.

We next obtain that the vector $(S^* + T)x'$ is orthogonal to y whenever x' is orthogonal to y. One can deduce that this implies that $(S^* + T)$ is a scalar multiple of the identity, $S^* = \lambda I - T$ holds for some scalar λ. Consequently, we have

$$\phi(x \otimes y) = Tx \otimes y - x \otimes yT + \lambda x \otimes y, \quad x, y \in H$$

and this implies that

$$\phi(A) = TA - AT + \lambda A, \quad A \in \mathbb{F}(H).$$

Again, we recall that $\operatorname{Tr}(\phi(A)B + A\phi(B)) = 0$ holds for all $A, B \in \mathbb{F}(H)$ from which we deduce

$$\operatorname{Tr}((TA - AT + \lambda A)B + A(TB - BT + \lambda B)) = 0$$

and then obtain that

$$\operatorname{Tr} \lambda AB = 0, \quad A, B \in \mathbb{F}(H).$$

It follows that $\lambda = 0$. Therefore,

$$\phi(A) = TA - AT, \quad A \in \mathbb{F}(H).$$

It remains to show that the equality above holds not only on $\mathbb{F}(H)$ but on $\mathcal{A}$, too. To see this, consider the map $\psi : A \mapsto \phi(A) - (TA - AT)$ on $\mathcal{A}$. Clearly, this map also has the property (11) and, furthermore, it vanishes on $\mathbb{F}(H)$. Since for any $A \in \mathcal{A}$ and $F \in \mathbb{F}(H)$ we have

$$\operatorname{Tr}(\psi(A)F + A\psi(F)) = 0,$$

it follows that $\operatorname{Tr} \psi(A)F = 0$ holds for all $F \in \mathbb{F}(H)$. It obviously implies that $\psi(A) = 0$ for any $A \in \mathcal{A}$, and this completes the proof. □

Having proved the previous statement, one can say that the collection of all linear derivations of a standard operator algebra is operationally reflexive. In [2] an important result was shown telling that on any von Neumann algebra any 2-local linear derivation is a linear derivation. Therefore, it is natural to raise the following

Problem 2.8. Does a statement similar to Theorem 2.7 hold for maps $\phi : \mathcal{A} \to \mathcal{A}$, where $\mathcal{A}$ is any von Neumann algebra?

Evidently, an affirmative answer to this question would be a serious strengthening of the main result in [2].

3. Concluding remarks, further directions and problems to study

We make some concluding remarks. We first emphasize that there is much room for further investigations, for the development of our recent observations. In fact, beside the problems formulated in the previous section, our general problem described in the introduction can be raised in a much wider variety of settings which we have not even touched in this paper. Let us mention only one single direction. Here we have considered mainly noncommutative structures of matrices and operators. But what about related questions concerning commutative structures of scalar valued functions? As for semisimple commutative Banach algebras, respectively, commutative C^*-algebras, we have the following two easy results. The first one shows that, for the operation of addition, the operational reflexive closure of all algebra automorphisms coincides with the collection of all linear local automorphisms (i.e., the classical reflexive closure of the group of all algebra automorphisms).

Proposition 3.1. *Let $\mathcal{A}$ be a semisimple commutative Banach algebra. Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map with the property that for any $f, g \in \mathcal{A}$ we have an algebra automorphism $\alpha_{f,g}$ of $\mathcal{A}$ such that*

$$\phi(f) + \phi(g) = \alpha_{f,g}(f + g). \tag{15}$$

Then ϕ is a linear local automorphism. The converse statement is also true.

Proof. The proof is a simple application of the famous Kowalski-Slodkowski theorem [15].

First observe that since $\phi(f) + \phi(-f) = 0$, we have $\phi(-f) = -\phi(f)$. It follows that for any $f, g \in \mathcal{A}$, we have $\phi(f) - \phi(g) = \alpha(f - g)$ for some algebra automorphism α of $\mathcal{A}$ (depending on f, g). Consider any character φ of $\mathcal{A}$. Then $\varphi(\alpha(.))$ is a character of $\mathcal{A}$, hence $\varphi(\phi(f)) - \varphi(\phi(g)) = \varphi(\alpha(f - g)) \in \sigma(f - g)$. Since $\phi(0) = 0$ obviously holds, by the Kowalski-Slodkowski theorem we deduce that $f \mapsto \varphi(\phi(f))$ is a multiplicative linear functional. Since this holds for any character φ of $\mathcal{A}$, it follows that ϕ is an algebra endomorphism of $\mathcal{A}$, in particular, ϕ is linear. We then obtain the first statement of the proposition. The converse statement is obvious. $\square$

Therefore, assuming that all linear local automorpisms of our semisimple commutative Banach algebra are automorphisms (i.e., the automorphism group is reflexive), we deduce that the collection of all algebra automorphisms of $\mathcal{A}$ is also operationally reflexive with respect to the operation of addition. For example, this is the case with the space of all continuous complex functions over a first countable compact Hausdorff space, see Theorem 3.2.1 in [22], or the original source, [28], Theorem 2.2.

If we change addition to multiplication, the situation is rather different, more difficult. Actually, we have a positive result only under a continuity assumption.

Proposition 3.2. *Let X be a first countable compact Hausdorff space and $\mathcal{A}$ be the commutative C^*-algebra of all continuous functions on X (any separable commutative C^*-algebra is isometrically isomorphic to such a function algebra). Let $\phi : \mathcal{A} \to \mathcal{A}$ be a map with the property that for any $f, g \in \mathcal{A}$ we have an algebra automorphism $\alpha_{f,g}$ of $\mathcal{A}$ such that*

$$\phi(f)\phi(g) = \alpha_{f,g}(fg).$$

If ϕ is continuous, then ϕ is an algebra automorphism multiplied by an element with square equal to 1. The converse statement also holds.

Proof. Here we apply a multiplicative version of the Kowalski-Slodkowski theorem due to Touré, Schulz and Brits, see Theorem 3.7 in [37]. Observe that $\phi(1)^2 = 1$. Hence considering the transformation $\phi(1)\phi(.)$ if necessary, we can assume that $\phi(1) = 1$. Since, as it is well known, the algebra automorphisms of $C(X)$ are the composition operators corresponding to homeomorphisms of X, it follows that ϕ preserves the ranges of functions. In particular, ϕ maps real functions to real functions.

Consider any character φ of $C(X)$. Then the map $\psi(.) = \varphi(\phi(.))$ satisfies the following conditions: $\psi(f)\psi(g) \in \sigma(fg)$ holds for all $f, g \in C(X)$, $\psi(1) = 1$, and ψ is continuous. By the mentioned result in [37], we obtain that $f \mapsto \varphi(\phi(f))$ is a real linear multiplicative functional on the real algebra $C_{\mathbb{R}}(X)$ consisting of all real functions in $C(X)$. From this, we can deduce that ϕ is a unital algebra endomorphism of $C_{\mathbb{R}}(X)$. It follows that we have a continuous function $\tau : X \to X$ such that $\phi(f)(t) = f(\tau(t))$, $f \in C_{\mathbb{R}}(X)$. Since ϕ clearly preserves the norms of functions, τ is necessarily surjective.

By the first countability of X, using Urysohn lemma, we can construct a continuous function f on X such that $0 \leq f \leq 1$ and f vanishes exactly at a given point $z \in X$ (see, e.g., p. 170 in [22]). Assuming that τ takes the same value at two different points t_1, t_2 in X, choosing a function f as above vanishing exactly at $\tau(t_1) = \tau(t_2)$, we would arrive at a contradiction: on the one hand, $\phi(f) = f \circ \tau$ vanishes at least two points, and, on the other hand, $\phi(f) = f \circ \omega$ holds for some homeomorphism ω of X which vanishes at one point only. Therefore, τ is injective as well.

We have that τ is bijective. Therefore, considering $\phi(.)\circ\tau^{-1}$, we can assume that $\phi(f) = f$ holds for every real function $f \in C(X)$. It is now easy to see that ϕ is the identity on the whole algebra $C(X)$. In fact, to verify this, we first check the following: if $f, g \in C(X)$ are such that $\sigma(fh) = \sigma(gh)$ holds for all real functions $h \in C(X)$, then $f = g$. Assume that $f(t_0) \neq g(t_0)$, $|f(t_0)| \geq |g(t_0)|$ for some $t_0 \in X$. Choose a positive number ϵ such that $f(t_0)$ does not belong to the ϵ-neighbourhood of $g(t_0)$. Let U be an open set in X containing t_0 on which g takes values in the ϵ-neighbourhood of $g(t_0)$. Let h be a continuous function on X which vanishes outside U, $0 \leq h \leq 1$ and $h(t_0) = 1$. Then $\sigma(fh) = \sigma(gh)$ does not hold since $f(t_0) \in \sigma(fh)$ but $f(t_0) \notin \sigma(gh)$, a contradiction. After this, selecting any $f \in C(X)$, for all $h \in C_{\mathbb{R}}(X)$ we have

$$\sigma(\phi(f)h) = \sigma(\phi(f)\phi(h)) = \sigma(fh).$$

By the previous observation, this implies $\phi(f) = f$ and the proof of the first statement in the proposition is complete. The converse statement is apparent. $\square$

Concerning the continuity assumption in the above proposition we remark that we conjecture that it can be dropped. In fact, as in the previous proof, one can see that without loss of generality we can assume that $\phi(1) = 1$. Then we obtain that ϕ sends nowhere zero functions to nowhere zero functions and $\phi(f)^{-1} = \phi(f^{-1})$ holds for any such element of $C(X)$. Next, the range of the function $\phi(f)/\phi(g)$ equals that of f/g for any nowhere zero g and arbitrary f in $C(X)$. It follows that $\|\phi(f)/\phi(g) - 1\| = \|f/g - 1\|$ holds for any such f, g from which one can derive that ϕ is continuous at g whenever $g \in C(X)$ is nowhere zero. How to prove continuity at the remaining points in $C(X)$, we unfortunately do not know.

On the other hand, using a different approach that we do not present here, we could avoid the use of continuity and would be able to come to the same conclusion as in the proposition above if we knew that the answer to the following problem was affirmative.

Problem 3.3. Let X be any compact Hausdorff space and $\psi : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$ be a map with the following property: for any $f, g \in C_{\mathbb{R}}(X)$ there is an algebra automorphism $\alpha_{f,g}$ of $C_{\mathbb{R}}(X)$ such that $\psi(f)+\psi(g) = \alpha_{f,g}(f+g)$. Does it follow that ϕ is linear (and hence a linear local automorphism)? To formulate it in a shorter way: Is the real version of Proposition 3.1 for $C_{\mathbb{R}}(X)$ true?

This does not seem to be an easy question. And let us formulate an even more difficult problem to which if we had an affirmative answer, that would imply an affirmative answer to Problem 3.3, too (at least for first countable spaces). For that question which seems to be interesting on its own right, we recall the following problem raised previously by the author: Is the full isometry group (the group of all distance preserving bijections without the assumption of linearity) of $C(X)$, X being a first countable Hausdorff space, 2-reflexive? Mori in Theorem 4.6 [29] and Oi in Corollary 4.3 in [30] gave affirmative answers to that question.

Problem 3.4. For a first countable compact Hausdorff space X, do we have the 2-reflexivity of the full isometry group of the real space $C_{\mathbb{R}}(X)$, too?

To see the connection to Problem 3.3, we note that, by the famous Mazur-Ulam theorem, the surjective isometries of $C_{\mathbb{R}}(X)$ are automatically affine and they are linear surjective isometries composed by translations. Therefore, by Banach-Stone theorem, a map ψ on $C_{\mathbb{R}}(X)$ is a 2-local surjective isometry exactly when it has the following property: for any $f, g \in C_{\mathbb{R}}(X)$ we have a homeomorphism $\tau_{f,g}$ and a continuous function $\epsilon_{f,g} : X \to \{-1, 1\}$ such that

$$\psi(f) - \psi(g) = \epsilon_{f,g}(f - g) \circ \tau_{f,g}.$$

It is now apparent to see that any map $\psi : C_{\mathbb{R}}(X) \to C_{\mathbb{R}}(X)$ in Problem 3.3 has this property. So, if Problem 3.4 has a positive answer then the maps in Proposition 3.3 (assuming X is first countable) are necessarily affine and hence linear.

Obviously, due to the generality of our basic problem, beside the above mentioned one particular direction (concerning function algebras), one could start investigations in very many different directions by considering different algebraic structures and/or different collections of transformations and study the new types of reflexive closures and operational reflexivity what we have introduced in the paper. We hope that several interesting new results will be obtained in the near future.

References

1. Al-Rawashdeh, A. Non-isomorphic C^*-algebras with isomorphic unitary groups. *Adv. Oper. Theory* **1** (2016), 160–163. MR3723617, Zbl 1367.46042, doi: 10.22034/aot.1609.1004 566
2. Ayupov, Sh.; Kudaybergenov, K. 2-local derivations on von Neumann algebras. *Positivity* **19** (2015), 445–455. MR3386119, Zbl 1344.46046, doi: 10.1007/s11117-014-0307-3 574
3. Beneduci, R.; Molnár, L. On the standard K-loop structure of positive invertible elements in a C^*-algebra. *J. Math. Anal. Appl.* **420** (2014), 551–562. MR3229839, Zbl 1308.46061, doi: 10.1016/j.jmaa.2014.05.009 560, 568
4. Botelho, F.; Jamison, J.; Molnár, L. Algebraic reflexivity of isometry groups and automorphism groups of some operator structures. *J. Math. Anal. Appl.* **408** (2013), 177–195. MR3079956, Zbl 1306.47080, doi: 10.1016/j.jmaa.2013.06.001 561, 567

5. Brešar, M.; Šemrl, P. Mappings which preserve idempotents, local automorphisms, and local derivations. *Canad. J. Math.* **45** (1993), 483–496. MR1222512, Zbl 0796.15001, doi: 10.4153/CJM-1993-025-4 558
6. Broise, M. Commutateurs dans le groupe unitaire d'un facteur. *J. Math. Pures Appl.* **46** (1967), 299–312. MR0223900, Zbl 0191.42703 566
7. Chernoff, P.R. Representations, automorphisms, and derivations of some operator algebras. *J. Funct. Anal.* **12** (1973), 275–289. MR0350442, Zbl 0252.46086, doi: 10.1016/0022-1236(73)90080-3 572
8. Daif, M.N. When is a multiplicative derivation additive? *Internat. J. Math. Math. Sci.* **14** (1991), 615–618. MR1110064, Zbl 0733.16013, doi: 10.1155/S0161171291000844 572
9. Dieudonné, J. On the automorphisms of the classical groups. With a supplement by Loo-Keng Hua. *Mem. Amer. Math. Soc.* **2** (1951), vi+122 pp. MR0045125, Zbl 0042.25603, doi: 10.1090/memo/0002 568
10. Dye, H.A. On the geometry of projections in certain operator algebras. *Ann. of Math.* **61** (1955), 73–89. MR0066568, Zbl 0064.11002, doi: 10.2307/1969620 566
11. Gyenti, B.; Molnár, L. On local automorphisms of some quantum mechanical structures of Hilbert space operators. *New York J. Math.* **26** (2020), 1444–1472. MR4184833, Zbl 7333611 560, 569
12. Hatori, O.; Molnár, L. Isometries of the unitary group. *Proc. Amer. Math. Soc.* **140** (2012), 2127–2140. MR2888199, Zbl 1244.47032, doi: 10.1090/S0002-9939-2011-11216-4 567
13. Hatori, O.; Molnár, L. Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^*-algebras. *J. Math. Anal. Appl.* **409** (2014), 158–167. MR3095026, Zbl 1306.46053, doi: 10.1016/j.jmaa.2013.06.065 561
14. Kadison, R.V. Local derivations. *J. Algebra* **130** (1990), 494–509. MR1051316, Zbl 0751.46041, doi: 10.1016/0021-8693(90)90095-6 557
15. Kowalski S.; Slodkowski, Z. A characterization of multiplicative linear functionals in Banach algebras. *Studia Math.* **67** (1980), 215–223. MR0592387, Zbl 0456.46041, doi: 10.4064/sm-67-3-215-223 558, 559, 575
16. Larson, D.R.; Sourour, A.R. Local derivations and local automorphisms of $B(X)$. Proc. Sympos. Pure Math. 51. Providence, Rhode Island 1990, Part 2, pp. 187–194. MR1077437, Zbl 0713.47045 557, 558
17. Lu, F. Derivations of CDC algebras. *J. Math. Anal. Appl.* **323** (2006), 179–189. MR2261159, Zbl 1105.47030, doi: 10.1016/j.jmaa.2005.10.021 573
18. Marcus, M.; Purves, R. Linear transformations on algebras of matrices: the invariance of the elementary symmetric functions. *Canadian J. Math.* **11** (1959), 383–396. MR0105425, Zbl 0086.01704, doi: 10.4153/CJM-1959-039-4 565
19. Molnár, L. 2-local isometries of some operator algebras. *Proc. Edinb. Math. Soc.* **45** (2002), 349–352. MR1912644, Zbl 1027.46062, doi: 10.1017/S0013091500000043 558
20. Molnár, L. Local automorphisms of operator algebras on Banach spaces. *Proc. Amer. Math. Soc.* **131** (2003), 1867–1874. MR1955275, Zbl 1043.47030, doi: 10.1090/S0002-9939-02-06786-2 561, 562, 564
21. Molnár, L. Non-linear Jordan triple automorphisms of sets of self-adjoint matrices and operators. *Studia Math.* **173** (2006), 39–48. MR2204461, Zbl 1236.47031, doi: 10.4064/sm173-1-3 568
22. Molnár, L. Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces. Lecture Notes in Mathematics, 1895. *Springer-Verlag, Berlin,* 2007. xiv+232 pp. MR2267033, Zbl 1119.47001, doi: 10.1007/3-540-39944-5 561, 570, 575, 576
23. Molnár, L. Thompson isometries of the space of invertible positive operators. *Proc. Amer. Math. Soc.* **137** (2009), 3849–3859. MR2529894, Zbl 1184.46021, doi: 10.1090/S0002-9939-09-09963-8 561
24. Molnár, L. General Mazur-Ulam type theorems and some applications. in Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics, W. Arendt,

R. Chill, Y. Tomilov (Eds.), Operator Theory: Advances and Applications, Vol. 250, pp. 311–342. *Birkhäuser,* 2015. MR3468225, Zbl 1372.47048, doi: 10.1007/978-3-319-18494-4_21 561, 568

25. Molnár, L. On the surjectivity of generalized isometries on the positive definite cone of matrices. *Mediterr. J. Math.* **14**:161 (2017). MR3689150, Zbl 6803982, doi: 10.1007/s00009-017-0959-x 561
26. Molnár, L. On 2-local *-automorphisms and 2-local isometries of $B(H)$. *J. Math. Anal. Appl.* **479** (2019), 569–580. MR3987046, Zbl 7096850, doi: 10.1016/j.jmaa.2019.06.038 558, 559, 560, 561
27. Molnár, L.; Šemrl, P. Local automorphisms of the unitary group and the general linear group on a Hilbert space. *Expo. Math.* **18** (2000), 231–238. MR1763889, Zbl 0963.46044 566
28. Molnár, L.; Zalar, B. Reflexivity of the group of surjective isometries on some Banach spaces. *Proc. Edinb. Math. Soc.* **42** (1999), 17–36. MR1669397, Zbl 0930.47015, doi: 10.1017/S0013091500019982 575
29. Mori, M. On 2-local nonlinear surjective isometries on normed spaces and C^*-algebras. *Proc. Amer. Math. Soc.* **148** (2020), 2477–2485. MR4080890, Zbl 1445.46012, doi: 10.1090/proc/14949 558, 577
30. Oi, S. A spherical version of the Kowalski-Slodkowski theorem and its applications. *J. Austral. Math. Soc.* **111** (2021), 386–411. MR4337945, Zbl 7426034, doi: 10.1017/S1446788720000452 558, 577
31. Sakai, S. On the group isomorphisms of unitary groups in AW^*-algebras. *Tohoku Math. J.* **7** (1955), 87–95. MR0073139 566
32. Schreier, O.; van der Waerden, B.L. Die Automorphismen der projektiven Gruppen. *Abh. Math. Sem. Univ. Hamburg* **9** (1928), 303–322. 565
33. Šemrl, P. Ring derivations on standard operator algebras. *J. Funct. Anal.* **112** (1993), 318–324. MR1213141, Zbl 0801.47024, doi: 10.1006/jfan.1993.1035 572
34. Šemrl, P. The functional equation of multiplicative derivation is superstable on standard operator algebras. *Integral Equations Operator Theory* **18** (1994), 118–122. MR1250762, Zbl 0810.47029, doi: 10.1007/BF01225216 572
35. Šemrl, P. Local automorphisms and derivations on $B(H)$. *Proc. Amer. Math. Soc.* **125** (1997), 2677–2680. MR1415338, Zbl 0887.47030, doi: 10.1090/S0002-9939-97-04073-2 558, 560, 564, 572
36. Šemrl, P. The optimal version of Hua's fundamental theorem of geometry of rectangular matrices. *Mem. Amer. Math. Soc.* **232** (2014), no. 1089, vi+74 pp. MR3244139, Zbl 1301.15005, doi: 10.1090/memo/1089 562, 563
37. Touré, C.; Schulz, F.; Brits, R. Some character generating functions on Banach algebras. *J. Math. Anal. Appl.* **468** (2018), 704–715. MR3852549, Zbl 1410.46032, doi: 10.1016/j.jmaa.2018.08.052 575
38. Waterhouse, W.C. Automorphisms of $GL_n(\mathbb{R})$. *Proc. Amer. Math. Soc.* **79** (1980), 347–351. MR0567969, Zbl 0442.20027, doi: 10.2307/2043063 565
39. Zhu J.; Xiong, C. Bilocal derivations of standard operator algebras. *Proc. Amer. Math. Soc.* **125** (1997), 1367–1370. MR1363442, Zbl 0870.47028, doi: 10.1090/S0002-9939-97-03722-2 573

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