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Periodic spanning surfaces of periodic knots

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ABSTRACT. It is a well-known result by Edmonds [1] that every periodic knot of genus g bounds an equivariant Seifert surface of genus g. We show that this is not true if one instead considers nonorientable spanning surfaces of a periodic knot. We demonstrate by example that the difference between the first Betti number of an equivariant and a nonequivariant nonorientable spanning surface of a periodic knot, can be arbitrarily large.

CONTENTS

1. Introduction and results

A knot K in S^3 is said to be *periodic* if there exists an integer $p \geq 2$, an orientation preserving diffeomorphism $f : S^3 \to S^3$ of order p that preserves the knot K, and whose fixed point set $\text{Fix}(f)$ is diffeomorphic to S^1. In this case we say that K is p-periodic, that p is a period of K, and we call $\text{Fix}(f)$ the *axis of f*. See [3] for more background on periodic knots.

In [1] Edmonds proved, using the theory of surfaces of least area, that if K is a p-periodic knot of genus g, then there exists a Seifert surface Σ for K of genus g that is invariant under the diffeomorphism f. Said differently, if we define the *p-periodic* (or *equivariant*) *3-genus $g_{3,p}(K)$ of a p-periodic knot K* as

$$g_{3,p}(K) = \min\{g \geq 0 \mid K \text{ possesses an } f\text{-invariant Seifert surface of genus } g\},$$

then Edmonds' theorem can be seen as saying that $g_3(K) = g_{3,p}(K)$ for every p-periodic knot K (with $g_3(K)$ being the Seifert genus of K).

The goal of this note is to show that if one considers nonorientable spanning surfaces for periodic knots instead, the analogue of Edmonds' theorem is not true. To state our result, we recall the definition of the *nonorientable*

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(nonequivariant) 3-genus $\gamma_3(K)$, and we define the *p-periodic* (or *equivariant*) *nonorientable 3-genus* $\gamma_{3,p}(K)$ of a p-periodic knot K as

$$\gamma_3(K) = \min\{b_1(\Sigma) \mid \Sigma \subset S^3 \text{ is a nonorientable spanning surface for } K\},$$

$$\gamma_{3,p}(K) = \min\left\{b_1(\Sigma) \quad \middle| \quad \begin{array}{l} \Sigma \subset S^3 \text{ is an } f\text{-invariant nonorienatble} \\ \text{spanning surface for } K \end{array} \right\}.$$

It is not hard to see that every p-periodic knot has an equivariant nonorientable spanning surface, and thus the definition of $\gamma_{3,p}(K)$ is well posed. Indeed, such a surface can be obtained from an equivariant Seifert surface by attaching p half-twisted bands along its boundary (thus effectively performing p Reidemeister moves of type I on the knot) in an equivariant manner. It is also not hard to show that $\gamma_{3,p}(K) \le 2g_3(K) + p$, if K is p-periodic.

Theorem 1.1. *Let K be a p-periodic knot with* $p \ge 2$ *and with* $\gamma_3(K) \ge 2$*. Then* $\gamma_{3,p}(K) \ge p$*.*

Proof. Let $f : S^3 \to S^3$ be an orientation preserving diffeomorphism that displays the p-periodicity of K and let $A = \text{Fix}(f)$ be its axis. Let further $\Sigma \subset S^3$ be a nonorientable f-invariant spanning surface for K and let $\overline{\Sigma} \subset S^3$ be the quotient of Σ by the action of $\mathbb{Z}_p$ generated by f, note that $\overline{\Sigma}$ is nonorientable, as it is being branch-covered by the nonorientable surface Σ. Then $\Sigma \to \overline{\Sigma}$ is a p-fold cyclic cover, branched along $\lambda \ge 0$ points, with λ being the number of points in $\Sigma \cap A$. A straightforward computation of Euler characteristics gives

$$\chi(\Sigma) = p \cdot \chi(\overline{\Sigma}) - (p-1)\lambda. \tag{1}$$

Write $b_1(\Sigma) = a$ and $b_1(\overline{\Sigma}) = b$. The assumption $\gamma_3(K) \ge 2$ forces $a \ge 2$, while by definition $b \ge 1$ and $\lambda \ge 0$. Equation (1) then becomes

$$a - 1 = p(b-1) + (p-1)\lambda. \tag{2}$$

If $b = 1$, we obtain $a - 1 = (p-1)\lambda$ forcing $\lambda > 0$ since $a \ge 2$. This in turn forces the inequality $a - 1 \ge p - 1$ or $a \ge p$. If $b \ge 2$ then (2) implies $a - 1 \ge p$. Thus, in either case we find $a \ge p$ and hence $\gamma_{3,p}(K_p) \ge p$, since Σ was an arbitrary equivariant nonorientable spanning surface for K. □

Remark 1.2. Both the proof and the validity of Theorem 1.1 break down for the case of a p-periodic knot K with $\gamma_3(K) = 1$. The proof comes to a halt at Equation (2) which in the event of $\gamma_3(K) = 1$ allows for the solution $a = 1 = b$, $\lambda = 0$. On the other hand, the p-periodic torus knots $T(2, p)$, with $p \ge 3$ and odd, satisfy $\gamma_3(T(2,p)) = 1 = \gamma_{3,p}(T(2,p))$.

2. Applications and examples

Corollary 2.1. *The difference between the equivariant and nonequivariant nonorientable 3-genera of a periodic knot can become arbitrarily large. Specifically, for every integer* $p \ge 3$ *there exists a p-periodic knot* K_p *with*

$$\gamma_3(K_p) = 2 \qquad \textit{and} \qquad \gamma_{3,p}(K_p) \ge p.$$

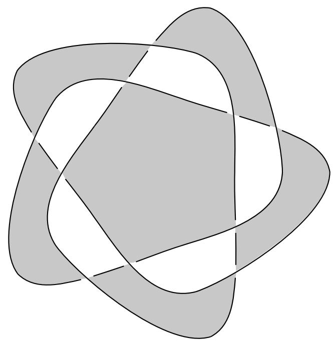

FIGURE 1. The torus knot $T(5,3)$ shown with an equivariant nonorientable spanning surface Σ with $b_1(\Sigma) = 5$.

Proof. Let K_p be the torus knot $T(4p, 2p-1)$. By [6] (see also [4]) we obtain $\gamma_3(K_p) = 2$ for all $p \geq 3$. The periods of a torus knot $T(a,b)$ are precisely the divisors of $|a|$ and $|b|$, showing that K_p is p-periodic. Theorem 1.1 implies that $\gamma_{3,p}(K_p) \geq p$. □

The preceding proof does not work for $p = 2$ as $\gamma_3(T(8,3)) = 1$, violating the hypothesis of Theorem 1.1. Nevertheless, each knot $T(4p, 2p-1)$ is of course 2-periodic, showing that $\gamma_{3,2}(K) - \gamma_3(K)$ can grow to arbitrary size for 2-periodic knots as well.

The next example shows that the inequality $\gamma_{3,p}(K) \geq p$ from Theorem 1.1 is sharp.

Example 2.2. Consider the 5-periodic torus knot $K = T(5,3)$. It follows from [6] that $\gamma_3(K) = 2$ (or use [5] where $T(5,3)$ is the knot 10_{124}), showing that K meets the hypothesis of Theorem 1.1 and thus $\gamma_{3,5}(K) \geq 5$. An equivariant spanning surface Σ for K with $b_1(\Sigma) = 5$ is shown in Figure 1, leading to $\gamma_{3,5}(K) = 5$. The values of a, b, λ from the proof of Theorem 1.1 are 5, 1, 1 respectively, and satisfy equation (2).

Another important result of Edmonds' [1] is the bound $p \leq 2g_3(K) + 1$ satisfied by any period p of the knot K. While it was known prior to Edmonds' work that a knot may only have finitely many periods (cf. Theorem 3 in [2]), the preceding inequality was the first quantitative bound on the number of possible periods of a knot. Corollary 2.1 shows, as yet another contrast to Edmonds' results, that no upper bound on the periods of a knot can exist by any polynomial function in the nonorientable 3-genus. This conclusion also follows from considering the p-periodic alternating torus knots $T(2,p)$ for which $\gamma_3(T(2,p)) = 1 = \gamma_{3,p}(T(2,p))$, with $p \geq 3$ odd.

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