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A class of prime fusion categories of dimension 2^N

Jingcheng Dong, Sonia Natale and Hua Sun

Abstract. We study a class of strictly weakly integral fusion categories $\mathfrak{I}_{N,\zeta}$, where $N \geq 1$ is a natural number and ζ is a 2^Nth root of unity, that we call N-Ising fusion categories. An N-Ising fusion category has Frobenius-Perron dimension 2^{N+1} and is a graded extension of a pointed fusion category of rank 2 by the cyclic group of order $\mathbb{Z}_{2^N}$. We show that every braided N-Ising fusion category is prime and also that there exists a slightly degenerate N-Ising braided fusion category for all $N > 2$. We also prove a structure result for braided extensions of a rank 2 pointed fusion category in terms of braided N-Ising fusion categories.

Contents

1. Introduction

Among the most basic examples of fusion categories, the *pointed fusion categories* are those whose simple objects are invertible. A pointed fusion category is determined by its group of invertible objects G and the cohomology class of a 3-cocycle ω on G, who is responsible for the associativity constraint. We denote by Vec_G^ω the pointed fusion category associated to the pair (G, ω).

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Let G be a finite group. A fusion category $\mathcal{C}$ is called a *G-extension* of a fusion category $\mathcal{D}$ if it admits a faithful grading by the group G,

$$\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g,$$

such that $\mathcal{C}_g \otimes \mathcal{C}_h \subseteq \mathcal{C}_{gh}$, for all $g, h \in G$, and the trivial homogeneous component is equivalent to $\mathcal{D}$ [10]. Thus, a fusion category $\mathcal{C}$ is pointed if and only if $\mathcal{C}$ is a G-extension of the fusion category Vec of finite dimensional vector spaces, for some finite group G.

An *Ising category* is a fusion category of Frobenius-Perron dimension 4 which is not pointed. Ising categories appear in Conformal Field Theory related to 2-dimensional Ising models.

Every Ising fusion category is a $\mathbb{Z}_2$-extension of the rank 2 pointed fusion category $\mathrm{Vec}_{\mathbb{Z}_2}$ and it belongs to the class of fusion categories classified by Tambara and Yamagami in [20]; in particular there exist exactly 2 Ising fusion categories up to equivalence, and they are a 3-cocycle twist of each other.

By the main result of [19], every Ising fusion category admits exactly 4 non-equivalent braidings. In particular all such braidings are non-degenerate. Several properties of Ising fusion categories are studied in [4, Appendix B]. See Subsection 2.4.

In this paper we study a family of examples of fusion categories that are obtained from Ising fusion categories and share some features with them. We call them *N-Ising fusion categories*. They are special instances of the cyclic extensions of adjoint categories of ADE type classified in [5] and are defined as follows: Let $\mathfrak{I}$ be the semisimplification of the representation category of $U_{-q}(\mathfrak{sl}_2)$, with $q = \exp(i\pi/4)$. Then $\mathfrak{I}$ is an Ising fusion category. Let Z be the non-invertible simple object of $\mathfrak{I}$. Then an N-Ising category is defined as a 3-cocycle twist of the fusion subcategory of $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2N}}$ generated by the simple object $Z \boxtimes 1$; c.f. Section 4. (The definition of a 3-cocycle twist of a group-graded fusion category is recalled in Subsection 2.2.)

A 1-Ising fusion category is thus an Ising fusion category. For every $N \geq 1$, an N-Ising fusion category has Frobenius-Perron dimension 2^{N+1} and is a graded extension of a pointed fusion category of rank 2 by the cyclic group of order $\mathbb{Z}_{2^N}$. In addition every N-Ising fusion category is strictly weakly integral. Its group of invertible objects is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{N-1}}$ and it has 2^{N-1} simple objects of Frobenius-Perron dimension $\sqrt{2}$, none of which is self-dual except in the case $N = 1$.

As graded extensions of $\mathrm{Vec}_{\mathbb{Z}_2}$, N-Ising fusion categories are parameterized by the integer N and a 2^Nth root of unity ζ. The corresponding category is denoted $\mathfrak{I}_{N,\zeta}$. We use the notation $\mathfrak{I}_N$ to indicate the category $\mathfrak{I}_{N,1}$.

Every N-Ising fusion category $\mathfrak{I}_{N,\pm 1}$ admits the structure of a braided fusion category. We show that a braided N-Ising fusion category is always prime (Corollary 4.8), that is, it does not contain any nontrivial non-degenerate fusion subcategory. We also show that with respect to any possible braiding, an N-Ising fusion category is non-degenerate if and only if $N = 1$. In addition, we prove that a slightly degenerate braided N-Ising category exists if $N > 2$. See Subsection 4.1. We point out that the classification of slightly degenerate fusion categories of Frobenius-Perron dimension 8 in [21, Proposition 4.6] implies that a 2-Ising fusion category cannot be slightly degenerate.

Observe that, as shown in [5], when $N \geq 2$ there is another family of non-pointed $\mathbb{Z}_{2^N}$-extensions of $\mathrm{Vec}_{\mathbb{Z}_2}$ which is not equivalent to any N-Ising fusion category. However, the fusion categories in this family do not admit any braiding (Theorem 5.3).

Our main result for braided extensions of a rank 2 pointed fusion category is the following theorem:

Theorem 5.5. Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. Then $\mathcal{C}$ is equivalent as a fusion category to $\mathcal{I}_N \boxtimes \mathcal{B}$, for some $N \geq 1$, where $\mathcal{I}_N$ is a braided N-Ising fusion category, and $\mathcal{B}$ is a pointed braided fusion category. Furthermore, the categories $\mathcal{I}_N$ and $\mathcal{B}$ projectively centralize each other in $\mathcal{C}$.

The notion of projective centralizer of a fusion subcategory, introduced in [4], is recalled in Subsection 2.2.

Theorem 5.5 is proved in Section 5. Its proof relies on the classification results of [5]. We point out that Theorem 5.5 applies in particular when $\mathcal{C}$ is a slightly degenerate braided fusion category with generalized Tambara-Yamagami fusion rules, that is, when $\mathcal{C}$ is slightly degenerate, not pointed, and the tensor product of two non-invertible simple objects decomposes as a sum of invertible objects.

The paper is organized as follows. In Section 2 we discuss some preliminary notions and results on fusion categories that will be relevant in the rest of the paper. Section 3 contains some basic results on the structure of a general group extension of a rank 2 pointed fusion category and on braided such extensions that will be needed in the sequel. In Section 4 we introduce N-Ising categories and study their main properties. In Section 5 we give a proof of our main result on braided extensions of a rank 2 pointed fusion category.

2. Preliminaries

We shall work over an algebraically closed field k of characteristic zero. A fusion category over k is a k-linear semisimple rigid tensor category with finitely many isomorphism classes of simple objects, finite-dimensional vector spaces of morphisms and such that the unit object $\mathbf{1}$ is simple. We refer the reader to [7], [4] for the main notions on fusion categories and braided fusion categories used throughout.

An object of a fusion category $\mathcal{C}$ is called *trivial* if it is isomorphic to $\mathbf{1}^{\oplus n}$ for some natural number n.

Let $\mathcal{C}$ be a fusion category. The tensor product in $\mathcal{C}$ induces a ring structure in the Grothendieck ring $K(\mathcal{C})$ of $\mathcal{C}$. By [7, Section 8], there is a unique ring homomorphism $\operatorname{FPdim} : K(\mathcal{C}) \to \mathbb{R}$ such that $\operatorname{FPdim}(X) \geq 1$ for all nonzero $X \in \mathcal{C}$. The number $\operatorname{FPdim}(X)$ is called the Frobenius-Perron dimension of X. The Frobenius-Perron dimension of $\mathcal{C}$ is defined by

$$\operatorname{FPdim}(\mathcal{C}) = \sum_{X \in \operatorname{Irr}(\mathcal{C})} \operatorname{FPdim}(X)^2,$$

where $\operatorname{Irr}(\mathcal{C})$ is the set of isomorphism classes of simple objects in $\mathcal{C}$.

A simple object $X \in \mathcal{C}$ is called invertible if $X \otimes X^* \cong \mathbf{1}$, where X^* is the dual of X. Thus X is invertible if and only if $\operatorname{FPdim}(X) = 1$. A fusion category $\mathcal{C}$ is called pointed if every simple object of $\mathcal{C}$ is invertible. Pointed fusion categories whose group of invertible objects is isomorphic to G are classified by the orbits of the action of the group $\operatorname{Out}(G)$ in $H^3(G, k^\times)$. The pointed fusion category corresponding to the class of a 3-cocycle ω will be denoted by $\operatorname{Vec}_G^\omega$.

The largest pointed subcategory of $\mathcal{C}$, denoted $\mathcal{C}_{pt}$, is the fusion subcategory generated by all invertible simple objects. The set $G = G(\mathcal{C})$ of isomorphism classes of invertible objects of $\mathcal{C}$ is a finite group with multiplication given by tensor product. The inverse of $X \in G$ is its dual X^*. The group G acts on the set $\operatorname{Irr}(\mathcal{C})$ by left tensor product multiplication. Let $G[X]$ be the stabilizer of $X \in \operatorname{Irr}(\mathcal{C})$ under this action. Then we have a decomposition

$$X \otimes X^* = \bigoplus_{g \in G[X]} g \oplus \sum_{Y \in \operatorname{Irr}(\mathcal{C}) - G[X]} \dim \operatorname{Hom}(Y, X \otimes X^*)\, Y. \tag{2.1}$$

2.1. Group extensions of fusion categories. Let G be a finite group. A fusion category $\mathcal{C}$ is graded by G if $\mathcal{C}$ has a direct sum decomposition into full abelian subcategories $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ such that $\mathcal{C}_g \otimes \mathcal{C}_h \subseteq \mathcal{C}_{gh}$, for all $g, h \in G$. If $\mathcal{C}_g \neq 0$, for all $g \in G$, then the grading is called faithful. When the grading is faithful, $\mathcal{C}$ is called a G-extension of the trivial component $\mathcal{C}_e$.

If $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ is a faithful grading of $\mathcal{C}$, then [7, Proposition 8.20] shows that

$$\operatorname{FPdim}(\mathcal{C}) = |G| \operatorname{FPdim}(\mathcal{C}_e), \quad \operatorname{FPdim}(\mathcal{C}_g) = \operatorname{FPdim}(\mathcal{C}_h), \quad \forall g, h \in G.$$

It follows from the results of [10] that every fusion category $\mathcal{C}$ has a canonical faithful grading $\mathcal{C} = \oplus_{g \in U(\mathcal{C})} \mathcal{C}_g$ with trivial component $\mathcal{C}_e = \mathcal{C}_{ad}$, where $\mathcal{C}_{ad}$ is the adjoint subcategory of $\mathcal{C}$, that is, the fusion subcategory generated by the simple constituents of $X \otimes X^*$, for all $X \in \mathrm{Irr}(\mathcal{C})$. This grading is called the universal grading of $\mathcal{C}$, and $U(\mathcal{C})$ is called the *universal grading group* of $\mathcal{C}$. Any other faithful grading $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ of $\mathcal{C}$ is determined by a surjective group homomorphism $\pi : U(\mathcal{C}) \to G$. Hence the trivial component $\mathcal{C}_e$ contains $\mathcal{C}_{ad}$.

Let G be a finite group and let $\mathcal{C}$ be a G-extension of a fusion category $\mathcal{D} \cong \mathcal{C}_e$. Let also $\omega \in Z^3(G, k^\times)$ be a 3-cocycle. We shall denote by $\mathcal{C}^\omega$ the fusion category obtained from $\mathcal{C}$ by twisting the associator with ω. For $\omega_1, \omega_2 \in Z^3(G, k^\times)$, the categories $\mathcal{C}^{\omega_1}$ and $\mathcal{C}^{\omega_2}$ are equivalent as G-extensions of $\mathcal{D}$ if and only if the classes of ω_1 and ω_2 coincide in $H^3(G, k^\times)$. See [8].

2.2. Braided fusion categories. A braided fusion category $\mathcal{C}$ is a fusion category admitting a braiding c, that is, a family of natural isomorphisms: $c_{X,Y}: X \otimes Y \to Y \otimes X$, $X, Y \in \mathcal{C}$, obeying the hexagon axioms.

Let $\mathcal{C}$ be a braided fusion category. Two objects $X, Y \in \mathcal{C}$ are said to centralize each other if $c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}$. The centralizer $\mathcal{D}'$ of a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ is the full subcategory of objects which centralize every object of $\mathcal{D}$, that is

$$\mathcal{D}' = \{X \in \mathcal{C} \mid c_{Y,X}c_{X,Y} = \mathrm{id}_{X\otimes Y}, \forall\, Y \in \mathcal{D}\}.$$

The Müger center $\mathcal{Z}_2(\mathcal{C})$ of a braided fusion category $\mathcal{C}$ is the centralizer $\mathcal{C}'$ of $\mathcal{C}$ itself. A braided fusion category $\mathcal{C}$ is called non-degenerate if $\mathcal{Z}_2(\mathcal{C})$ is equivalent to the category Vec of finite-dimensional vector spaces. A braided fusion category $\mathcal{C}$ is called slightly degenerate if $\mathcal{Z}_2(\mathcal{C})$ is equivalent to the category sVec of finite-dimensional super-vector spaces.

Two full subcategories $\mathcal{D}$ and $\tilde{\mathcal{D}}$ of $\mathcal{C}$ are said to *projectively centralize each other* if for all simple objects $X \in \mathcal{D}$ and $Y \in \tilde{\mathcal{D}}$, the squared braiding $c_{Y,X}c_{X,Y}$ is a scalar multiple of the identity $\mathrm{id}_{X\otimes Y}$. See [4, Subsection 3.3].

Suppose that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are fusion subcategories of $\mathcal{C}$ that projectively centralize each other. Then [4, Proposition 3.32] shows that there exist finite groups G and $\tilde{G}$ endowed with a non-degenerate pairing $b : G \times \tilde{G} \to k^\times$ and faithful gradings $\mathcal{D} = \bigoplus_{g\in G} \mathcal{D}_g$, $\tilde{\mathcal{D}} = \bigoplus_{g\in \tilde{G}} \tilde{\mathcal{D}}_g$, such that $\mathcal{D}_0 = \mathcal{D} \cap \tilde{\mathcal{D}}'$, $\tilde{\mathcal{D}}_0 = \mathcal{D}' \cap \tilde{\mathcal{D}}$, and for all homogeneous simple objects $X \in \mathcal{D}_g$, $Y \in \tilde{\mathcal{D}}_h$, $g \in G$, $h \in \tilde{G}$, the squared braiding $c_{Y,X}c_{X,Y}$ is given by

$$c_{Y,X}c_{X,Y} = b(g, h)\, \mathrm{id}_{X\otimes Y}\,.$$

A braided fusion category $\mathcal{C}$ is called symmetric if $\mathcal{Z}_2(\mathcal{C}) = \mathcal{C}$. Hence the Müger center of a braided fusion category is a symmetric fusion category.

A symmetric fusion category $\mathcal{C}$ is called Tannakian if it is equivalent to the category $\mathrm{Rep}(G)$ of finite-dimensional representations of a finite group G, as braided fusion categories.

Let $\mathcal{C}$ be a symmetric fusion category. Deligne proved that there exist a finite group G and a central element u of order 2, such that $\mathcal{C}$ is equivalent to the category $\mathrm{Rep}(G,u)$ of representations of G on finite-dimensional super vector spaces, where u acts as the parity operator [3].

The symmetric category $\mathcal{C}$ is either Tannakian or a $\mathbb{Z}_2$-extension of a Tannakian subcategory. Therefore, if $\mathrm{FPdim}(\mathcal{C})$ is odd, then $\mathcal{C}$ is Tannakian. Moreover if $\mathrm{FPdim}(\mathcal{C})$ is bigger than 2 then $\mathcal{C}$ necessarily contains a Tannakian subcategory. Also, a non-Tannakian symmetric fusion category of Frobenius-Perron dimension 2 is equivalent to the category sVec. See [4, Subsection 2.12].

The following proposition is a special case of Corollary 3.26 of [4].

Proposition 2.1. *Let $\mathcal{C}$ be a braided fusion category. Then $\mathcal{C}_{ad} \subseteq (\mathcal{C}_{pt})'$.*

The following theorem is due to Drinfeld et al. In the case when $\mathcal{C}$ is modular, it is due to Müger [16, Theorem 4.2].

Theorem 2.2. [4, Theorem 3.13] *Let $\mathcal{C}$ be a braided fusion category and let $\mathcal{D}$ be a non-degenerate subcategory of $\mathcal{C}$. Then $\mathcal{C}$ is braided equivalent to $\mathcal{D} \boxtimes \mathcal{D}'$, where $\mathcal{D}'$ is the centralizer of $\mathcal{D}$ in $\mathcal{C}$.*

For a pair of fusion subcategories $\mathcal{A}, \mathcal{B}$ of $\mathcal{D}$, we use the notation $\mathcal{A} \vee \mathcal{B}$ to indicate the smallest fusion subcategory of $\mathcal{C}$ containing $\mathcal{A}$ and $\mathcal{B}$. The following result will be used frequently.

Lemma 2.3. [4, Corollary 3.11] *Let $\mathcal{C}$ be a braided fusion category. If $\mathcal{D}$ is any fusion subcategory of $\mathcal{C}$ then $\mathcal{D}'' = \mathcal{D} \vee \mathcal{Z}_2(\mathcal{C})$.*

2.3. Pointed braided fusion categories. We recall in this subsection some facts related to the classification of pointed braided fusion categories. We refer the reader to [12], [18], [4] for a detailed exposition.

Let G be a finite abelian group. An *abelian 3-cocycle* on G with values in $k^\times$ is a pair (ω, σ), where $\omega : G \times G \times G \to k^\times$ is a normalized 3-cocycle and $\sigma : G \times G \to k^\times$ is a 2-cochain such that

$$\omega(a,b,c)\,\omega(b,c,a)\,\sigma(a,bc) = \omega(b,a,c)\,\sigma(a,b)\,\sigma(a,c),$$

for all $a,b,c \in G$. Abelian 3-cocycles form an abelian group $Z^3_{ab}(G,k^\times)$. Let $B^3_{ab}(G,k^\times) \subseteq Z^3_{ab}(G,k^\times)$ be the subgroup of abelian coboundaries, that is, abelian 3-cocycles of the form $(du, u(u_{21})^{-1})$ where $u : G \times G \to k^\times$ is a normalized 2-cochain, $du(a,b,c) = u(b,c)\,u(ab,c)^{-1}\,u(a,bc)\,u(a,b)^{-1}$, and u_{21} is defined as $u_{21}(a,b) = u(b,a)$, for all $a,b,c \in G$.

The quotient $H^3_{ab}(G,k^\times) = Z^3_{ab}(G,k^\times)/B^3_{ab}(G,k^\times)$ is called the *abelian cohomology group* of G with coefficients in $k^\times$. Every braiding of a pointed fusion category with group G of invertible objects corresponds to an element of the group $H^3_{ab}(G,k^\times)$. In particular, given a normalized 3-cocycle ω and a 2-cochain σ on G, we have that the rule

$$\sigma_{a,b}\,\mathrm{id}_{ab} : a \otimes b \to b \otimes a, \qquad a,b \in G,$$

defines a braiding in the fusion category Vec_G^ω if and only if $(\omega,\sigma) \in Z^3(G,k^\times)$.

A *quadratic form* on G with values in $k^\times$ is a map $q : G \to k^\times$ satisfying $q(g) = q(g^{-1})$, for all $g \in G$, and such that the map $\beta : G \times G \to k^\times$ defined by $\beta(a, b) = q(ab)q(a)^{-1}q(b)^{-1}$ is a symmetric bicharacter on G. If q is a quadratic form on G, then the pair (G, q) is called a *pre-metric group*.

To every abelian 3-cocycle (ω, σ) on G one can associate a quadratic form on G defined by

$$q(g) = \sigma(g, g), \quad g \in G. \tag{2.2}$$

A result of Eilenberg and Mac Lane states that this correspondence defines a group isomorphism between the abelian cohomology group $H^3_{ab}(G, k^\times)$ and the abelian group of quadratic forms on G.

Moreover, the functor that associates to every pointed fusion category $\mathcal{C}$ the pre-metric group (G, q), where G is the group of invertible objects of $\mathcal{C}$ and q is the quadratic form (2.2), where σ is the braiding of $\mathcal{C}$, defines an equivalence between the category of pointed fusion categories and braided functors up to braided isomorphism and the category of pre-metric groups.

Thus, two braided fusion categories $\mathcal{C}(G, q)$ and $\mathcal{C}(G, q')$ associated to the quadratic forms q and q' on G are equivalent if and only if there exists an automorphism φ of G such that $q'(\varphi(g)) = q(g)$, for all $g \in G$.

The squared braiding of the braided fusion category $\mathcal{C}(G, q)$ associated to a quadratic form q is given by the symmetric bilinear form $\beta : G \times G \to k^\times$ associated to q.

Let M be a natural number and let $G = \mathbb{Z}_M$ be the cyclic group of order M. Let also $\zeta \in k^\times$ be an Mth root of 1. Then ζ determines a 3-cocycle ω_ζ on $\mathbb{Z}_M$ where, for all $0 \leq i, j, \ell \leq M - 1$,

$$\omega_\zeta(i, j, \ell) = \begin{cases} 1, & \text{if } j + \ell < M, \\ \zeta^i, & \text{if } j + \ell \geq M. \end{cases} \tag{2.3}$$

The assignment $\zeta \mapsto \omega_\zeta$ gives rise to a group isomorphism between the group $\mathbb{G}_M$ of Mth roots of 1 in $k^\times$ and the group $H^3(\mathbb{Z}_M, k^\times)$. In particular $H^3(\mathbb{Z}_M, k^\times) \cong \mathbb{Z}_M$.

We shall denote by $\mathrm{Vec}^\zeta_{\mathbb{Z}_M}$ the pointed fusion category corresponding to the 3-cocycle ω_ζ. Thus $\mathrm{Vec}^1_{\mathbb{Z}_M} = \mathrm{Vec}_{\mathbb{Z}_M}$ and, if M is even, $\mathrm{Vec}^{-1}_{\mathbb{Z}_M}$ is the pointed fusion category corresponding to the 3-cocycle ω_{-1} associated to $\zeta = -1 \in \mathbb{G}_M$.

Let $\xi \in k^\times$ such that $\xi^{M^2} = 1 = \xi^{2M}$. Then the pair $(\omega_{\xi^M}, \sigma_\xi)$ is an abelian 3-cocycle on G where, for all $0 \leq i, j, \ell \leq M - 1$,

$$\sigma_\xi(i, j) = \xi^{ij}. \tag{2.4}$$

Furthermore, this gives rise to a group isomorphism between $H^3_{ab}(\mathbb{Z}_M, k^\times)$ and the group $\mathbb{G}_d$ of dth roots of 1 in $k^\times$, where $d = \gcd(M^2, 2M)$. See [12, pp. 49], [18, Subsection 2.5.2].

Thus $\mathrm{Vec}_{\mathbb{Z}_M}^{\xi^M}$ is a braided fusion category whose squared braiding is given by $\beta_\xi(i,j)\ \mathrm{id}_{i+j} : i+j \to i+j$, where $\beta_\xi : \mathbb{Z}_M \times \mathbb{Z}_M \to k^\times$ is the bilinear form defined as

$$\beta_\xi(i,j) = \xi^{2ij}, \quad 0 \leq i,j < M.$$

The quadratic form $q : \mathbb{Z}_M \to k^\times$ and the corresponding symmetric bilinear form on $\mathbb{Z}_M$ associated to the braiding (2.4) are given, respectively, by the formulas

$$q(j) = \xi^{j^2}, \qquad \beta(i,j) = \xi^{2ij}, \tag{2.5}$$

for all $0 \leq i,j \leq M-1$.

Note that the condition $\xi^{2M} = 1$ forces $\xi^M = \pm 1$. In particular, for a fixed value of $\zeta = \pm 1$, there are exactly M choices for ξ. Thus we obtain:

Lemma 2.4. *If the pointed fusion category* $\mathrm{Vec}_{\mathbb{Z}_M}^{\zeta}$ *admits a braiding then* $\zeta = \pm 1$. *In addition we have:*

(1) If M is odd, $\mathrm{Vec}_{\mathbb{Z}_M}^{\zeta}$ *does not admit any braiding unless $\zeta = 1$, and in this case, it admits exactly M braidings up to equivalence.*
(2) If M is even, then each of the categories $\mathrm{Vec}_{\mathbb{Z}_M}$ *and* $\mathrm{Vec}_{\mathbb{Z}_M}^{-1}$ *admits exactly M braidings, up to equivalence.*

Example 2.5. Let $N \geq 1$ and let $\xi \in k^\times$ be a 2^{N+1}th root of 1. It follows from formulas (2.5) that the braided fusion category associated to ξ is non-degenerate if and only if ξ is primitive. If this is the case, then the underlying fusion category is $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$.

Let $\xi \in k^\times$ be a primitive 8th root of 1. Let $\mathcal{C} = \mathcal{C}(\mathbb{Z}_4, \xi)$ be the corresponding (non-degenerate) braided fusion category. We get from formulas (2.5) that $q(2) = \xi^4 = -1$. Hence in this case the subcategory $\langle 2 \rangle \subseteq \mathcal{C}$ is equivalent to sVec.

2.4. Ising categories. An Ising category is a fusion category of Frobenius-Perron dimension 4 which is not pointed. Let $\mathcal{I}$ be an Ising fusion category. Then, up to isomorphism, $\mathcal{I}$ has a unique nontrivial invertible object δ and a unique non-invertible simple object Z. Thus $\mathrm{FPdim}\, Z = \sqrt{2}$ and the fusion rules of $\mathcal{I}$ are determined by the relation

$$Z^{\otimes 2} \cong \mathbf{1} \oplus \delta. \tag{2.6}$$

In view of the results of [20], there exist exactly 2 non-equivalent Ising fusion categories. The universal grading group of $\mathcal{I}$ is isomorphic to $\mathbb{Z}_2$. The explicit formulas for the associators of Ising categories in [20] imply that if $\mathcal{I}^+$ and $\mathcal{I}^-$ are two non-equivalent Ising categories then, up to an equivalence of fusion categories, any of them is obtained from the other by twisting the associator by the 3-cocycle ω_{-1} on $\mathbb{Z}_2$ determined by the relation $\omega_{-1}(1,1,1) = -1$.

Every Ising fusion category admits a braiding and all possible braidings are classified by the main result of [19] (see also [4]); in particular all such braidings are non-degenerate. The category $\mathcal{I}_{pt}$ is equivalent to the category sVec of super-vector spaces as a braided fusion category.

2.5. Equivariantizations and de-equivariantizations. Let $\mathcal{C}$ be a fusion category with an action by tensor autoequivalences $\rho : \underline{G} \to \mathrm{Aut}_{\otimes}(\mathcal{C})$ of a finite group G. The equivariantization $\mathcal{C}^G$ of $\mathcal{C}$ under the action of G is defined as the category of G-equivariant objects and G-equivariant morphisms of $\mathcal{C}$. Thus, an object of $\mathcal{C}^G$ is a pair $(X, (u_g)_{g\in G})$, where X is an object of $\mathcal{C}$, $u_g : \rho^g(X) \to X$, $g \in G$, is an isomorphism such that

$$u_{gh} \circ \rho^2_{g,h} = u_g \circ \rho^g(u_h),$$

for all $g, h \in G$, where $\rho^2_{g,h} : \rho^g(\rho^h(X)) \to \rho^{gh}(X)$ is the monoidal structure of the action ρ. The tensor product of equivariant objects is defined by means of the monoidal structure of the action.

Let $\mathcal{C}$ be a fusion category and let $\mathcal{E} = \mathrm{Rep}(G) \subseteq \mathcal{Z}(\mathcal{C})$ be a Tannakian subcategory of the Drinfeld center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ that embeds into $\mathcal{C}$ via the forgetful functor $\mathcal{Z}(\mathcal{C}) \to \mathcal{C}$. Then the algebra $A = k^G$ of k-valued functions on G is a commutative algebra in $\mathcal{Z}(\mathcal{C})$. The de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathcal{E}$ is the fusion category defined as the category of left A-modules in $\mathcal{C}$. See [4] for details on equivariantizations and de-equivariantizations.

The operations of equivariantization and de-equivariantization are inverse to each other: $(\mathcal{C}_G)^G \cong \mathcal{C} \cong (\mathcal{C}^G)_G$. As for their Frobenius-Perron dimensions, we have

$$\mathrm{FPdim}(\mathcal{C}) = |G|\,\mathrm{FPdim}(\mathcal{C}_G), \quad \mathrm{FPdim}(\mathcal{C}^G) = |G|\,\mathrm{FPdim}(\mathcal{C}).$$

Given a Tannakian subcategory $\mathrm{Rep}(G)$ of a braided fusion category $\mathcal{C}$, we have an exact sequence of fusion categories (see [2, Section 1]):

$$\mathrm{Rep}(G) \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}_G,$$

where $\mathcal{C}_G$ is the de-equivariantization of $\mathcal{C}$ by $\mathrm{Rep}(G)$ and F is the forgetful functor. Hence $\mathrm{Rep}(G)$ is the kernel of F, that is, the subcategory of $\mathcal{C}$ whose objects have trivial image under F.

3. Extensions of a rank 2 pointed fusion category

3.1. General Results. Recall that a *generalized Tambara-Yamagami fusion category* is a fusion category $\mathcal{C}$ which is not pointed and such that the tensor product of two non-invertible simple objects of $\mathcal{C}$ is a sum of invertible objects. See [13].

Theorem 3.1. *Let $\mathcal{C}$ be a G-extension of a pointed fusion category* $\mathrm{Vec}^{\omega}_{\mathbb{Z}_2}$. *Then the following hold:*

(1) If $\omega = -1$, then $\mathcal{C}$ is pointed.

(2) If $\omega = 1$, then $\mathcal{C}$ is either pointed or a generalized Tambara-Yamagami fusion category. If the last possibility holds, then:
   (i) Up to isomorphism, $\mathcal{C}$ has $2n$ invertible objects and n simple objects of Frobenius-Perron dimension $\sqrt{2}$, for some $n \geq 1$.
   (ii) $\mathcal{C}_{ad} \cong \mathrm{Vec}_{\mathbb{Z}_2}$ as fusion categories, and $U(\mathcal{C}) = G$ is of order $2n$.

Proof. Let $\mathcal{C} = \oplus_{g\in G}\mathcal{C}_g$ be a faithful grading such that $\mathcal{C}_e = \mathrm{Vec}^{\omega}_{\mathbb{Z}_2}$. Since this grading is faithful, every component $\mathcal{C}_g$ has Frobenius-Perron dimension 2. Since $\mathcal{C}$ is weakly integral, the Frobenius-Perron dimension of every simple object is a square root of some integer [7, Proposition 8.27]. This implies that every component $\mathcal{C}_g$ either contains 2 non-isomorphic invertible objects, or it contains a unique $\sqrt{2}$-dimensional simple object. If $\mathcal{C}$ is not pointed, then the trivial component $\mathcal{C}_e$ is pointed and there exists a component $\mathcal{C}_g$ containing a unique $\sqrt{2}$-dimensional simple object. It follows from [11, Lemma 2.6] that ω is trivial. Then (1) holds.

Suppose that $\mathcal{C}$ is not pointed. By [10, Theorem 3.10], $\mathcal{C}$ is endowed with a faithful $\mathbb{Z}_2$-grading $\mathcal{C} = \oplus_{h\in\mathbb{Z}_2}\mathcal{C}^h$, where the trivial component $\mathcal{C}^0$ is $\mathcal{C}_{pt}$ and $\mathcal{C}^1$ contains all $\sqrt{2}$-dimensional simple objects. Let X, Y be non-invertible simple objects of $\mathcal{C}$. Then $X, Y \in \mathcal{C}^1$ and hence $X \otimes Y \in \mathcal{C}^0$, which implies that $X \otimes Y$ is a direct sum of invertible objects. Hence $\mathcal{C}$ is a generalized Tambara-Yamagami fusion category and (2) holds.

Assume that the number of non-isomorphic $\sqrt{2}$-dimensional simple objects is $n \geq 1$. Then $2n = \mathrm{FPdim}(\mathcal{C}^1) = \mathrm{FPdim}(\mathcal{C}^0)$. Hence $|G| = 2n$ and we get part (i).

Since $\mathcal{C}_{ad} \subseteq \mathcal{C}_e \cong \mathrm{Vec}_{\mathbb{Z}_2}$, we know $\mathcal{C}_{ad} = \mathrm{Vec}$ or $\mathrm{Vec}_{\mathbb{Z}_2}$. Since $\mathcal{C}$ is not pointed, then $\mathcal{C}_{ad}$ cannot be Vec. Therefore $\mathcal{C}_{ad} = \mathcal{C}_e$ and $G = U(\mathcal{C})$. In particular the order of $U(\mathcal{C})$ is $2n$. This proves part (ii). □

For a fusion category $\mathcal{C}$, let $\mathrm{cd}(\mathcal{C})$ denote the set of Frobenius-Perron dimensions of simple objects of $\mathcal{C}$.

Corollary 3.2. *Let $\mathcal{C}$ be a non-pointed fusion category. Then $\mathcal{C}$ is an extension of a rank 2 pointed fusion category if and only if $\mathrm{cd}(\mathcal{C}) = \{1, \sqrt{2}\}$.*

Proof. In view of Theorem 3.1, it will be enough to show that the condition $\mathrm{cd}(\mathcal{C}) = \{1, \sqrt{2}\}$ implies that $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. So assume that $\mathrm{cd}(\mathcal{C}) = \{1, \sqrt{2}\}$.

As in the proof of Theorem 3.1 we get that $\mathcal{C}$ is a generalized Tambara-Yamagami fusion category. Then, by [17, Proposition 5.2], the adjoint subcategory $\mathcal{C}_{ad}$ coincides with the fusion subcategory generated by $G[X]$, for any $\sqrt{2}$-dimensional simple object X. Hence $\mathrm{FPdim}(\mathcal{C}_{ad}) = 2$ and $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. □

Corollary 3.3. *Let $\mathcal{C}$ be a G-extension of $\mathrm{Vec}_{\mathbb{Z}_2}$. Assume that $\mathcal{C}$ is not pointed. Then the following hold:*

(1) *The action of the group $G(\mathcal{C})$ by left (or right) tensor multiplication on the set of non-invertible simple objects of $\mathcal{C}$ is transitive.*
(2) *The group $\mathbb{Z}_2$ is a normal subgroup of $G(\mathcal{C})$.*

Proof. Since $\mathcal{C}$ is not pointed, Theorem 3.1 implies that $\mathcal{C}$ is a generalized Tambara-Yamagami fusion category. The corollary then follows from [17, Lemma 5.1]. □

3.2. Braided extensions of $\mathbf{Vec}_{\mathbb{Z}_2}$. Throughout this subsection $\mathcal{C}$ will be an extension of $\mathrm{Vec}_{\mathbb{Z}_2}$. In addition, we assume that $\mathcal{C}$ is braided and not pointed.

Lemma 3.4. *The adjoint subcategory $\mathcal{C}_{ad}$ is equivalent to* sVec *as braided fusion categories.*

Proof. By Theorem 3.1, we know that $\mathcal{C}_{ad} \cong \mathrm{Vec}_{\mathbb{Z}_2}$. By [6, Lemma 2.5], $\mathcal{C}_{ad} = \mathcal{C}_{ad} \cap \mathcal{C}_{pt}$ is symmetric. Suppose on the contrary that $\mathcal{C}_{ad}$ is Tannakian. Then $\mathcal{C}_{ad} \cong \mathrm{Rep}(\mathbb{Z}_2)$ as braided fusion categories and $\mathcal{C}$ is a $\mathbb{Z}_2$-equivariantization of a fusion category $\mathcal{C}_{\mathbb{Z}_2}$.

The forgetful functor $F : \mathcal{C} \to \mathcal{C}_{\mathbb{Z}_2}$ is a tensor functor and the image of every object in $\mathcal{C}_{ad}$ under F is a trivial object of $\mathcal{C}_{\mathbb{Z}_2}$. Let δ be the unique nontrivial simple object of $\mathcal{C}_{ad}$. If X is a non-invertible simple object of $\mathcal{C}$ then $X \otimes X^* \cong \mathbf{1} \oplus \delta$. Hence $F(X \otimes X^*) \cong F(X) \otimes F(X)^* \cong \mathbf{1} \oplus \mathbf{1}$, which implies that $F(X)$ is not simple. Then the decomposition of $F(X) \otimes F(X)^*$ must contain at least four simple direct summands. This contradiction shows that $\mathcal{C}_{ad}$ cannot be Tannakian, and therefore $\mathcal{C}_{ad} \cong \mathrm{sVec}$, as claimed. □

Recall that if $\mathcal{D}$ is a fusion category with commutative Grothendieck ring and $\mathcal{A}$ is a fusion subcategory of $\mathcal{D}$, the *commutator* of $\mathcal{A}$ in $\mathcal{D}$, denoted by $\mathcal{A}^{co}$, is the fusion subcategory of $\mathcal{D}$ generated by all simple objects X of $\mathcal{D}$ such that $X \otimes X^*$ is contained in $\mathcal{A}$ [10].

Lemma 3.5. *The following relations hold:*
(1) $(\mathcal{C}_{ad})' = \mathcal{C}_{pt}$ *and* $\mathcal{Z}_2(\mathcal{C}) \subseteq \mathcal{C}_{pt}$.
(2) $\mathcal{Z}_2(\mathcal{C}_{pt}) = \mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C})$.

Proof. (1) By [4, Proposition 3.25], a simple object $X \in \mathcal{C}$ belongs to $(\mathcal{C}_{ad})'$ if and only if it belongs to $\mathcal{Z}_2(\mathcal{C})^{co}$; that is, if and only if $X \otimes X^* \in \mathcal{Z}_2(\mathcal{C})$. If X is not invertible then $X \otimes X^* \cong \mathbf{1} \oplus \delta$ and hence $\delta \otimes X \cong X$, where δ is unique nontrivial simple object of $\mathcal{C}_{ad}$. Hence $\mathrm{sVec} \subseteq \mathcal{Z}_2(\mathcal{C})$. But by Lemma 3.4, $\mathcal{C}_{ad} \cong \mathrm{sVec}$. This is impossible by [14, Lemma 5.4] which says that if $\mathrm{sVec} \subseteq \mathcal{Z}_2(\mathcal{C})$ then $\delta \otimes Y \ncong Y$ for any $Y \in \mathcal{C}$. Therefore, $(\mathcal{C}_{ad})' \subseteq \mathcal{C}_{pt}$ is pointed. By Proposition 2.1, $(\mathcal{C}_{ad})' \supseteq (\mathcal{C}_{pt})'' = \mathcal{C}_{pt} \vee \mathcal{Z}_2(\mathcal{C})$. Hence we have

$$\mathcal{C}_{pt} \supseteq (\mathcal{C}_{ad})' \supseteq \mathcal{C}_{pt} \vee \mathcal{Z}_2(\mathcal{C}) \supseteq \mathcal{C}_{pt},$$

which shows that $(\mathcal{C}_{ad})' = \mathcal{C}_{pt}$ and $\mathcal{Z}_2(\mathcal{C}) \subseteq \mathcal{C}_{pt}$.

(2) By part (1), we have

$$\mathcal{Z}_2(\mathcal{C}_{pt}) = \mathcal{C}_{pt} \cap (\mathcal{C}_{pt})' = \mathcal{C}_{pt} \cap (\mathcal{C}_{ad})'' = \mathcal{C}_{pt} \cap (\mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C})) = \mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C}),$$

the third equality by Lemma 2.3. This proves part (2). □

4. N-Ising categories

In what follows we shall denote by $\mathfrak{I}$ the semisimplification of the representation category of $U_{-q}(\mathfrak{sl}_2)$, where $q = \exp(i\pi/4)$. Then $\mathfrak{I}$ is an Ising fusion category; see Subsection 2.4.

Recall that there exist exactly 2 non-equivalent such fusion categories, say $\mathfrak{I}$ and $\mathfrak{I}^-$. So that $\mathfrak{I}^-$ is obtained from $\mathfrak{I}$ by twisting the associator by the 3-cocycle α on $\mathbb{Z}_2$ such that $\alpha(1,1,1) = -1$.

We shall use the notation $\mathcal{I}$ to indicate either of the categories $\mathfrak{I}$ or $\mathfrak{I}^-$. As in Subsection 2.4 we shall denote by δ the unique nontrivial invertible object of $\mathcal{I}$ and Z the unique non-invertible simple object.

Let $M \geq 2$ be an even natural number. Consider the fusion subcategory $\mathcal{C}_M$ of $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_M}$ generated by the object $Z \boxtimes 1$. The relation (2.6) implies that $\mathcal{C}_M$ has $M/2$ non-invertible simple objects:

$$Z_j = Z \boxtimes (2j+1), \quad 0 \leq j \leq \frac{M}{2} - 1, \tag{4.1}$$

and M invertible objects:

$$\delta^i \boxtimes (2j), \quad 0 \leq i \leq 1,\ 0 \leq j \leq \frac{M}{2} - 1. \tag{4.2}$$

Thus $\mathrm{FPdim}\, Z_j = \sqrt{2}$, for all $j = 0, \dots, M/2 - 1$ and $\mathrm{FPdim}\, \mathcal{C}_M = 2M$.

Remark 4.1. Every fusion category $\mathcal{C}_M$, $M \geq 2$, admits a braiding; to see this it suffices to consider any braiding in $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_M}$ and restrict it to $\mathcal{C}_M$.

The categories $\mathcal{C}_M$ have generalized Tambara-Yamagami fusion rules. Let us denote by $a = \mathbf{1} \boxtimes 2 \in \mathcal{C}_M$. Explicitly, the fusion rules of $\mathcal{C}_M$ are determined as follows: the group of invertible objects is a direct product $\langle \delta \rangle \boxtimes \langle a \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{M/2}$ and

$$Z_j \otimes Z_\ell \cong a^{j+\ell+1} \oplus \delta\, a^{j+\ell+1}, \quad 0 \leq j, \ell \leq \frac{M}{2} - 1. \tag{4.3}$$

Remark 4.2. The categories $\mathcal{C}_M$ are particular cases of the construction in [5] of fusion categories which are cyclic extensions of fusion categories of adjoint ADE type. Note that the adjoint subcategory of $\mathcal{C}_M$ coincides with the subcategory generated by δ. In particular, $\mathcal{C}_M$ is a $\mathbb{Z}_M$-extension of the fusion category of adjoint $A_3^{(1)}$ type $\mathfrak{I}_{ad} \cong \mathrm{Vec}_{\mathbb{Z}_2}$.

Remark 4.3. The construction of the categories $\mathcal{C}_M$ can be generalized replacing the cyclic group $\mathbb{Z}_M$ by any finite Abelian group A as follows: We may suppose that $A = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$, where $d_1, \dots, d_r \geq 1$. Let $e_1, \dots, e_r$ be the canonical generators of A. Then the fusion subcategory of $\mathfrak{I} \boxtimes A$ generated by the simple objects $Z \boxtimes e_j$, $1 \leq j \leq r$, is an A-graded extension of $\mathrm{Vec}_{\mathbb{Z}_2}$. Observe that all the fusion categories arising in this way admit a braiding (c.f. Remark 4.1). In fact, the examples arising from this construction boil down to the ones obtained from cyclic groups, in view of Theorem 5.5 below.

Let $N \geq 1$. In what follows we shall use the notation $\mathfrak{I}_N$ to indicate the fusion category $\mathcal{C}_{2^N}$ defined above.

Example 4.4. As pointed out before, the category $\mathfrak{I}_1 = \mathfrak{I}$ is an Ising fusion category. In particular, it is non-degenerate. The category $\mathfrak{I}_2$ has two non-isomorphic simple objects Z_1 and Z_2 of Frobenius-Perron dimension $\sqrt{2}$. The group of invertible objects is $\langle\delta\rangle \times \langle a\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and we have the fusion rules

$$Z_1^* \cong Z_2, \quad Z_1^{\otimes 2} \cong a \oplus \delta a \cong Z_2^{\otimes 2}.$$

In particular, $\mathfrak{I}_2$ does not contain any Ising fusion subcategory.

More generally, the fusion rules (4.3) imply that $\mathcal{C}_M$ contains a non-invertible self-dual simple object if and only if $M/2$ is odd. If this is the case, such self-dual simple object must generate an Ising fusion subcategory. From the non-degeneracy of Ising fusion categories we obtain, for each M such that $M/2$ is odd, an equivalence fusion categories $\mathcal{C}_M \cong \mathfrak{I} \boxtimes \mathcal{B}$ or $\mathcal{C}_M \cong \mathfrak{I}^- \boxtimes \mathcal{B}$, where $\mathcal{B}$ is a pointed fusion category. Furthermore, these are equivalences of braided fusion categories regardless of the choice of the braiding in the category $\mathcal{C}_M$. This feature is generalized in Theorem 4.5 below.

Theorem 4.5. *Let $M \geq 2$ be an even natural number. Suppose that $M = 2^N m$, where $N \geq 1$ and $m \geq 1$ is odd. Then there is an equivalence of fusion categories $\mathcal{C}_M \cong \mathfrak{I}_N \boxtimes \mathcal{B}$, where $\mathcal{B}$ is a pointed fusion category. Moreover, with respect to any braiding in $\mathcal{C}_M$, this is an equivalence of braided fusion categories for an appropriate braiding in $\mathfrak{I}_N$.*

Proof. It will be enough to show that $\mathcal{C}_M \cong \mathfrak{I}_N \boxtimes \mathcal{B}$ as fusion categories. Indeed, if this is the case, then regardless of the braiding we consider in $\mathcal{C}_M$, the fusion subcategories $\mathfrak{I}_N$ and $\mathcal{B}$ must centralize each other, since their Frobenius-Perron dimensions are coprime; see [4, Proposition 3.32].

By assumption, $\mathbb{Z}_M$ is the direct sum of the subgroup generated by m and the subgroup $S \cong \mathbb{Z}_m$ generated by 2^N. Let $\mathcal{D}_1 \cong \mathrm{Vec}_{\mathbb{Z}_m}$ denote the fusion subcategory of $\mathcal{C}_M$ generated by $\mathbf{1} \boxtimes S$.

We have an equivalence of fusion categories $\mathrm{Vec}_{\mathbb{Z}_{2^N}} \cong \langle m\rangle \subseteq \mathrm{Vec}_{\mathbb{Z}_M}$, where $\langle m\rangle$ is the fusion subcategory generated by m in $\mathrm{Vec}_{\mathbb{Z}_M}$. Thus the non-invertible simple object $Z \boxtimes m$ of $\mathcal{C}_M$ generates a fusion subcategory $\mathcal{D}_2$ equivalent to $\mathfrak{I}_N$.

Consider the braiding on $\mathcal{C}_M$ induced by some braiding in $\mathfrak{I}$ and the trivial half-braiding in $\mathrm{Vec}_{\mathbb{Z}_M}$. With respect to such braiding, the fusion subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$ centralize each other. In addition, since $\mathrm{FPdim}\,\mathcal{D}_1 = m$ and $\mathrm{FPdim}\,\mathcal{D}_2 = 2^{N+1}$ are coprime, then $\mathcal{D}_1 \cap \mathcal{D}_2 \cong \mathrm{Vec}$. Therefore, $\mathcal{D}_1 \vee \mathcal{D}_2 \cong \mathcal{D}_1 \boxtimes \mathcal{D}_2$, by [15, Proposition 7.7]. Since $\mathrm{FPdim}(\mathcal{D}_1 \boxtimes \mathcal{D}_2) =$

$2^{N+1}m = \operatorname{FPdim} \mathcal{C}_M$, then $\mathcal{C}_M = \mathcal{D}_1 \vee \mathcal{D}_2 \cong \mathcal{D}_1 \boxtimes \mathcal{D}_2 \cong \mathfrak{I}_N \boxtimes \operatorname{Vec}_{\mathbb{Z}_m}$, as was to be shown. □

Let ω be a 3-cocycle on $\mathbb{Z}_M$. Recall from Subsection 2.2 that $\mathcal{C}_M^\omega$ denotes the fusion category obtained from $\mathcal{C}_M$ by twisting the associator with ω.

It follows from [5, Lemma 2.12] that, for every 3 cocycle ω on $\mathbb{Z}_M$, the fusion category $\mathcal{C}_M^\omega$ has a concrete realization as the fusion subcategory of $\mathfrak{I} \otimes \operatorname{Vec}_{\mathbb{Z}_M}^\omega$ generated by the simple object $Z \boxtimes 1$.

For every Mth root of 1, $\zeta \in k^\times$, we shall denote by $\mathcal{C}_{M,\zeta}$ the fusion category obtained from $\mathcal{C}_M$ by twisting the associator with the 3-cocycle ω_ζ defined by formula (2.3). Letting $M = 2^N$, we obtain 2^N fusion categories $\mathfrak{I}_{N,\zeta}$ which are 3-cocycle twists of $\mathfrak{I}_N = \mathfrak{I}_{N,1}$. For $\zeta_1 \neq \zeta_2$, the fusion categories $\mathfrak{I}_{N,\zeta_1}$ and $\mathfrak{I}_{N,\zeta_2}$ are non-equivalent as $\mathbb{Z}_{2^N}$-extensions of $\operatorname{Vec}_{\mathbb{Z}_2}$. We stress that, for fixed N, all the categories $\mathfrak{I}_{N,\zeta}$ share the same fusion rules.

Definition 4.6. For $N \geq 1$, $\zeta \in \mathbb{G}_{2^N}$, the category $\mathfrak{I}_{N,\zeta}$ will be called an *N-Ising fusion category.*

Recall that a fusion category $\mathcal{C}$ has an *exact factorization* into a product of two fusion subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$ if every simple object of $\mathcal{C}$ has a unique expression of the form $X \otimes Y$, where X and Y are simple objects of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. See [9].

It follows from Theorem 4.5 that every fusion category $\mathcal{C}_{M,\zeta}$ has an exact factorization into a product of a pointed fusion subcategory and an N-Ising fusion subcategory. The next theorem shows that this decomposition is sharp.

Theorem 4.7. *Let $N \geq 1$ and let $\zeta \in k^\times$ be a 2^Nth root of 1. Then every proper fusion subcategory of $\mathfrak{I}_{N,\zeta}$ is pointed. In particular, the category $\mathfrak{I}_{N,\zeta}$ does not admit any proper exact factorization.*

Proof. It is enough to show the first statement. Let $\mathcal{C} = \mathfrak{I}_{N,\zeta}$. Let us identify the universal grading group of $\mathcal{C}$ with the cyclic group $\mathbb{Z}_{2^N}$ of order 2^N. Let $X = Z \boxtimes 1 \in \mathcal{C}_1$, so that X is a faithful simple object of $\mathcal{C}$. Then the rank of $\mathcal{C}_{2m-1}$ is 1 and the rank of $\mathcal{C}_{2m}$ is 2, for all $m \geq 1$. Since $2m-1$ is also a generator of $U(\mathcal{C})$, we have that every non-invertible simple object of $\mathcal{C}$ is faithful. This implies that $\mathcal{C}$ contains no proper non-pointed fusion subcategories, as claimed. □

Recall that a braided fusion category is called *prime* if it contains no nontrivial non-degenerate fusion subcategories.

As a consequence of Theorem 4.7 we obtain the primeness of the braided N-Ising categories:

Corollary 4.8. *Let $N \geq 1$ and let $\mathcal{I}_N$ be an N-Ising fusion category. Assume that $\mathcal{I}_N$ admits a braiding. Then $\mathcal{I}_N$ is prime.*

4.1. Braidings on N-Ising categories. In this subsection we discuss braidings on N-Ising fusion categories. If $N = 1$, then $\mathfrak{I}_{N,\pm 1}$ are Ising fusion categories and therefore they admit (necessarily non-degenerate) braidings.

Remark 4.9. Observe that if a non-degenerate braided fusion category is equivalent to a 3-cocycle twist of one the categories $\mathcal{C}_M$, then $M/2$ must be odd. In fact, by [17, Lemma 5.4 (ii)], every non-degenerate fusion category with generalized Tambara-Yamagami fusion rules has a non-invertible self-dual simple object. In particular, with respect to any possible braiding, an N-Ising fusion category is non-degenerate if and only if $N = 1$.

Let $M \geq 1$ be any even natural number. Consider the braiding in $\mathcal{C}_M$ induced by some fixed braiding in $\mathfrak{I}$ and the trivial braiding in $\mathrm{Vec}_{\mathbb{Z}_M}$. Then the Müger center $\mathcal{Z}_2(\mathcal{C}_M)$ is $\mathcal{C}_M \cap \mathcal{C}'_M$, where $\mathcal{C}'_M$ is the Müger centralizer of $\mathcal{C}_M$ in $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_M}$. Since $\mathcal{C}_M$ is generated by the simple object $Z \boxtimes 1$, then $\mathcal{C}'_M = \mathbf{1} \boxtimes \mathrm{Vec}_{\mathbb{Z}_M}$ and therefore $\mathcal{Z}_2(\mathcal{C}_M) \cong \mathrm{Vec}_{\mathbb{Z}_{M/2}}$ is Tannakian. Hence for this particular braiding, the category $\mathcal{C}_M$ is not slightly degenerate neither.

Note that, by Lemma 2.4, each of the categories $\mathrm{Vec}_{\mathbb{Z}_{2^N}}$ and $\mathrm{Vec}^{-1}_{\mathbb{Z}_{2^N}}$ admits a braiding. Hence $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}$ and $\mathfrak{I} \boxtimes \mathrm{Vec}^{-1}_{\mathbb{Z}_{2^N}}$ admit a braiding and therefore the same holds for their fusion subcategories $\mathfrak{I}_{N,1}$ and $\mathfrak{I}_{N,-1}$.

Remark 4.10. Let $N \geq 1$ and let $\zeta \in \mathbb{G}_{2^N}$. Suppose that $\mathfrak{I}_{N,\zeta}$ admits a braiding. Then $\zeta = \pm 1$ or $\zeta = \pm\sqrt{-1}$.

Indeed, the pointed fusion subcategory $(\mathfrak{I}_{N,\zeta})_{pt}$ is equivalent to $\langle \delta \rangle \boxtimes \langle 2 \rangle \cong \mathrm{Vec}_{\mathbb{Z}_2} \boxtimes \mathrm{Vec}^{\bar{\omega}}_{\mathbb{Z}_{2^{N-1}}}$, where $\bar{\omega}$ is the 3-cocycle on $\mathbb{Z}_{2^{N-1}} \cong \langle 2 \rangle$ corresponding to the restriction of ω_ζ. Thus $\bar{\omega} = \omega_{\zeta^2}$. Since $\mathrm{Vec}^{\bar{\omega}}_{\mathbb{Z}_{2^{N-1}}}$ admits a braiding, Lemma 2.4 implies that $\zeta^2 = \pm 1$. Therefore $\zeta = \pm 1$ or $\zeta = \pm\sqrt{-1}$, as claimed.

In addition, Lemma 3.4 implies that the adjoint subcategory $(\mathfrak{I}_{N,\zeta})_{ad}$ is equivalent to sVec as braided fusion categories.

Lemma 4.11. *Let $\zeta \in \mathbb{G}_4$. Then a 2-Ising fusion category $\mathfrak{I}_{2,\zeta}$ admits a braiding if and only if $\zeta = \pm 1$.*

Proof. As observed in Remark 4.10, both $\mathfrak{I}_{2,1}$ and $\mathfrak{I}_{2,-1}$ admit a braiding.

Suppose conversely that $\mathfrak{I}_{2,\zeta}$ admits a braiding. As pointed out in Remark 4.10, $\zeta = \pm 1$ or $\zeta = \pm\sqrt{-1}$. If $\zeta = \pm\sqrt{-1}$, then the pointed subcategory $\langle 2 \rangle$ must be equivalent as a fusion category to $\mathrm{Vec}^{-1}_{\mathbb{Z}_2}$. In particular, $\langle 2 \rangle$ is non-degenerate, which contradicts the primeness of $\mathfrak{I}_{2,\zeta}$ (see Corollary 4.8). Then we get that $\zeta = \pm 1$. □

Lemma 4.12. *Suppose that $\mathcal{I}_N$, $N \geq 1$, is a braided N-Ising fusion category such that its Müger center contains a fusion subcategory braided equivalent to the category* sVec *of super-vector spaces. Then $\mathcal{I}_N$ is slightly degenerate.*

Proof. Let $\mathcal{C} = \mathcal{I}_N$. Then the Müger center $\mathcal{Z}_2(\mathcal{C})$ is a pointed fusion category. Since the group of invertible objects of $\mathcal{C}$ coincides with $\langle\delta\rangle \boxtimes \langle 2\rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{N-1}}$ and $\mathcal{Z}_2(\mathcal{C}) \cap \mathcal{C}_{ad} \cong \mathrm{Vec}$, then the group of invertible objects of $\mathcal{Z}_2(\mathcal{C})$ is cyclic. Combined with Lemma 5.1 below, the assumption implies that $\mathcal{Z}_2(\mathcal{C}) \cong \mathrm{sVec}$ as braided fusion categories. Thus $\mathcal{C}$ is slightly degenerate. □

It was shown in [21, Proposition 4.6] that every slightly degenerate fusion category of Frobenius-Perron dimension 8 is equivalent to a tensor product $\mathrm{sVec} \boxtimes \mathcal{D}$, for some non-degenerate fusion category $\mathcal{D}$ of dimension 4. In view of Theorem 4.7, this implies that a 2-Ising fusion category cannot be slightly degenerate.

The next example shows that, for all $N > 2$, the categories $\mathfrak{I}_{N,-1}$ admit slightly degenerate braidings.

Example 4.13. Suppose that $N > 2$. Recall from Example 2.5 that the fusion category $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$ admits a non-degenerate braiding if and only if $\zeta = -1$.

Consider the braiding in $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$ induced by any fixed braiding in $\mathfrak{I}$ and a non-degenerate braiding in $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$. Then $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$ is non-degenerate.

Regard $\mathcal{C} = \mathfrak{I}_{N,-1}$ as a braided fusion category with the braiding induced from $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$. Hence $\mathcal{Z}_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$. Moreover, since $\mathrm{FPdim}\, \mathfrak{I}_{N,-1} = 2^{N+1}$ and $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$ is non-degenerate, then $\mathrm{FPdim}\, \mathcal{C}' = 2$. Since $\mathcal{C}$ is degenerate, then $\mathcal{C}' \subseteq \mathcal{C}$.

Since $\mathfrak{I}$ is non-degenerate, then the nontrivial simple object of $\mathcal{C}'$ must be of the form $Y \boxtimes a$, where $a \in \mathbb{Z}_{2^N}$ is the unique element of order 2 and $Y = \mathbf{1}$ or $Y = \delta$. Suppose that $Y = \mathbf{1}$. Then $\mathbf{1} \boxtimes a$ centralizes $Z \boxtimes 1$ and therefore a centralizes $1 \in \mathbb{Z}_{2^N}$. This implies that a centralizes $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$, which contradicts the non-degeneracy of $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{-1}$. Thefore $Y = \delta$.

Let q be the quadratic form on $\langle\delta\rangle \boxtimes \mathbb{Z}_{2^{N-1}}$ associated to the induced braiding in $\mathcal{C}_{pt}$. The observations in Example 2.5, imply that $q(a) = 1$. Since $\delta \boxtimes 0$ is the only nontrivial object of $\mathcal{C}_{ad} \cong \mathrm{sVec}$, then $q(\delta \boxtimes 0) = -1$. Using that $\delta \boxtimes 0$ centralizes $\mathcal{C}_{pt}$, we get that $q(\delta \boxtimes a) = q(\delta \boxtimes 0) q(\mathbf{1} \boxtimes a) = -1$. This implies that $\mathcal{Z}_2(\mathcal{C}) \cong \mathrm{sVec}$. Then $\mathcal{C} = \mathfrak{I}_{N,-1}$ is slightly degenerate.

If $N = 2$ then $a = 2$ and, as observed in Example 2.5, $\langle a\rangle \cong \mathrm{sVec}$. Hence $\mathcal{Z}_2(\mathfrak{I}_{2,-1}) = \langle\delta \boxtimes a\rangle \cong \mathrm{Rep}\, \mathbb{Z}_2$ is a Tannakian subcategory.

Observe that in these examples the pointed subcategory of $\mathfrak{I}_{N,-1}$ is $\langle\delta\rangle \boxtimes \langle 2\rangle \cong \mathrm{sVec} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^{N-1}}}$.

Lemma 4.14. *Let $N > 2$. Consider a braiding in $\mathfrak{I}_{N,\zeta}$ induced from a braiding in $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$. Then $\mathfrak{I}_{N,\zeta}$ is slightly degenerate if and only if the* *induced braiding in* $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$ *is non-degenerate. If this is the case, then* $\zeta = -1$.

Proof. By Lemma 2.4, $\zeta = \pm 1$. In view of Example 2.5, it will be enough to prove the first statement. The 'if' direction was shown in Example 4.13. Suppose conversely that $\mathfrak{I}_{N,\zeta}$ is slightly degenerate. Note that with respect to any braiding in $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$, the subcategory $\mathbf{1} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$ must centralize $\mathfrak{I} \boxtimes 0$ projectively. In view of [4, Proposition 3.32], this implies that if $a = 2^{N-1}$ is the unique element of order 2 of $\mathbb{Z}_{2^N}$, then $\mathbf{1} \boxtimes a$ centralizes $Z \boxtimes 0$.

If $\mathbf{1} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$ is degenerate, then its Müger center must contain $\mathbf{1} \boxtimes a$ and therefore $\mathbf{1} \boxtimes a$ centralizes $Z \boxtimes 1$. Since $\mathbf{1} \boxtimes a \in \mathfrak{I}_{N,\zeta} = \langle Z \boxtimes 1 \rangle$, then $\mathbf{1} \boxtimes a \in \mathcal{Z}_2(\mathfrak{I}_{N,\zeta})$. Hence $\mathcal{Z}_2(\mathfrak{I}_{N,\zeta}) = \langle \mathbf{1} \boxtimes a \rangle$. But, from Formula (2.5), $q(a) = 1$, where q is the quadratic form in $\mathbb{Z}_{2^N}$ corresponding to the induced braiding in $\mathbf{1} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$. Then $\mathcal{Z}_2(\mathfrak{I}_{N,\zeta})$ is Tannakian against the assumption.

This shows that $\mathrm{Vec}_{\mathbb{Z}_{2^N}}^{\zeta}$ must be non-degenerate and finishes the proof of the lemma. □

Remark 4.15. Suppose $\mathcal{C}$ is a slightly degenerate N-Ising fusion category and $N > 2$. We have $\mathcal{C}_{pt} \cong \mathcal{C}_{ad} \boxtimes \mathcal{D}$, where $\mathcal{D} = \langle \mathbf{1} \boxtimes 2 \rangle$ is a pointed fusion category whose group of invertible objects is cyclic of order 2^{N-1}. This is in fact an equivalence of braided fusion categories since, by Lemma 3.5, $\mathcal{C}_{ad}$ centralizes $\mathcal{C}_{pt}$. Therefore

$$\mathcal{Z}_2(\mathcal{C}_{pt}) \cong \mathcal{C}_{ad} \boxtimes \mathcal{Z}_2(\mathcal{D}). \tag{4.4}$$

On the other hand, using again Lemma 3.5 and [15, Proposition 7.7], we find

$$\mathcal{Z}_2(\mathcal{C}_{pt}) = \mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C}) \cong \mathcal{C}_{ad} \boxtimes \mathcal{Z}_2(\mathcal{C}). \tag{4.5}$$

From (4.4) and (4.5) we obtain that $\operatorname{FPdim} \mathcal{Z}_2(\mathcal{D}) = 2$. Furthermore, if $\mathcal{Z}_2(\mathcal{D}) \cong \mathrm{sVec}$, then Lemma 5.1 implies that sVec is a direct factor of $\mathcal{D}$. This is possible only if $N = 2$.

Since $N > 2$, then $\mathcal{Z}_2(\langle \mathbf{1} \boxtimes 2 \rangle)$ is Tannakian of dimension 2. Hence $\mathcal{Z}_2(\langle \mathbf{1} \boxtimes 2 \rangle) \cong \langle \mathbf{1} \boxtimes 2^{N-2} \rangle \cong \operatorname{Rep} \mathbb{Z}_2$ and the nontrivial object of $\mathcal{Z}_2(\mathcal{C})$ is $\delta \boxtimes 2^{N-2}$.

5. The structure of braided extensions of $\mathbf{Vec}_{\mathbb{Z}_2}$

Suppose that $\mathcal{B}$ is a pointed braided fusion category. Corollary A. 19 of [4] states that if the Müger center $\mathcal{Z}_2(\mathcal{B})$ of $\mathcal{B}$ coincides with the category sVec of super-vector spaces, then the Müger center is a direct factor of $\mathcal{B}$, that is, $\mathcal{B} \cong \mathrm{sVec} \boxtimes \mathcal{B}_0$, for some pointed (necessarily non-degenerate in this case) braided fusion category $\mathcal{B}_0$. However, the proof of [4, Corollary A. 19] only uses the fact that $\mathrm{sVec} \subseteq \mathcal{Z}_2(\mathcal{B})$, in other words, it actually proves the following:

Lemma 5.1. *Let $\mathcal{B}$ be a pointed braided fusion category. Suppose that the Müger center of $\mathcal{B}$ contains a fusion subcategory $\mathcal{D}$ braided equivalent to the category* sVec *of super-vector spaces. Then $\mathcal{B} \cong \mathcal{D} \boxtimes \mathcal{B}_0$, for some pointed braided fusion category $\mathcal{B}_0$.*

Let $\mathrm{Vec}_{\mathbb{Z}_{2M}}^{\alpha}$ be the pointed fusion category with associativity constraint given by the 3-cocycle α, where

$$\alpha(a,b,c) = \begin{cases} 1, & b+c < 2M, \\ \exp(\frac{2i\pi a}{M}), & b+c \geq 2M. \end{cases}$$

Consider the fusion category $\mathcal{D}_{2M}$ of $\mathfrak{I} \boxtimes \mathrm{Vec}_{\mathbb{Z}_{2M}}^{\alpha}$ generated by the simple object $Z \boxtimes 1$. Let $(\mathcal{D}_{2M})_{\mathcal{E}}$ be the de-equivariantization of the fusion category $\mathcal{D}_{2M}$ by its (central) subcategory $\mathcal{E}$ generated by the invertible object $\delta \boxtimes M$.

The following result is a special instance of the classification of cyclic extensions of fusion categories of adjoint ADE type in [5].

Theorem 5.2. ([5, Lemma 3.10].) *Up to twisting the associator by a 3-cocycle ω on $\mathbb{Z}_M$, every $\mathbb{Z}_M$-extension of $\mathrm{Vec}_{\mathbb{Z}_2}$, $\otimes$-generated by a simple object of Frobenius-Perron dimension less than 2, is equivalent as a fusion category to some of the categories $\mathcal{C}_M$ or, if* 4 *divides M, to some of the categories $(\mathcal{D}_{2M})_{\mathcal{E}}$.*

As an application of Theorem 5.2, we obtain:

Theorem 5.3. *Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is a $\mathbb{Z}_M$-extension of the fusion category $\mathrm{Vec}_{\mathbb{Z}_2}$. Then $\mathcal{C}$ is equivalent as a fusion category to $\mathcal{C}_M^{\omega}$, for some 3-cocycle ω on $\mathbb{Z}_M$.*

Proof. By assumption the braided fusion category $\mathcal{C}$ is nilpotent. Since $\mathcal{C}$ is not pointed, then $\mathcal{C}_{ad} \cong \mathrm{Vec}_{\mathbb{Z}_2}$ and therefore $U(\mathcal{C}) \cong \mathbb{Z}_M$. Then [17, Theorem 4.7] implies that $\mathcal{C}$ has a faithful simple object X and in addition X is not invertible. Since the homogeneous components of the $\mathbb{Z}_M$-grading of $\mathcal{C}$ have dimension 2, then $\mathrm{FPdim}\, X = \sqrt{2}$ (see Theorem 3.1). Hence $\mathcal{C}$ is $\otimes$-generated by a simple object of Frobenius-Perron dimension less than 2.

In view of Theorem 5.2 we may assume that $\mathcal{C}$ is equivalent to a 3-cocycle twist of one of the fusion categories $(\mathcal{D}_{2M})_{\mathcal{E}}$, where M is divisible by 4.

Consider the canonical dominant tensor functor $F : \mathcal{D}_{2M} \to (\mathcal{D}_{2M})_{\mathcal{E}}$, that is, the functor F is the 'free A-module functor', where A is the regular algebra determined by the Tannakian category $\mathcal{E}$.

The functor F takes a simple object of Frobenius-Perron dimension $\sqrt{2}$ of $\mathcal{D}_{2M}$ to a simple object (of the same Frobenius-Perron dimension) of $(\mathcal{D}_{2M})_{\mathcal{E}}$. Then F induces a surjective group homomorphism $G(\mathcal{D}_{2M}) \to G((\mathcal{D}_{2M})_{\mathcal{E}})$ whose kernel is the subgroup $\langle \delta \boxtimes M \rangle$ generated by $\delta \boxtimes M$. Hence we obtain a group isomorphism $G((\mathcal{D}_{2M})_{\mathcal{E}}) \cong G(\mathcal{D}_{2M})/\langle \delta \boxtimes M \rangle$. But $G(\mathcal{D}_{2M}) = \langle \delta \rangle \boxtimes \langle 2 \rangle$, so that $G((\mathcal{D}_{2M})_{\mathcal{E}}) \cong \mathbb{Z}_M$ is cyclic of order M.

Then the group of invertible objects of $\mathcal{C}$ is cyclic of order M. Since $\mathcal{C}$ is not pointed, then $\mathcal{C}$ has generalized Tambara-Yamagami fusion rules. Then the group of invertible objects of $\mathcal{C}$, being cyclic, must contain a unique subgroup of order 2. This subgroup is necessarily the group of invertible objects of the adjoint subcategory $\mathcal{C}_{ad} \cong \mathrm{Vec}_{\mathbb{Z}_2}$.

By Lemmas 3.4 and 3.5, $\mathcal{C}_{ad} \cong$ sVec as braided fusion categories and $\mathcal{Z}_2(\mathcal{C}_{pt}) = \mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C})$. Then, by Lemma 5.1, $\mathcal{C}_{pt} \cong \mathcal{C}_{ad} \boxtimes \mathcal{B}$, for some pointed fusion category $\mathcal{B}$. Since $G(\mathcal{C})$ is cyclic, we obtain that $\mathcal{B}$ has odd dimension n. This implies that $M/2 = n$ is odd, against the assumption. The proof of the theorem is now complete. □

Remark 5.4. The proof of Theorem 5.3 shows that (twistings of) the fusion categories $(\mathcal{D}_{2M})_{\mathcal{E}}$ are not braided unless $M/2$ is odd, in which case they are equivalent to a twisting of the fusion category $\mathcal{C}_M$. When $M = 4$, $(\mathcal{D}_{2M})_{\mathcal{E}}$ has Fermionic Moore-Reed fusion rules. It is known that there are four fusion categories admitting these fusion rules and none of them is braided; see [1], [13].

The following is the main result of this section:

Theorem 5.5. *Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. Then $\mathcal{C}$ is equivalent as a fusion category to $\mathcal{I}_N \boxtimes \mathcal{B}$, for some $N \geq 1$, where $\mathcal{I}_N$ is a braided N-Ising fusion category, and $\mathcal{B}$ is a pointed braided fusion category. Furthermore, the categories $\mathcal{I}_N$ and $\mathcal{B}$ projectively centralize each other in $\mathcal{C}$.*

Proof. Let $U(\mathcal{C})$ be the universal grading group of $\mathcal{C}$, denoted additively. Then $U(\mathcal{C})$ is an Abelian group and $\mathcal{C} = \bigoplus_{a \in U(\mathcal{C})} \mathcal{C}_a$, with $\mathcal{C}_0 = \mathcal{C}_{ad} \cong \mathrm{Vec}_{\mathbb{Z}_2}$. Then $\mathcal{C}_{ad} \cong$ sVec as braided fusion categories. We shall denote by δ the unique non-invertible simple object of $\mathcal{C}_{ad}$.

Let us identify $U(\mathcal{C})$ with a direct sum of cyclic groups $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}$, where the integers $2 \leq d_1, \dots, d_r$ are such that $d_j | d_{j+1}$, for all $j = 1, \dots, r-1$. Let $e_i \in U(\mathcal{C})$, $1 \leq i \leq r$, be the canonical generators: e_i has 1 in the ith component and 0 in the remaining components.

For each $1 \leq i \leq r$, let $\mathcal{C}_{e_i}$ be the homogeneous component of degree e_i of $\mathcal{C}$. Write the set $\{1, \dots, r\}$ as a disjoint union $\{i_1, \dots, i_p\} \cup \{j_1, \dots, j_q\}$, where $p + q = r$ and the indices $i_1, \dots, i_p, j_1, \dots, j_q$ are such that

$$i_1 \leq \cdots \leq i_p, \quad j_1 \leq \cdots \leq j_q, \tag{5.1}$$

the homogeneous components $\mathcal{C}_{e_{i_\ell}}$, $1 \leq \ell \leq p$, contain a non-invertible simple object Z_{i_ℓ}, and the components $\mathcal{C}_{e_{j_s}}$, $1 \leq s \leq q$, contain two non-isomorphic invertible objects a_{j_s} and b_{j_s}.

Claim 5.6. The $p + 2q$ simple objects

$$Z_{i_1}, \dots, Z_{i_p}, a_{j_1}, b_{j_1}, \dots, a_{j_q}, b_{j_q}, \tag{5.2}$$

generate the fusion category $\mathcal{C}$.

Proof of the claim. Let X be a simple object of $\mathcal{C}$ and suppose that $X \in \mathcal{C}_a$, $a \in U(\mathcal{C})$. Since $e_1, \dots, e_r$ generate $U(\mathcal{C})$, then $a = t_1 e_1 + \cdots + t_r e_r$, for some non-negative integers $t_1, \dots, t_r$. Then the tensor product

$$Z_{i_1}^{\otimes t_{i_1}} \otimes \cdots \otimes Z_{i_p}^{\otimes t_{i_p}} \otimes x_{j_1}^{t_{j_1}} \dots x_{j_q}^{t_{j_q}} \tag{5.3}$$

belongs to $\mathcal{C}_a$, where, for all $1 \leq s \leq q$, $x_{j_s} = a_{j_s}$ or b_{j_s}.

If X is the unique simple object of $\mathcal{C}_a$ up to isomorphism, then the tensor product (5.3) must be isomorphic to a direct sum of copies of X. In particular X is a simple constituent of (5.3) and therefore it belongs to the fusion subcategory generated by (5.2).

Note in addition that such a non-invertible simple object X of $\mathcal{C}$ must exist, because $\mathcal{C}$ is not pointed. Thus if $t_1, \dots, t_r$ are chosen as above, then (5.3) does not contain any invertible constituent. Hence some of the simple objects in (5.2) must be non-invertible, that is, $p \geq 1$. Since $Z_{i_1} \otimes Z_{i_1}^* \cong \mathbf{1} \oplus \delta$, then we find that δ belongs to the fusion subcategory generated by (5.2).

Suppose next that the simple object $X \in \mathcal{C}_a$ is invertible. Then the only simple objects of $\mathcal{C}_a$ are, up to isomorphism, X and δX. Also in this case, at least one of the objects X or δX is a simple constituent of (5.3) and therefore it belongs to the fusion subcategory generated by (5.2). Then so does the other, because δ belongs to this subcategory. This proves the claim. $\square$

By Corollary 3.3 (1), the action of the group of invertible objects of $\mathcal{C}$ on the isomorphism classes of non-invertible simple objects is transitive. Then, for all $1 \leq \ell \leq p$,

$$Z_{i_1} \otimes Z_{i_\ell}^* \cong g_\ell \oplus \delta g_\ell,$$

for some invertible object $\mathbf{1} \neq g_\ell$ such that

$$g_\ell \otimes Z_{i_1} \cong Z_{i_\ell}. \tag{5.4}$$

In particular $g_1 = \delta$. Then g_ℓ and δg_ℓ are, up to isomorphism, the unique simple objects of $\mathcal{C}_{e_{i_1} - e_{i_\ell}}$.

Let $\tilde{\mathcal{B}}$ be the pointed fusion subcategory of $\mathcal{C}$ generated by the invertible objects

$$a_{j_1}, b_{j_1}, \dots, a_{j_q}, b_{j_q}, g_1, g_2, \dots, g_p. \tag{5.5}$$

Since $\delta = g_1$ generates $\mathcal{C}_{ad}$, then $\operatorname{sVec} \cong \mathcal{C}_{ad} \subseteq \tilde{\mathcal{B}}$. But by Lemma 3.5, $\mathcal{C}_{ad}$ centralizes $\tilde{\mathcal{B}}$. Lemma 5.1 implies that $\tilde{\mathcal{B}} \cong \mathcal{C}_{ad} \boxtimes \mathcal{B}_0$ for some pointed fusion category $\mathcal{B}_0$. Note that the degree of homogeneity b of a simple object of $\mathcal{B}_0$ is of the form

$$\begin{aligned} b &= s_2(e_{i_1} - e_{i_2}) + \cdots + s_p(e_{i_1} - e_{i_p}) + n_1 e_{j_1} + \cdots + n_q e_{j_q} \\ &= h e_{i_1} - s_2 e_{i_2} - \cdots - s_p e_{i_p} + n_1 e_{j_1} + \cdots + n_q e_{j_q}, \end{aligned}$$

for some non-negative integers $s_2, \dots, s_p, n_1, \dots, n_q$, where $h = s_2 + \cdots + s_p$.

Let $Z = Z_{i_1}$. Relation (5.4) and Claim 5.6 imply that the fusion subcategory $\langle Z \rangle$ generated by Z and $\mathcal{B}_0$ generate $\mathcal{C}$. By commutativity of the fusion rules of $\mathcal{C}$, we obtain that every simple object Y of $\mathcal{C}$ decomposes in the form

$$Y \cong X \otimes g, \tag{5.6}$$

for some simple object X of $\langle Z \rangle$ and some invertible object $g \in \mathcal{B}_0$.

Suppose that $X, X' \in \langle Z \rangle$ and $g, g' \in \mathcal{B}_0$ are simple objects such that

$$X \otimes g \cong X' \otimes g'. \tag{5.7}$$

Then $X \otimes g(g')^{-1} \in \langle Z \rangle$ and thus $g(g')^{-1}$ is a simple constituent of $Z^{\otimes m}$, for some $m \geq 0$. In particular, $g(g')^{-1}$ is homogeneous of degree me_{i_1}.

On the other hand, $g(g')^{-1} \in \mathcal{B}_0$. Then

$$me_{i_1} = he_{i_1} - s_2 e_{i_2} - \cdots - s_p e_{i_p} + n_1 e_{j_1} + \cdots + n_q e_{i_q},$$

for some non-negative integers $s_2, \ldots, s_p, n_1, \ldots, n_q$, and $h = s_2 + \cdots + s_p$. Therefore $d_{i_2} | s_2, \ldots, d_{i_p} | s_p$ and $d_{j_1} | n_1, \ldots, d_{j_q} | n_q$. From condition (5.1), we have that $d_{i_1} | d_{i_2} | \ldots | d_{i_p}$. Hence $d_{i_1} | h$ and $g(g')^{-1} \in \mathcal{C}_0 = \mathcal{C}_{ad}$. Therefore $g(g')^{-1} \cong \mathbf{1}$, by the definition of $\mathcal{B}_0$. Then $g \cong g'$ and also $X \cong X'$, by (5.7).

We have thus shown that the factorization (5.6) of a simple object of $\mathcal{C}$ is unique up to isomorphism. By [9, Theorem 3.8], $\mathcal{C}$ has an exact factorization into a product of its fusion subcategories $\langle Z \rangle$ and $\mathcal{B}_0$. Since $\mathcal{C}$ is braided, then $\mathcal{C} \cong \langle Z \rangle \boxtimes \mathcal{B}_0$ as fusion categories and the categories $\langle Z \rangle$ and $\mathcal{B}_0$ projectively centralize each other, by [9, Corollary 3.9]. Since $\langle Z \rangle$ is a cyclic extension of $\mathrm{Vec}_{\mathbb{Z}_2}$, then Theorems 5.3 and 4.5 imply that $\langle Z \rangle \cong \mathfrak{I}_{N,\zeta} \boxtimes \mathcal{D}$, for some $N \geq 1$, where ζ is a 2^Nth root of 1, and $\mathcal{D}$ is a pointed braided fusion category, such that $\mathfrak{I}_{N,\zeta}$ and $\mathcal{D}$ centralize each other. Letting $\mathcal{B} = \mathcal{D} \boxtimes \mathcal{B}_0$, we obtain the theorem. □

Keep the notation in Theorem 5.5. Observe that the equivalence stated in the theorem is in principle a tensor equivalence, rather than a braided equivalence. The following question was asked by the referee:

Question 5.7. *Is there an explicit example where such an equivalence is actually not braided?*

For instance, the answer to this question is negative if $\mathrm{FPdim}\, \mathcal{C} = 4m$, with m odd. Indeed in this case we must have $N = 1$ and therefore the category $\mathcal{I}_N$ would be non-degenerate, forcing $\mathcal{C}$ to be braided equivalent to a tensor product of $\mathcal{I}_N$ and the pointed braided fusion category $\mathcal{I}_N'$ (see Theorem 2.2).

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(Jingcheng Dong) College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
jcdong@nuist.edu.cn

(Sonia Natale) Facultad de Matemática, Astronomía, Física y Computación, Universidad Nacional de Córdoba, CIEM – CONICET, (5000) Ciudad Universitaria, Córdoba, Argentina
natale@famaf.unc.edu.ar

(Hua Sun) Department of Mathematics, Yangzhou University, Yangzhou, Jiangsu 225002, China
d160028@yzu.edu.cn

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