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Atomic decomposition of product Hardy spaces via wavelet bases on spaces of homogeneous type

Yongsheng Han, Ji Li, M. Cristina Pereyra and Lesley A. Ward

ABSTRACT. We provide an atomic decomposition of the product Hardy spaces $H^p(\widetilde{X})$ which were recently developed by Han, Li, and Ward in the setting of product spaces of homogeneous type $\widetilde{X} = X_1 \times X_2$. Here each factor (X_i, d_i, μ_i), for $i = 1, 2$, is a space of homogeneous type in the sense of Coifman and Weiss. These Hardy spaces make use of the orthogonal wavelet bases of Auscher and Hytönen and their underlying reference dyadic grids. However, no additional assumptions on the quasi-metric or on the doubling measure for each factor space are made. To carry out this program, we introduce product (p, q)-atoms on $\widetilde{X}$ and product atomic Hardy spaces $H^{p,q}_{\rm at}(\widetilde{X})$. As consequences of the atomic decomposition of $H^p(\widetilde{X})$, we show that for all $q > 1$ the product atomic Hardy spaces coincide with the product Hardy spaces, and we show that the product Hardy spaces are independent of the particular choices of both the wavelet bases and the reference dyadic grids. Likewise, the product Carleson measure spaces $\mathrm{CMO}^p(\widetilde{X})$, the bounded mean oscillation space $\mathrm{BMO}(\widetilde{X})$, and the vanishing mean oscillation space $\mathrm{VMO}(\widetilde{X})$, as defined by Han, Li, and Ward, are also independent of the particular choices of both wavelets and reference dyadic grids.

CONTENTS

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1. Introduction

The product Hardy spaces $H^p(\widetilde{X})$ were recently developed in [HLW] in the setting of product spaces of homogeneous type $\widetilde{X} = X_1 \times X_2$, where each factor (X_i, d_i, μ_i), $i = 1, 2$, is a space of homogeneous type in the sense of Coifman and Weiss. In this paper we provide an atomic decomposition of these product Hardy spaces $H^p(\widetilde{X})$.

Spaces of homogeneous type were introduced by Coifman and Weiss in the early 1970s [CW1]. We say that (X, d, μ) is a *space of homogeneous type in the sense of Coifman and Weiss* if X is a set, d is a quasi-metric on X, and μ is a nonzero Borel-regular measure on X satisfying the doubling condition. A *quasi-metric* d on a set X is a function $d : X \times X \longrightarrow [0, \infty)$ satisfying (i) $d(x, y) = d(y, x) \geq 0$ for all $x, y \in X$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0 \in [1, \infty)$ such that,

$$d(x, y) \leq A_0\big[d(x, z) + d(z, y)\big] \quad \text{for all } x, y, z \in X. \tag{1.1}$$

The quasi-metric ball is defined by $B(x, r) := \{y \in X : d(x, y) < r\}$ for $x \in X$ and $r > 0$. Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open[1]. We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant $C_\mu \geq 1$ such that for all $x \in X$ and $r > 0$,

$$0 < \mu\big(B(x, 2r)\big) \leq C_\mu\, \mu\big(B(x, r)\big) < \infty. \tag{1.2}$$

We say a measure μ is *Borel regular* if for each measurable set A there is a Borel set B such that $B \subset A$ and $\mu(B) = \mu(A)$. This Borel regularity ensures that the Lebesgue Differentiation Theorem holds on (X, d, μ) and that step functions are dense in $L^2(X, \mu)$ [AlM, AuH2].

We point out that the doubling condition (1.2) implies that there exist positive constants C and ω (known as an *upper dimension* of X) such that for all $x \in X$, $\lambda \geq 1$ and $r > 0$,

$$\mu\big(B(x, \lambda r)\big) \leq C\lambda^\omega \mu\big(B(x, r)\big). \tag{1.3}$$

We can express C and ω in condition (1.3) in terms of the doubling constant C_μ of the measure. In fact we can and will choose $C = C_\mu \geq 1$ and $\omega = \log_2 C_\mu$.

Throughout this paper we assume that $\mu(X) = \infty$. Given a space of homogeneous type (X, d, μ), the quasi-triangle constant A_0, the doubling constant C_μ, and an upper dimension ω are referred to as the *geometric constants* of the space X.

In the classical theory, the Hardy spaces H^p can be defined via maximal functions, via approximations of the identity and Littlewood-Paley theory, via

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[1] Any quasi-metric defines a topology, for which the balls $B(x, r)$ form a base. However when $A_0 > 1$ the balls need not be open. The measure μ is assumed to be defined on a σ-algebra that contains all balls $B(x, r)$ and all Borel sets induced by this topology.

square functions, or via atomic decompositions, and all these definitions coincide. When moving to more exotic settings one can start with any of the equivalent definitions and then hope to show that they all define the same space. In the one-parameter setting of spaces of homogeneous type this program was carried out, but additional conditions were required on the quasi-metric or on the measure. The first author was involved in many of these developments. For more details see Section 2.

A natural question arises: can one develop the theory of the spaces H^p and BMO on spaces of homogeneous type in the sense of Coifman and Weiss, with only the original quasi-metric d and a Borel-regular doubling measure μ?

This question was posed, and answered in the affirmative, in [HLW], in both the one-parameter and product settings. The key idea used in [HLW] was to employ the remarkable orthonormal wavelet basis constructed by Auscher and Hytönen for spaces of homogeneous type [AuH1] to define appropriate product square functions and Hardy spaces. Note that it is in the construction of the wavelets that the Borel regularity of the measure is required [AuH2]. In the current paper we provide an atomic decomposition in the product setting and, as a consequence of our main result, we show that the $H^p(\widetilde{X})$ spaces defined via a wavelet basis in [HLW] are independent not only of the chosen wavelet basis, but also of the choice of underlying reference dyadic grids.

In the one-parameter setting the Hardy space $H^p(X)$ was built in [HLW] using the Hytönen-Auscher wavelets (themselves built upon a fixed reference dyadic grid). Using the Plancherel-Pólya inequalities proved in [HLW] (see also [Han2]), one can observe that the spaces $H^p(X)$ are well defined, meaning they are independent of the choice of wavelet basis (built upon the same reference dyadic grid). Later, in [HHL1], the atomic and molecular characterizations of the one-parameter Hardy space were studied; it was shown that $H^p(X)$ is equivalent to $H^p_{\rm at}(X)$, the Coifman-Weiss atomic Hardy space, and therefore the definition of $H^p(X)$ is independent of the choice of the wavelets and of the underlying reference dyadic grid. See also the work in [FY] for characterizing the atomic Hardy space via wavelet bases. More recently, in [HeHLLYY], the authors provided a complete real-variable theory of one-parameter Hardy spaces on spaces of homogeneous type, especially for proving the radial maximal characterization of $H^p_{\rm at}(X)$, which answered completely a question asked by Coifman and Weiss [CW2, p.642].

We now turn to the product case. As in the one-parameter case, the product Plancherel-Pólya inequalities proved in [HLW] would imply that $H^p(\widetilde{X})$ is independent of the choice of wavelet basis (built upon fixed reference dyadic grids on each component of the product $\widetilde{X}$ of spaces of homogenenous type). In this paper, instead we introduce the product (p,q)-atoms for $0 < p \leq 1 < q$ and corresponding atomic product Hardy spaces $H^{p,q}_{\rm at}(\widetilde{X})$, whose definition is independent of any wavelet bases and also of the reference dyadic grids. As a direct application, we deduce that the product Hardy spaces $H^p(\widetilde{X})$ are independent of the choices of wavelets and of underlying reference dyadic grids. This result is consistent with the product theory on the Euclidean setting $\mathbb{R}^n \times \mathbb{R}^m$, and parallel to the one-parameter theory on spaces of homogenenous type (X, d, μ) as presented in [HHL1].

Important features in the one-parameter case, treated in [HHL1], are that $H^p(X) \cap L^2(X)$ is dense in $H^p(X)$ and functions in $H^p(X) \cap L^2(X)$ have a nice atomic decomposition which converges in both $L^2(X)$ and $H^p(X)$. These features allow a linear operator bounded on $L^2(X)$ to pass through the sum in an atomic decomposition, hence reducing the proof of the boundedness of the operator to verifying uniform boundedness on atoms. See the discussion in [HHL1, p.3431–3432] regarding applications of these features to prove $T(1)$ theorems. Similar density features hold in the product case, as shown in [HLLin]; to be more precise, $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ is dense in $H^p(\widetilde{X})$ for all $q > 1$. In this paper, we will show in addition that for all $q > 1$ and all p with $0 < p \le 1$, $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ is a subset of $L^p(\widetilde{X})$, with the L^p-(semi)norm controlled by the H^p-(semi)norm. These facts will be an important cornerstone in proving the atomic decomposition for $H^p(\widetilde{X})$.

The product Carleson measure space $\mathrm{CMO}^p(\widetilde{X})$ was introduced in [HLW]. It was shown in the same paper that $\mathrm{CMO}^p(\widetilde{X})$ is the dual of $H^p(\widetilde{X})$, that the space of bounded mean oscillation $\mathrm{BMO}(\widetilde{X})$ coincides with $\mathrm{CMO}^1(\widetilde{X})$ and hence is the dual of $H^1(\widetilde{X})$, and that the vanishing mean oscillation space $\mathrm{VMO}(\widetilde{X})$ is the predual of $H^1(\widetilde{X})$. As a consequence of our result for the product Hardy spaces, we see that the spaces $\mathrm{CMO}^p(\widetilde{X})$, $\mathrm{BMO}(\widetilde{X})$, and $\mathrm{VMO}(\widetilde{X})$ are also independent not only of the chosen wavelet basis, but also of the chosen reference dyadic grids. Note that in the one-parameter case it was shown in [HHL1, Proposition 4.3] that $\mathrm{CMO}^p(X)$ coincides with the Campanato space $\mathcal{C}_{\frac{1}{p}-1}(X)$, which is the dual of the Coifman-Weiss atomic Hardy space $H^p_{\mathrm{at}}(X)$, and is a space defined independently of any wavelets and their reference dyadic grids.

When $\widetilde{X} = X_1 \times \cdots \times X_n$, the spaces $H^p(\widetilde{X})$ constructed in [HLW] are defined for all $p > \max\{\frac{\omega_i}{\omega_i+\eta_i} : i = 1, 2, \cdots, n\}$. Here $\eta_i \in (0, 1)$ is the exponent of Hölder regularity of the Auscher-Hytönen wavelets, defined on the spaces of homogeneous type (X_i, d_i, μ_i), that are used in the construction of $H^p(\widetilde{X})$, and $\omega_i > 0$ is an upper dimension of X_i, for $i = 1, \ldots, n$.

Our main result is the following.

Main Theorem. *Let $\widetilde{X} = X_1 \times X_2$, where for $i = 1, 2$, (X_i, d_i, μ_i) are spaces of homogeneous type in the sense of Coifman and Weiss as described above, with quasi-metrics d_i and Borel-regular doubling measures μ_i. Let ω_i be an upper dimension for X_i, and let η_i be the exponent of regularity of the Auscher-Hytönen wavelets used in the construction of the Hardy space $H^p(\widetilde{X})$. Suppose that* $\max\{\frac{\omega_i}{\omega_i+\eta_i} : i = 1, 2\} < p \le 1 < q < \infty$, *and $f \in L^q(\widetilde{X})$. Then $f \in H^p(\widetilde{X})$ if and only if f*

has an atomic decomposition:

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \tag{1.4}$$

where the a_j *are* (p,q)*-atoms,* $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$*, and the series converges in* $L^q(\widetilde{X})$*. Moreover, the series also converges in* $H^p(\widetilde{X})$ *and*

$$\|f\|_{H^p(\widetilde{X})} \sim \inf\Big\{\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{\frac{1}{p}} : f = \sum_{j=-\infty}^{\infty} \lambda_j a_j\Big\},$$

where the infimum is taken over all decompositions as in (1.4)*. The implicit constants are independent of the* $L^q(\widetilde{X})$*-norm and the* $H^p(\widetilde{X})$*-(semi)norm of* f*. They depend only on the geometric constants of the spaces* X_i *for* $i = 1, 2$*.*

For simplicity we work in the case of two factors: $\widetilde{X} = X_1 \times X_2$. However, we expect our results and proofs to go through for arbitrarily many factors; in particular one would need a n-parameter version of Journé's Lemma on spaces of homogeneous type, which would generalise both Pipher's n-parameter Euclidean version [P] and Han, Li and Lin's two-parameter version on spaces of homogeneous type [HLLin].

Remark 1.1. Using an approximation argument and the fact that $L^q(\widetilde{X}) \cap H^p(\widetilde{X})$ is dense in $H^p(\widetilde{X})$ for all $q > 1$, we will deduce that the atomic decomposition and norm characterization hold for all distributions in $H^p(\widetilde{X})$, not just those in $L^q(\widetilde{X})$. That is the content of Corollary A.

We deduce three corollaries from the Main Theorem. First, the atomic product spaces $H^{p,q}_{\text{at}}$ we define coincide, for all $q > 1$, with the product Hardy spaces H^p defined in [HLW].

Corollary A. *For all* q *with* $1 < q < \infty$ *and* p *with* $\max\{\frac{\omega_i}{\omega_i+\eta_i} : i = 1, 2\} < p \leq 1$*, we have*

$$H^{p,q}_{\text{at}}(\widetilde{X}) = H^p(\widetilde{X}).$$

Thus, we can define $H^p_{\text{at}}(\widetilde{X})$ to be $H^{p,q}_{\text{at}}(\widetilde{X})$ for any $q > 1$.

Second, as a consequence, we deduce that the product Hardy spaces are independent both of wavelets and of reference dyadic grids.

Corollary B. *Let* $\widetilde{X}$ *and* p *with* $p > \max\{\frac{\omega_i}{\omega_i+\eta_i} : i = 1, 2\}$ *be as in the Main Theorem. Then the Hardy spaces* $H^p(\widetilde{X})$ *as defined in* [HLW] *are independent of the particular choices of the Auscher-Hytönen wavelets and of the reference dyadic grids used in their construction.*

Third, the Carleson measure spaces and the spaces of bounded mean oscillation and of vanishing mean oscillation are also independent of both wavelets and reference dyadic grids.

Corollary C. *Let $\widetilde{X}$ and $\max\{\frac{\omega_i}{\omega_i+\eta_i} : i = 1,2\} < p \leq 1$ be as in the Main Theorem. Then the Carleson measure spaces $\mathrm{CMO}^p(\widetilde{X})$, the space of bounded mean oscillation $\mathrm{BMO}(\widetilde{X})$, and the space of vanishing mean oscillation $\mathrm{VMO}(\widetilde{X})$, as defined in* [HLW], *are independent of the particular choices of the Auscher-Hytönen wavelets and of the reference dyadic grids used in their construction.*

In the special case when $p = 1$ and $q = 2$, the (p, q)-atoms defined in this paper, and the corresponding atomic decomposition found for $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$, were used in establishing dyadic structure theorems for $H^1(\widetilde{X})$ and $\mathrm{BMO}(\widetilde{X})$ [KLPW, Definition 5.3 and Theorem 5.4]. To achieve this goal, corresponding dyadic atomic Hardy spaces were introduced in [KLPW, Definition 6.3 and Theorem 6.5].

We would like to mention that Fu and Yang [FY] present a characterization of the Coifman and Weiss atomic Hardy space $H^1_{\mathrm{at}}(X)$ in the one-parameter case, using the Auscher-Hytönen wavelets, under the assumptions that (X, d, μ) is a metric measure space of homogeneous type, $\mathrm{diam}(X) = \infty$, and X is a non-atomic space, meaning that $\mu(\{x\}) = 0$ for all $x \in X$. They prove that the Auscher-Hytönen wavelets form an unconditional basis in $H^1(X)$ and from there they deduce that a function being in $H^1_{\mathrm{at}}(X)$ is equivalent to the unconditional convergence in $L^1(X)$ of the function's wavelet expansion, and equivalent to the boundedness on $L^1(X)$ of each of three different discrete square functions, one of them coinciding with that in the definition of $H^1(X)$ presented in [HLW]. All these one-parameter Hardy spaces $H^1(X)$ coincide when the conditions assumed in [FY] are met. Fu and Yang did not address the case $p < 1$, nor the product case, which are the focus of this article.

The paper is organized as follows. In Section 2 we place our work in historical context, describing some of the progress made to date, from the original work of Coifman and Weiss until the present setting, mostly in the one-parameter case.

In Section 3 we recall the basic ingredients involved in the definition of product Hardy and BMO spaces, on spaces of homogeneous type in the sense of Coifman and Weiss with only the original quasi-metric and a Borel-regular doubling measure μ, as introduced in [HLW]. These preliminaries include the Hytönen-Kairema systems of dyadic cubes [HyK], the Auscher-Hytönen orthonormal basis and reference dyadic grids [AuH1, AuH2], and the test functions and distributions in both the one-parameter and product settings [HLW].

In Section 4 we recall the definitions in [HLW] of product Hardy spaces $H^p(\widetilde{X})$; their duals and the Carleson measure spaces $\mathrm{CMO}^p(\widetilde{X})$; the space of bounded mean oscillation $\mathrm{BMO}(\widetilde{X})$; and the space of vanishing mean oscillation $\mathrm{VMO}(\widetilde{X})$, which turns out to be the predual of $H^1(\widetilde{X})$. These definitions are based on product square functions, themselves defined using the Auscher-Hytönen wavelets and the reference dyadic grids used in their construction in [HLW]. We prove a key new lemma in Section 4 that allows us to decompose the Auscher-Hytönen wavelets into compactly supported building blocks rescaled as needed and with appropriate size, smoothness, and cancellation properties, following the approach in Nagel and Stein [NS]. In turn this lemma allows us to show that, within the allowed range of p dictated by the geometric constants and the Hölder-continuity parameters of the wavelets, functions in $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ for $1 < q < \infty$ are L^p-integrable, with L^p-(semi)norm controlled by their H^p-(semi)norm.

In Section 5 we introduce the product (p,q)-atoms and atomic product Hardy spaces $H^{p,q}_{\mathrm{at}}(\widetilde{X})$ for $1 < q < \infty$ and for p in the same range for which the product Hardy spaces $H^p(\widetilde{X})$ are defined. We restate the Main Theorem, and use it to prove Corollaries A, B, and C, thus establishing that the atomic product Hardy spaces $H^{p,q}_{\mathrm{at}}(\widetilde{X})$ coincide with the product Hardy spaces $H^p(\widetilde{X})$ for all $q > 1$, and that the spaces $\mathrm{CMO}^p(\widetilde{X})$, $\mathrm{BMO}(\widetilde{X})$, and $\mathrm{VMO}(\widetilde{X})$ are independent of the choices of wavelet bases and of reference dyadic grids on X_1 and X_2 used in their construction. Finally we prove the Main Theorem, yielding an atomic decomposition for $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ in terms of (p,q)-atoms for each q with $1 < q < \infty$, with convergence in both H^p and L^q and showing that (p,q)-atoms are uniformly in $H^p(\widetilde{X})$. Key in this decomposition is the use of a Journé-type covering lemma in the product setting, which was proved in [HLLin].

Throughout the paper the following notation is used. First, $A \lesssim B$ means there is a constant $C > 0$ depending only on the geometric constants (quasi-triangle constants of the quasi-metrics, doubling constants of the measures, and upper dimensions of X_i for $i = 1, 2$) such that $A \le CB$. Second, $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. Third, the value of a constant $C > 0$ may change from line to line within a string of inequalities. If the constant C depends on some other parameter(s), for example on $q > 1$ and $\delta > 0$, then we may denote it by $C_{q,\delta}$. Likewise, the notation $\lesssim_{q,\delta}$ indicates that the implied constant in the inequality depends also on the parameters q and δ. We denote by χ_A the characteristic function of a set $A \subset X$, that is, $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

2. Context and significance

In this section we discuss the developments in the theory of one-parameter Hardy spaces that led to the results presented in this paper. This is by no means a comprehensive historical survey, rather a series of snapshots that will give some perspective to our work. For a more complete survey see [HHL2].

We recall the atomic Hardy space $H^p_{\mathrm{at}}(X)$ on a space of homogeneous type, following [CW2]. Given (X, d, μ), a space of homogeneous type in the sense of Coifman and Weiss, as presented in the Introduction, the atomic Hardy space $H^p_{\mathrm{at}}(X)$ is defined to be a certain subcollection of the bounded linear functionals on the Campanato space $\mathcal{C}_\alpha(X)$ with $\alpha = \frac{1}{p} - 1$, $0 < p \le 1$. Namely, $H^p_{\mathrm{at}}(X)$ is defined to be those bounded linear functionals on $\mathcal{C}_\alpha(X)$ that admit an atomic decomposition

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{2.1}$$

where the functions a_j are $(p,2)$-atoms, $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$, and the series in (2.1) converges in the dual space of $\mathcal{C}_\alpha(X)$. The quasi-norm of f in $H^p_{\rm at}(X)$ is defined by

$$\|f\|_{H^p_{\rm at}(X)} := \inf\Big\{\Big(\sum_{j=1}^{\infty} |\lambda_j|^p\Big)^{\frac{1}{p}}\Big\},$$

where the infimum is taken over all such atomic representations of f.

Here a function $a(x)$ is said to be a $(p,2)$-*atom* if the following conditions hold:

(i) (Support condition) the support of $a(x)$ is contained in a ball $B(x_0,r)$ for some $x_0 \in X$ and $r > 0$;

(ii) (Size condition) $\|a\|_{L^2(X)} \le \mu\big(B(x_0,r)\big)^{\frac{1}{2}-\frac{1}{p}}$; and

(iii) (Cancellation condition) $\int_X a(x)\, d\mu(x) = 0$.

Recall that the Campanato space $\mathcal{C}_\alpha(X)$, $\alpha \ge 0$, consists of those functions f for which

$$\left\{\frac{1}{\mu(B)}\int_B |f(x) - f_B|^2 d\mu(x)\right\}^{\frac{1}{2}} \le C[\mu(B)]^\alpha, \tag{2.2}$$

where B is any quasi-metric ball, $f_B := \frac{1}{\mu(B)}\int_B f(x)\, d\mu(x)$, and the constant $C > 0$ is independent of the ball B. Let $\|f\|_{\mathcal{C}_\alpha(X)}$ be the infimum of all C for which (2.2) holds. On $\mathbb{R}^n$ the Campanato spaces $\mathcal{C}_\alpha(\mathbb{R}^n)$ coincide with the α-Lipschitz class when $0 < \alpha \le 1$ and with BMO when $\alpha = 0$, thanks to the John-Nirenberg inequality.

The Coifman-Weiss definition of the atomic Hardy space $H^p_{\rm at}(X)$ does not require any regularity on the quasi-metric d, and requires only the doubling property on the Borel-regular measure μ. Moreover, for each atomic decomposition $\sum_{j=1}^{\infty} \lambda_j a_j$ where the functions a_j are $(p,2)$-atoms with $\sum_{j=1}^{\infty} |\lambda_j|^p < \infty$, the series automatically converges in the dual space of $\mathcal{C}_\alpha(X)$ with $\alpha = \frac{1}{p} - 1$. Indeed, if a is a $(p,2)$-atom and $g \in \mathcal{C}_\alpha(X)$ with $\alpha = \frac{1}{p} - 1$, then, applying first the support and cancellation conditions on the atom a and second Hölder's inequality together with the size condition on the atom a, we obtain

$$\begin{aligned}
\Big|\int_B a(x)g(x)\, d\mu(x)\Big| &= \Big|\int_B a(x)[g(x) - g_B]\, d\mu(x)\Big| \\
&\le \|a\|_2 \Big(\int_B [g(x) - g_B]^2\, d\mu(x)\Big)^{\frac{1}{2}} \\
&\le \|g\|_{\mathcal{C}_\alpha(X)},
\end{aligned}$$

where $B = B(x_0, r)$.

Therefore, if $\sum_{j=1}^{\infty} \lambda_j a_j$ is an atomic decomposition, $g \in \mathcal{C}_\alpha(X)$, and $\alpha = \frac{1}{p} - 1$, then

$$\Big|\Big\langle \sum_{j=1}^{\infty} \lambda_j a_j, g\Big\rangle\Big| \le \sum_{j=1}^{\infty} |\lambda_j| \, \|g\|_{\mathcal{C}_\alpha(X)} \le \Big\{ \sum_{j=1}^{\infty} |\lambda_j|^p \Big\}^{\frac{1}{p}} \|g\|_{\mathcal{C}_\alpha(X)},$$

which implies that the atomic decomposition $\sum_{j=1}^{\infty} \lambda_j a_j$ converges in the dual space of $\mathcal{C}_\alpha(X)$.

In fact, in [CW2, Theorem A, p.592], Coifman and Weiss define (p, q)-atoms, replacing 2 by $q > 1$ in the definition above, and define corresponding atomic Hardy spaces $H^{p,q}_{\rm at}(X)$. They show that for each fixed $p \le 1$, the spaces $H^{p,q}_{\rm at}(X)$ for $q > 1$ all coincide. We will show in Section 5 that the analogue of this result holds for appropriately defined product (p, q)-atoms and product atomic spaces $H^{p,q}_{\rm at}(\widetilde{X})$ in the bi-parameter case $\widetilde{X} = X_1 \times X_2$.

The atomic Hardy spaces have many applications. For example, if an operator T is bounded on $L^2(X)$ and from $H^p_{\rm at}(X)$ to $L^p(X)$ for some $p \le 1$, then T is bounded on $L^q(X)$ for $1 < q \le 2$. See [CW2] for this and for more applications.

We would like to point out that Coifman and Weiss introduced the atomic Hardy spaces $H^p_{\rm at}(X)$ on spaces of homogeneous type (X, d, μ) where the quasi-metric balls were required to be open; see [CW2] for more details. To establish the maximal function characterization of the atomic Hardy space of Coifman and Weiss, some additional geometrical considerations on the quasi-metric d and the measure μ were imposed. For this purpose, Macías and Segovia [MS1] proved the following fundamental results. The first pertains to quasi-metric spaces; the second to spaces of homogeneous type.

First, suppose that (X, d) is a space endowed with a quasi-metric d that may have no regularity. Then there exists a quasi-metric d' that is topologically equivalent to d such that $d(x, y) \sim d'(x, y)$ for all $x, y \in X$ and there exist constants $\theta \in (0, 1)$ and $C > 0$ so that d' has the following regularity:

$$|d'(x, y) - d'(x', y)| \le C\, d'(x, x')^\theta \,[d'(x, y) + d'(x', y)]^{1-\theta} \tag{2.3}$$

for all $x, x', y \in X$. Moreover, if the quasi-metric balls are defined by this new quasi-metric d', that is, $B'(x, r) := \{y \in X : d'(x, y) < r\}$ for $r > 0$, then these balls are open in the topology induced by d'. See [MS1, Theorem 2, p.259]. Second, suppose that (X, d, μ) is a space of homogeneous type in the sense of Coifman and Weiss, with the property that the balls are open subsets. Then the function $d'' : X \times X \to \mathbb{R}$ defined by

$$d''(x, y) := \inf\{\mu(B) : x, y \in B, B \text{ is a } d\text{-ball}\}$$

if $x \ne y$, and $d''(x, y) = 0$ if $x = y$, is a quasi-metric topologically equivalent to d. Furthermore, the measure μ satisfies the following property for all d''-balls $B''(x, r)$, where $x \in X$ and $r > 0$:

$$\mu\big(B''(x, r)\big) \sim r. \tag{2.4}$$

See [MS1, Theorem 3, p.259]. Spaces satisfying property (2.4) are called 1-*Ahlfors regular quasi-metric spaces*[2]. Note that property (2.4) is much stronger than the doubling condition.

Starting with a quasi-metric d for which the balls are not necessarily open, we can obtain d', and we can then pass to its topologically equivalent quasi-metric $d''(x,y) := \inf\{\mu(B') : x,y \in B', B' \text{ is a } d'\text{-ball}\}$ to obtain a quasi-metric satisfying (2.3) and with the measure μ satisfying (2.4).

Macías and Segovia obtained a grand maximal function characterization for the atomic Hardy spaces $H^p(X)$ on spaces of homogeneous type (X,d,μ) that satisfy the regularity condition (2.3) on the quasi-metric d, and property (2.4) on the measure μ, with $1/(1+\theta) < p \leq 1$, where θ is the regularity exponent of the quasi-metric [MS2, Theorem (5.9), p.306].

For an authoritative modern account of Hardy spaces on n-Ahlfors regular quasi-metric spaces, see the book by Alvarado and Mitrea [AlM]. Given a quasi-metric d, the authors work with an equivalence class of quasi-metrics that includes d and the Macías-Segovia quasi-metric. In contrast, the approach in the present paper is to keep the original quasi-metric d untouched but to allow for a certain randomness in the cubes that enter into the construction of the wavelets.

To develop the Littlewood-Paley characterization of Hardy spaces on *normal spaces of homogeneous type* (X,d,μ) *of order* θ, in other words, spaces satisfying the regularity condition (2.3) on the quasi-metric d and property (2.4) on the measure μ, a suitable approximation to the identity was required. The construction of such an approximation to the identity is due to Coifman [DaJS], and this construction leads to a corresponding Calderón-type reproducing formula and Littlewood-Paley theory [DeH, p.3–4]. A further discretization of this Calderón reproducing formula is needed, and it was achieved, using the dyadic cubes of Christ [Chr], by the first author and Sawyer. See [Han1, Han2, HaS] for more details. In the present paper, a further discretization will also be needed; we will instead use the dyadic cubes of Hytönen and Kairema [HyK] on which the wavelets of Auscher and Hytönen [AuH1, AuH2] are based.

To carry out the Littlewood-Paley characterization of the atomic Hardy space on a normal space (X,d,μ) of order θ, the following test function spaces were introduced in [HaS].

Definition 2.1 (Test functions [HaS])**.** Let (X,d,μ) be a normal space of homogeneous type of order θ. Fix $x_0 \in X$, $r > 0$, $\beta \in (0,\theta]$ where θ is the regularity exponent of d, and $\gamma > 0$. A function f defined on X is said to be a *test function of type* (x_0,r,β,γ) *centered at* $x_0 \in X$ if f satisfies the following three conditions:

(i) (Size condition) For all $x \in X$,

$$|f(x)| \leq C \frac{r^\gamma}{\big(r + d(x,x_0)\big)^{1+\gamma}};$$

[2]A quasi-metric Borel measure space (X,d,μ) is *n-Ahlfors regular* if $\mu\big(B(x,r)\big) \sim r^n$.

(ii) (Hölder regularity condition) For all $x, y \in X$ with $d(x, y) < (2A_0)^{-1}\big(r + d(x, x_0)\big)$,

$$|f(x) - f(y)| \le C\Big(\frac{d(x, y)}{r + d(x, x_0)}\Big)^{\beta} \frac{r^{\gamma}}{\big(r + d(x, x_0)\big)^{1+\gamma}}; \quad \text{and}$$

(iii) (Cancellation condition)

$$\int_X f(x)\, d\mu(x) = 0.$$

Denote by $\mathcal{M}(x_0, r, \beta, \gamma)$ the set of all test functions of type (x_0, r, β, γ). The norm of f in $\mathcal{M}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0, r, \beta, \gamma)} := \inf\{C > 0 : \text{ (i) and (ii) hold}\}.$$

For each fixed x_0, let $\mathcal{M}(\beta, \gamma) := \mathcal{M}(x_0, 1, \beta, \gamma)$. It is easy to check that for each fixed $x_0' \in X$ and $r > 0$, we have $\mathcal{M}(x_0', r, \beta, \gamma) = \mathcal{M}(\beta, \gamma)$ with equivalent norms. Furthermore, it is also easy to see that $\mathcal{M}(\beta, \gamma)$ is a Banach space with respect to the norm on $\mathcal{M}(\beta, \gamma)$.

We remark that the above test function space $\mathcal{M}(\beta, \gamma)$ on (X, d, μ) offers the same service as the Schwartz test function space $\mathcal{S}_\infty = \{f \in \mathcal{S} : \int f(x) x^{\alpha}\, dx = 0, |\alpha| \ge 0\}$ does on $\mathbb{R}^n$, and as the Campanato space $\mathcal{C}_\alpha(X)$ does on a space X of homogenenous type in the sense of Coifman and Weiss.

In [NS], Nagel and Stein developed the product L^p-theory $(1 < p < \infty)$ in the setting of Carnot-Carathéodory spaces formed by vector fields satisfying Hörmander's m-finite rank condition, where $m \ge 2$ is a positive integer. The Carnot-Carathéodory spaces studied in [NS] are spaces of homogeneous type with a regular quasi-metric d and a measure μ satisfying the conditions $\mu\big(B(x, sr)\big) \sim s^{m+2}\mu\big(B(x, r)\big)$ for $s \ge 1$ and $\mu\big(B(x, sr)\big) \sim s^4\mu\big(B(x, r)\big)$ for $s \le 1$. These conditions on the measure are weaker than property (2.4) but are still stronger than the original doubling condition (1.2).

Motivated by the work of Nagel and Stein, Hardy spaces via Littlewood-Paley theory were developed by the first author, Müller and Yang [HMY2, HMY1] on spaces of homogeneous type with a regular quasi-metric and a measure satisfying some additional conditions. To be precise, let (X, d, μ) be a space of homogeneous type where the quasi-metric d satisfies the Hölder regularity property (2.3), and the measure μ satisfies the doubling condition (1.2) and the *reverse doubling condition*; that is, there are constants $\kappa \in (0, \omega]$ and $c \in (0, 1]$ such that

$$c\lambda^{\kappa}\mu\big(B(x, r)\big) \le \mu\big(B(x, \lambda r)\big) \tag{2.5}$$

for all $x \in X$, r with

$$0 < r < \sup_{x, y \in X} d(x, y)/2,$$

and λ with

$$1 \le \lambda < \sup_{x, y \in X} d(x, y)/2r.$$

The first author, Müller, and Yang observed in [HMY2, HMY1] that Coifman's construction of an approximation to the identity still works on spaces of homogeneous type (X, d, μ) with these properties.

They also showed how to define the corresponding test functions of type (x_0, r, β, γ). Their definition is very similar to Definition 2.1 above, except that one power of $\big(r + d(x, x_0)\big)$ in the denominator is replaced by $\big(\mu\big(B(x, r)\big) + \mu\big(B(x, d(x, x_0))\big)\big)$. Also, their definition is identical to the definition of test functions needed in our setting, Definition 3.5, except that in their case $\beta \in [0, \theta]$ where θ is the regularity exponent of the metric, while in our case $\beta \in [0, \eta]$ where η is the Hölder exponent of the wavelets.

Applying Coifman's approximation to the identity and a proof similar to the one in [Han1, Han2, HaS], the first author, Müller, and Yang proved that a discrete Calderón reproducing formula still holds on (X, d, μ) when the quasi-metric d satisfies the regularity condition (2.3) and the measure μ satisfies the doubling condition (1.2) and the reverse doubling condition (2.5). As a consequence, the Hardy spaces defined via the Littlewood-Paley theory were established for such spaces of homogeneous type and, moreover, these Hardy spaces have atomic decompositions. See [HMY2] for more details.

However, there are settings for which the reverse doubling condition is not available. One specific example of such a space of homogeneous type appears in the Bessel setting treated by Muckenhoupt and Stein [MuS]. They studied the Bessel operator

$$\Delta_\lambda = -\frac{d}{dx^2} - \frac{2\lambda}{x}\frac{d}{dx}, \qquad \lambda \in \big(-\frac{1}{2}, \infty\big),\ x \in (0, \infty),$$

with the underlying space $(X, d, \mu) = \big((0, \infty), |\cdot|, x^{2\lambda}\, dx\big)$. The corresponding Hardy space was studied in [BDT] and the weak factorization was obtained in [DLWY]. We note that the measure $x^{2\lambda}\, dx$ is doubling when $\lambda \in (-\frac{1}{2}, \infty)$, however when $\lambda \in (-\frac{1}{2}, 0)$ the measure does not satisfy a reverse doubling condition. We also note that we cannot change the metric twice as in [MS1], for if we did we would be changing the whole setting, including the Bessel operator in question.

In [HLW], the first, second and fourth authors developed a theory of Hardy spaces H^p and BMO on spaces of homogeneous type in the sense of Coifman and Weiss, with only the original quasi-metric d and a (Borel-regular) doubling measure μ, in both the one-parameter and product settings. A crucial idea in [HLW] was to use a square-function characterization where the square function was built using the Auscher-Hytönen orthonormal wavelet basis on spaces of homogeneous type [AuH1, AuH2]. In the current paper we provide an atomic decomposition for $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ for each q with $1 < q < \infty$, for $\widetilde{X} = X_1 \times X_2$ with X_i a space of homogenenous type in the sense of Coifman and Weiss for $i = 1, 2$. This atomic decomposition is completely independent of any wavelet bases and reference dyadic grids on X_i for $i = 1, 2$ used to define $H^p(\widetilde{X})$. As a consequence of the main result of this paper, the $H^p(\widetilde{X})$ spaces defined in [HLW] via a particular Auscher-Hytönen wavelet basis are independent not only of the chosen wavelet bases, but also of the choice of reference dyadic grids.

3. Preliminaries

In this section, we will recall first Hytönen and Kairema's systems of dyadic cubes [HyK], second Auscher and Hytönen's orthonormal basis [AuH1] paying close attention to their underlying reference dyadic grids, and third the sets of test functions and distributions developed in [HLW] in both one-parameter and the product settings. We recall that the Auscher and Hytönen wavelets in both one-parameter and product settings are suitable test functions. These are all necessary ingredients in the definition of product Hardy spaces introduced in [HLW] that we present in Section 4.

3.1. Systems of dyadic cubes. We now describe the Hytönen and Kairema [HyK] families of dyadic "cubes" built on geometrically doubling quasi-metric spaces. A quasi-metric space (X, d) is *geometrically doubling* if there exists a natural number N such that any quasi-metric ball $B(x, r)$ can be covered with no more than N balls of half the radius. Coifman and Weiss [CW1] showed that spaces of homogeneous type (X, d, μ) are geometrically doubling quasi-metric spaces. The Hytönen-Kairema construction builds on seminal work of Guy David [Da], Christ [Chr], and Sawyer and Wheeden [SW].

Theorem 3.1 ([HyK], Theorem 2.2)**.** *Given a geometrically doubling quasi-metric space (X, d), let $A_0 > 0$ denote the quasi-triangle constant for the metric d. Given constants c_0 and C_0 with $0 < c_0 \le C_0 < \infty$, and constant $\delta \in (0, 1)$ satisfying*

$$12A_0^3C_0\delta \le c_0. \tag{3.1}$$

Given a set of points $\{z_\alpha^k\}_{\alpha\in\mathscr{A}_k}$, where $\mathscr{A}_k$ is a countable set of indices for each $k \in \mathbb{Z}$, with the properties that

$$d(z_\alpha^k, z_\beta^k) \ge c_0\delta^k \ (\alpha \ne \beta), \qquad \min_{\alpha\in\mathscr{A}_k} d(x, z_\alpha^k) < C_0\delta^k \quad \textit{for all } x \in X, \tag{3.2}$$

(*called a* (c_0, C_0)-maximal set of δ^k-separated points)*, we can construct families of sets* $\widetilde{Q}_\alpha^k \subseteq Q_\alpha^k \subseteq \overline{Q}_\alpha^k$ (*called open, half-open and closed* dyadic cubes)*, such that:*

$$\widetilde{Q}_\alpha^k \textit{ and } \overline{Q}_\alpha^k \textit{ are the interior and closure of } Q_\alpha^k, \textit{ respectively}; \tag{3.3}$$

$$\text{(Nested family)} \ \textit{if } \ell \ge k, \textit{ then either } Q_\beta^\ell \subseteq Q_\alpha^k \textit{ or } Q_\alpha^k \cap Q_\beta^\ell = \emptyset; \tag{3.4}$$

$$\text{(Disjoint union)} \ X = \bigcup_{\alpha\in\mathscr{A}_k} Q_\alpha^k \qquad \textit{for all } k \in \mathbb{Z}; \tag{3.5}$$

$$\text{(Inner and outer balls)} \ B(z_\alpha^k, c_1\delta^k) \subseteq Q_\alpha^k \subseteq B(z_\alpha^k, C_1\delta^k), \tag{3.6}$$

where $c_1 := (3A_0^2)^{-1}c_0$ *and* $C_1 := 2A_0C_0$;

$$\textit{if } \ell \geq k \textit{ and } Q_\beta^\ell \subseteq Q_\alpha^k, \textit{ then } B(z_\beta^\ell, C_1\delta^\ell) \subseteq B(z_\alpha^k, C_1\delta^k). \tag{3.7}$$

The open and closed cubes $\widetilde{Q}_\alpha^k$ *and* $\overline{Q}_\alpha^k$ *depend only on the points* z_β^ℓ *for* $\ell \geq k$. *The half-open cubes* Q_α^k *depend on* z_β^ℓ *for* $\ell \geq \min(k, k_0)$, *where* $k_0 \in \mathbb{Z}$ *is a preassigned number entering the construction.*

We denote by $\mathscr{D}$ the family of dyadic cubes $\{Q_\alpha^k\}_{k\in\mathbb{Z},\mathscr{A}_k}$ as in Theorem 3.1. We will refer to $\mathscr{D}$ as a *Hytönen-Kairema dyadic system* or *grid on* X. We will refer to any cube $Q_\beta^{k+1} \in \mathscr{D}$ that is contained in $Q_\alpha^k \in \mathscr{D}$ as a *child of* Q_α^k. Note that every cube has at least one child and no more than M children, where M is a uniform bound determined by the geometric doubling condition.

The existence of countable sets of separated points as in (3.2) is ensured by the geometric doubling property of the quasi-metric space (X, d). For a given Hytönen-Kairema dyadic system of cubes, we will call c_0 and C_0 the *separation constants* of the system, c_1 and C_1 the *dilation constants* of the system, and δ the *base side length* of the cube. Collectively these will be called *structural constants* of the dyadic system or of the dyadic grid. Note that in (3.6), as it should be, the dilation constants c_1 and C_1, determining the radii of the inner and outer balls for each cube, satisfy $0 < c_1 < C_1$, since by hypothesis the separation constants satisfy $0 < c_0 \leq C_0$, but *a priori* C_1 is not necessarily greater than one. We will sometimes denote by B_Q' and B_Q'' the inner and outer balls of a dyadic cube Q.

Given a cube Q_α^k, we denote the quantity δ^k by $\ell(Q_\alpha^k)$, by analogy with the sidelength of a Euclidean cube. We define the dilate λQ_α^k of a dyadic cube to be the λ-dilate of its outer ball. That is, for $\lambda > 0$,

$$\lambda Q_\alpha^k := B(z_\alpha^k, \lambda C_1 \delta^k).$$

By construction, the cubes are unions of quasi-metric balls, hence in the setting of a space of homogeneous type, the cubes are measurable. In the presence of a doubling measure μ (doubling with respect to balls) the measure μ is "doubling" with respect to Hytönen-Kairema cubes. More precisely,

$$\mu(\lambda Q_\alpha^k) \leq \Big(\lambda \frac{C_1}{c_1}\Big)^\omega \mu\big(B(z_1^k, c_1\delta^k)\big) \leq \lambda^\omega \Big(\frac{C_1}{c_1}\Big)^\omega \mu(Q_\alpha^k), \tag{3.8}$$

where the first inequality is a consequence of the doubling property (1.3), and the second holds simply because the inner ball of a cube sits inside the cube. Also note that by construction, specifically properties (3.6) and (3.1), the ratio $C_1/c_1 = 6A_0^3(C_0/c_0) \leq \delta^{-1}/2$, where $\delta \in (0,1)$ is the base side length of the cubes. Potentially the base side length parameter δ can be arbitrarily small, therefore making the upper bound in (3.8) arbitrarily large. Also, the ratio C_1/c_1 may be under control, but that does not imply the outer dilation constant cannot be arbitrarily large, since a priori we could allow the inner dilation constant to also be arbitrarily large. These facts can be problematic, therefore we single out the dyadic systems that do not suffer from these problems, and we call them *regular families of dyadic systems* or *grids*.

Definition 3.2 (Regular families of dyadic systems)**.** Given a geometric doubling quasi-metric space (X,d). A family $\{\mathscr{D}^b\}_{b\in\mathscr{B}}$ of Hytönen-Kairema dyadic systems on X is *regular* if the outer dilation constants $\{C_1^b\}_{b\in\mathscr{B}}$ and the ratio of the outer and inner dilation constants $\{C_1^b/c_1^b\}_{b\in\mathscr{B}}$ of the systems in the family are uniformly bounded by constants depending only on the quasi-triangle constant A_0 of the quasi-metric d.

In the proof of the main theorem in Section 5.4 we will have atomic decompositions in the setting of a product of spaces of homogenenous type, $X_1 \times X_2$, with atoms a associated to dyadic grids $\mathcal{D}_i^a$ belonging to regular families on (X_i, d_i, μ_i) for $i = 1, 2$. Often we will estimate the measure of dilates of cubes $Q_i \in \mathscr{D}_i^a$ as in inequality (3.8), and will simply say "by doubling"

$$\mu_i(\lambda Q_i) \lesssim \lambda^{\omega_i} \mu_i(Q_i). \tag{3.9}$$

The $\lesssim$ will only depend on the geometric constants of the spaces X_i for $i = 1, 2$, but not on the structural constants of the dyadic grids, because $\mathcal{D}_i^a$ belong to a regular family of dyadic systems. Elsewhere in the proof of the main theorem the outer dilation constants C_1^i will come into the estimates, and we will need them also to be uniformly bounded by a constant depending only on the geometric constants of X_i for $i = 1, 2$.

3.2. Orthonormal basis, reproducing formula, and cut-off functions.

Auscher and Hytönen [AuH1] constructed a remarkable orthonormal basis of $L^2(X)$, where (X, d, μ) is a space of homogeneous type. To state their result, we first recall the *reference dyadic points* x_α^k as follows.

Let δ be a fixed small positive parameter ($\delta \le 10^{-3} A_0^{-10}$, where A_0 is the quasi-triangle constant of the quasi-metric d). For $k = 0$, let $\mathscr{X}^0 := \{x_\alpha^0\}_{\alpha\in\mathscr{A}_0}$ be a maximal set of 1-separated points in X. Inductively, for $k \in \mathbb{Z}_+$, let $\mathscr{X}^k := \{x_\alpha^k\}_{\alpha\in\mathscr{A}_k} \supseteq \mathscr{X}^{k-1}$ and $\mathscr{X}^{-k} := \{x_\alpha^{-k}\}_{\alpha\in\mathscr{A}_{-k}} \subseteq \mathscr{X}^{-(k-1)}$ be maximal δ^k- and δ^{-k}-separated collections in $\mathscr{X}^{k-1}$ and $\mathscr{X}^{-(k-1)}$, respectively. The families $\mathscr{X}^k$ have the separation properties required in Theorem 3.1 for the construction of cubes, with separation constants $c_0 = 1$, $C_0 = 2A_0$, base side length the given $\delta \in (0, 1)$, and with the additional property that $\mathscr{X}^k \subseteq \mathscr{X}^{k+1}$ for $k \in \mathbb{Z}$. We denote the corresponding cubes by Q_α^k, and the dyadic system $\mathscr{D}^W$. We will call $\mathscr{D}^W$ the *reference dyadic system* or *grid* underlying the wavelets.

A randomization $\mathscr{X}^k(\omega)$ of the families $\mathscr{X}^k$, as discussed in [HyK, HyM], has the separation properties for each random parameter ω (in a certain space Ω equipped with a probability measure $\mathbb{P}_\omega$) needed to construct the dyadic cubes $Q_\alpha^k(\omega)$ according to Theorem 3.1. However, in [AuH1, Theorem 2.11]) they modify the construction so that the randomized dyadic cubes $Q_\alpha^k(\omega)$ have uniform (in the random parameter $\omega \in \Omega$) dilation constants (in fact $c_1(\omega) = \frac{1}{6} A_0^{-5}$ and $C_1(\omega) = 6A_0^4 > 1$ for all $\omega \in \Omega$), and an additional "small boundary layer property" on average with respect to the probability measure introduced by the randomization [AuH1, Equation (2.3)]. It is in measuring the smallness of the boundary layer that a small parameter $\eta > 0$ appears, dependent only on the geometric constants of the space X. This parameter η is the Hölder regularity of the wavelets defined in Theorem 3.3. In this randomized construction, the reference dyadic point x^k_α may also be viewed as the center of the random cubes $Q^k_\alpha(\omega)$ for all ω belonging to the parameter space Ω. For the details of this beautiful construction see [AuH1, Section 2].

Now denote $\mathscr{Y}^k := \mathscr{X}^{k+1} \backslash \mathscr{X}^k$, and relabel the points x^k_α that belong to $\mathscr{Y}^k$ as y^k_α, where $\alpha \in \mathscr{A}_{k+1} \backslash \mathscr{A}_k$ and $k \in \mathbb{Z}$. To each such point y^k_α, Auscher and Hytönen associate a function ψ^k_α that is almost supported near y^k_α at scale δ^k (these functions are not compactly supported, but have exponential decay). Also note that to each Hytönen-Kairema cube Q^k_α there corresponds the point x^k_α and to each of the children of Q^k_α there correspond other points x^{k+1}_β, one of which coincides by construction with x^k_α. Thus the number of indices α in $\mathscr{A}_{k+1} \backslash \mathscr{A}_k$ corresponding to Q^k_α is exactly $N(Q^k_\alpha) - 1$, where $N(Q^k_\alpha)$ denotes the number of children of Q^k_α. This is the right number of wavelets we will need per cube if our intuition is guided by tensor product wavelets in $\mathbb{R}^n$, or Haar functions on spaces of homogeneous type based on Hytönen-Kairema cubes, as constructed for example in [KLPW]. Later on we will write $\alpha \in \mathscr{Y}^k$ meaning $\alpha \in \mathscr{A}_{k+1} \backslash \mathscr{A}_k$.

We now state the theorem describing precisely the wavelets of Auscher and Hytönen.

Theorem 3.3 ([AuH1], Theorem 7.1)**.** *Let (X, d, μ) be a space of homogeneous type with quasi-triangle constant A_0, with reference dyadic system of cubes $\mathscr{D}^W = \{Q^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{A}^k}$ that has base side length $\delta \in (0,1)$ and small boundary layer parameter $\eta \in (0,1]$. Let*

$$a := (1 + 2\log_2 A_0)^{-1}. \tag{3.10}$$

There exist an orthonormal basis $\{\psi^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{A}_{k+1}\backslash\mathscr{A}_k}$ of $L^2(X)$ and finite constants $C > 0$ and $\nu > 0$ such that for all $k \in \mathbb{Z}$ and $\alpha \in \mathscr{A}_{k+1} \setminus \mathscr{A}_k$ each function ψ^k_α satisfies the following conditions:

(i) *ψ^k_α is centered at $y^k_\alpha \in \mathscr{Y}^k$;*
(ii) *ψ^k_α has exponential decay determined by parameters a and ν, namely for all $x \in X$,*

$$|\psi^k_\alpha(x)| \le \frac{C}{\sqrt{\mu\big(B(y^k_\alpha, \delta^k)\big)}} \exp\Big(-\nu\Big(\frac{d(y^k_\alpha, x)}{\delta^k}\Big)^a\Big); \tag{3.11}$$

(iii) *ψ^k_α has* (local) *Hölder regularity with Hölder exponent η, namely for all $x, y \in X$ such that $d(x,y) \le \delta^k$,*

$$|\psi^k_\alpha(x) - \psi^k_\alpha(y)| \le \frac{C}{\sqrt{\mu\big(B(y^k_\alpha, \delta^k)\big)}} \Big(\frac{d(x,y)}{\delta^k}\Big)^\eta \exp\Big(-\nu\Big(\frac{d(y^k_\alpha, x)}{\delta^k}\Big)^a\Big); \tag{3.12}$$

(iv) ψ_α^k *has vanishing mean, namely*

$$\int_X \psi_\alpha^k(x)\, d\mu(x) = 0. \tag{3.13}$$

In Theorem 3.3, the constants C, ν, η, and δ are independent of k, α, and y_α^k. They depend only on the geometric constants of the space X: quasi-triangle inequality, doubling constant, and upper dimension. The constant $\delta \in (0,1)$, determining the side length of the reference dyadic cubes, is a fixed small parameter, more precisely, $\delta \leq 10^{-3}A_0^{-10}$.

In what follows, we refer to the functions ψ_α^k as *Auscher-Hytönen wavelets* or simply *wavelets*. The wavelet expansion, convergent in the sense of $L^2(X)$, is given by

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\alpha\in\mathscr{Y}^k} \langle f, \psi_\alpha^k\rangle \psi_\alpha^k(x). \tag{3.14}$$

Here $\langle f, g\rangle := \int_X f(x)\overline{g(x)}d\mu(x)$ denotes the L^2-pairing. The Auscher-Hytönen wavelets $\{\psi_\alpha^k\}_{k\in\mathbb{Z},\alpha\in\mathscr{Y}_k}$ form an unconditional basis of $L^q(X)$ for all q with $1 < q < \infty$; see [AuH1, Corollary 10.4]. Therefore, the reproducing formula (3.14) also holds for $f \in L^q(X)$. Note that for the reproducing formula (3.14) to hold, it suffices that the measure μ is Borel regular; see addendum [AuH2]. Also note that it is possible to build different wavelets based on the same reference dyadic points [AuH1].

In the Auscher-Hytönen construction of wavelets, the reference dyadic grids $\mathscr{D}^W$ form a regular family of dyadic systems according to Definition 3.2, because the outer dilation constants and the ratio of the outer and inner dilation constants are respectively, $C_1 = 6A_0^4 > 1$ and $C_1/c_1 = 36A_0^9$, for all the systems in the family.

For a general space of homogeneous type, the Hölder exponent η of the wavelets is bounded above by a constant η_0 $(0 < \eta < \eta_0)$ that only depends on the geometric parameters of the geometrically doubling space (X, d) [AuH1]. The constant η_0 is usually much smaller than one, even in the case of metric spaces. In [HyT], Hytönen and Tapiola presented a different construction of the metric wavelets that allows one to obtain Hölder-regularity for any exponent $\eta < 1$, strictly below but arbitrarily close to one.

The wavelets' regularity parameter η enters into the definition of the Hardy spaces $H^p(X)$ on spaces of homogeneous type (X, d, μ). In particular, η together with an upper dimension ω of the doubling measure μ determines the range of p for which the Hardy space is defined, namely $\omega/(\eta + \omega) < p \leq 1$. The larger η is, the smaller p can be chosen. A similar phenomenon occurs for the Hardy spaces on product spaces of homogeneous type, as pointed out in [HLW], see also Section 4. This is parallel to the theory on $\mathbb{R}^n$ where the theory of H^p-spaces with just the cancellation property is limited to $n/(n+1) < p \leq 1$, and to access smaller values of p, the test functions must have larger number of vanishing moments, unavailable in general spaces of homogeneous type.

The construction of wavelets hinges on the construction of certain "splines" on X defined using the probability measure $\mathbb{P}_\omega$ on the space Ω. For every $(k,\alpha) \in \mathbb{Z}\times\mathscr{Y}^k$ Auscher and Hytönen [AuH1, Equation (3.1)] define the spline function $s_\alpha^k : X \to [0,1]$ by

$$s_\alpha^k(x) := \mathbb{P}_\omega\big(x \in \overline{Q}_\alpha^k(\omega)\big).$$

The spline functions s_α^k are bumps supported on a ball centered at x_α^k and radius roughly δ^k, and they satisfy some interpolation, reproducing, and Hölder-continuity properties, described precisely in [AuH1, Theorem 3.1].

The splines in turn were used in [HLW] to construct smooth cut-off functions.

Lemma 3.4 ([HLW], Lemma 3.8)**.** *For each fixed $x_0 \in X$ and $R_0 \in (0,\infty)$, there exists a smooth cut-off function $h(x)$ such that $0 \le h(x) \le 1$,*

$$h(x) \equiv 1 \ \text{ when } x \in B(x_0, R_0/4), \qquad h(x) \equiv 0 \ \text{ when } x \in B(x_0, A_0^2R_0)^c,$$

and there exists a positive constant C, independent of x_0, R_0, x, and y (dependent only on geometric constants of the space X) such that

$$|h(x) - h(y)| \le C\Big(\frac{d(x,y)}{R_0}\Big)^\eta.$$

Note that the cut-off functions satisfy a global Hölder regularity condition with the same exponent η as the wavelets in Theorem 3.3. We will use these smooth cut-off functions on X in the proof of the key decomposition Lemma 4.8 for the wavelets.

3.3. Test function spaces and distributions. We now recall the definition of the test functions and distributions on (X,d,μ) that will enter into the definition of the Hardy spaces on product of spaces of homogeneous type. In particular, we observe that the normalized Auscher-Hytönen wavelets are test functions.

Let $V_r(x) := \mu\big(B(x,r)\big)$ for $x \in X$, $r > 0$ and $V(x,y) := \mu\big(B(x,d(x,y))\big)$ for $x, y \in X$.

Definition 3.5 (Test functions [HLW], Definition 3.1)**.** Fix $x_0 \in X$, $r > 0$, $\beta \in (0,\eta]$ where $\eta \le 1$ is the Hölder regularity exponent from Theorem 3.3, and $\gamma > 0$. A μ-measurable function f defined on X is said to be a *test function of type* (x_0, r, β, γ) *centered at* $x_0 \in X$ if f satisfies the following three conditions.

(i) (Size condition) For all $x \in X$ there is a constant $C > 0$ such that

$$|f(x)| \le C\,\frac{1}{V_r(x_0) + V(x,x_0)}\Big(\frac{r}{r + d(x,x_0)}\Big)^\gamma.$$

(ii) (Local Hölder regularity condition) For all x, $y \in X$ with $d(x,y) < (2A_0)^{-1}(r + d(x,x_0))$ there is a constant $C > 0$ such that

$$|f(x) - f(y)| \le C\Big(\frac{d(x,y)}{r + d(x,x_0)}\Big)^\beta \frac{1}{V_r(x_0) + V(x,x_0)}\Big(\frac{r}{r + d(x,x_0)}\Big)^\gamma.$$

(iii) (Cancellation condition)

$$\int_X f(x)\,d\mu(x) = 0.$$

These test functions generalize the test functions in Definition 2.1, which applies to the case when $\mu(B'(x,r)) \sim r$ and the quasi-metric d' has the Hölder regularity (2.3) with exponent θ. Notice that in this case $\big(V_r(x_0) + V(x,x_0)\big) \sim \big(r + d'(x,x_0)\big)$, and both definitions coincide. One can also compare to corresponding definitions in [HMY2, HMY1] in the case when the quasi-metric d satisfies the Hölder regularity (2.3) with exponent θ and the measure satisfies the doubling condition (1.2) and the reverse doubling condition (2.5). In these cases the only difference is that β is in $(0,\theta]$ instead of being in $(0,\eta]$; otherwise the definitions are identical.

Let $G(x_0,r,\beta,\gamma)$ denote the set of all test functions of type (x_0,r,β,γ). The norm on $G(x_0,r,\beta,\gamma)$ is defined by

$$\|f\|_{G(x_0,r,\beta,\gamma)} := \inf\{C > 0 : \text{ (i) and (ii) hold}\}.$$

Now fix $x_0 \in X$. Let $G(\beta,\gamma) := G(x_0,1,\beta,\gamma)$. It is easy to check that $G(x_1,r,\beta,\gamma) = G(\beta,\gamma)$ with equivalent norms for each fixed $x_1 \in X$ and $r > 0$. Furthermore, it is also easy to see that if $0 < \beta \le \eta$ then $G(\eta,\gamma) \subset G(\beta,\gamma)$ and $G(\eta,\gamma)$ is a Banach space with respect to the norm on $G(\eta,\gamma)$.

For $0 < \beta \le \eta$, let $\overset{\circ}{G}_\eta(\beta,\gamma)$ be the completion of the space $G(\eta,\gamma)$ in the norm of $G(\beta,\gamma)$. For $f \in \overset{\circ}{G}_\eta(\beta,\gamma)$, we define $\|f\|_{\overset{\circ}{G}_\eta(\beta,\gamma)} := \|f\|_{G(\beta,\gamma)}$. The spaces $\overset{\circ}{G}_\eta(\beta,\gamma)$ are nested; if $0 < \beta \le \beta'$ and $0 < \gamma \le \gamma'$ then $\overset{\circ}{G}_\eta(\beta',\gamma') \subset \overset{\circ}{G}_\eta(\beta,\gamma)$.

The distribution space $(\overset{\circ}{G}_\eta(\beta,\gamma))'$ is the set of all bounded linear functionals on $\overset{\circ}{G}_\eta(\beta,\gamma)$. We denote by $\langle f, h\rangle$ the natural pairing of elements $h \in \overset{\circ}{G}_\eta(\beta,\gamma)$ and $f \in (\overset{\circ}{G}_\eta(\beta,\gamma))'$.

The normalized Auscher-Hytönen wavelets are test functions in $G(\eta,\gamma)$ for any $\gamma > 0$. Later on we will take advantage of this fact, inherited from the exponential decay of the wavelets, and choose γ to be large enough.

The reproducing formula holds in the space of test functions and distributions with parameters $\beta',\gamma' \in (0,\eta)$. More precisely, the following propositions hold.

Proposition 3.6 ([HLW], Theorem 3.3). *Suppose $\{\psi^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{Y}_k}$ is an orthonormal basis as in Theorem* 3.3*, with Hölder regularity of order η. Then the normalized wavelet $\psi^k_\alpha(x)/\sqrt{\mu\big(B(y^k_\alpha,\delta^k)\big)}$ belongs to the set $G(y^k_\alpha,\delta^k,\eta,\gamma)$ of test functions of type $(y^k_\alpha,\delta^k,\eta,\gamma)$ centered at $y^k_\alpha \in X$ for each $k \in \mathbb{Z}$, $\alpha \in \mathcal{Y}^k$, and $\gamma > 0$.*

Proposition 3.7 ([HLW], Theorem 3.4)**.** *Suppose that* $f \in \overset{\circ}{G}_\eta(\beta,\gamma)$ *with* $\beta,\gamma \in (0,\eta)$*. Then the reproducing formula* (3.14) *holds in* $\overset{\circ}{G}_\eta(\beta',\gamma')$ *for each* $\beta' \in (0,\beta)$ *and* $\gamma' \in (0,\gamma)$*.*

As a consequence, the reproducing formula also holds for distributions.

Corollary 3.8 ([HLW], Corollary 3.5)**.** *The reproducing formula* (3.14) *holds in* $(\overset{\circ}{G}_\eta(\beta',\gamma'))'$*, when* $\beta',\gamma' \in (0,\eta)$*.*

3.4. Product setting. Consider the product setting $\widetilde{X} = X_1 \times X_2$, where each (X_i, d_i, μ_i), $i = 1, 2$, is a space of homogeneous type as defined in Section 1. For $i = 1, 2$, let $A_0^{(i)}$ be the constant in the quasi-triangle inequality (1.1), let C_{μ_i} be the doubling constant as in inequality (1.2), and let ω_i be an upper dimension of X_i as in inequality (1.3). By Theorem 3.3, on each space of homogeneous type (X_i, d_i, μ_i) for $i = 1,2$, there is a wavelet basis $\{\psi_{\alpha_i}^{k_i}\}_{k_i\in\mathbb{Z},\alpha_i\in\mathscr{Y}^{k_i}}$, with Hölder regularity exponent $\eta_i \in (0,1]$ as in inequality (3.12), and reference dyadic grid $\mathscr{D}_i^W$ with dilation constants c_1^i, C_1^i and their ratio C_1^i/c_1^i depending uniformly on $A_0^{(i)}$.

The spaces of product test functions and distributions on the product space $\widetilde{X}$ are defined as follows.

Definition 3.9 (Product test functions and distributions [HLW], Section 3)**.** Let $\widetilde{X} = X_1 \times X_2$ where (X_i, d_i, μ_i) is a space of homogenenous type for each $i = 1,2$. Suppose $\widetilde{x}_0 = (x_0, y_0) \in \widetilde{X}$ and $r_i > 0$, take β_i so that $0 < \beta_i \le \eta_i$, and take $\gamma_i > 0$, for $i = 1, 2$. Denote $\widetilde{r} = (r_1, r_2)$, $\widetilde{\beta} = (\beta_1, \beta_2)$, and $\widetilde{\gamma} = (\gamma_1, \gamma_2)$. A function $f(x, y)$ defined on $\widetilde{X}$ is said to be a *test function of type* $(\widetilde{x}_0, \widetilde{r}, \widetilde{\beta}, \widetilde{\gamma})$ if the following conditions hold. First, for each fixed $y \in X_2$, $f(x, y)$, as a function of the variable x, is a test function in $G(x_0, r_1, \beta_1, \gamma_1)$ on X_1. Second, for each fixed $x \in X_1$, $f(x, y)$, as a function of the variable y, is a test function in $G(y_0, r_2, \beta_2, \gamma_2)$ on X_2. Third, the following mixed conditions are satisfied, where $V_{2,r_2}(y_0) := \mu_2(B_{X_2}(y_0, r_2))$, and $V_2(y_0, y) := \mu_2\big(B_{X_2}(y_0, d_2(y, y_0))\big)$:

(i) (Size condition in y variable) For all $y \in X_2$,

$$\|f(\cdot, y)\|_{G(x_0,r_1,\beta_1,\gamma_1)} \le C \frac{1}{V_{2,r_2}(y_0) + V_2(y_0, y)} \Big(\frac{r_2}{r_2 + d_2(y, y_0)}\Big)^{\gamma_2}.$$

(ii) (Hölder regularity condition in y variable) For all $y, y' \in X_2$ with

$$d_2(y, y') \le \big(r_2 + d_2(y, y_0)\big)/2A_0^{(2)},$$

we have

$$\begin{aligned}\|f(\cdot, y) - f(\cdot, y')\|_{G(x_0,r_1,\beta_1,\gamma_1)} \;\le\;& C\Big(\frac{d_2(y, y')}{r_2 + d_2(y, y_0)}\Big)^{\beta_2} \\ &\times \frac{1}{V_{2,r_2}(y_0) + V_2(y_0, y)} \Big(\frac{r_2}{r_2 + d_2(y, y_0)}\Big)^{\gamma_2}.\end{aligned}$$

(iii) (Size and regularity conditions in x variable) Properties (i) and (ii) also hold interchanging the roles of x and y.

When f is a test function of type $(\widetilde{x}_0, \widetilde{r}, \widetilde{\beta}, \widetilde{\gamma})$, we write $f \in G(\widetilde{x}_0, \widetilde{r}, \widetilde{\beta}, \widetilde{\gamma})$. The expression $\|f\|_{G(\widetilde{x}_0,\widetilde{r},\widetilde{\beta},\widetilde{\gamma})} := \inf\{C :$ (i), (ii) and (iii) hold$\}$ defines a norm on $G(\widetilde{x}_0, \widetilde{r}, \widetilde{\beta}, \widetilde{\gamma})$.

We denote by $G(\widetilde{\beta}, \widetilde{\gamma})$ the class $G(\widetilde{x}_0, \widetilde{1}, \widetilde{\beta}, \widetilde{\gamma})$ for any fixed $\widetilde{x}_0 \in \widetilde{X}$ and where $\widetilde{1} = (1, 1)$. Then $G(\widetilde{x}_0, \widetilde{r}, \widetilde{\beta}, \widetilde{\gamma}) = G(\widetilde{\beta}, \widetilde{\gamma})$, with equivalent norms, for all $\widetilde{x}_0 \in \widetilde{X}$ and $r_1 > 0$, $r_2 > 0$. Furthermore, $G(\widetilde{\beta}, \widetilde{\gamma})$ is a Banach space with respect to the norm on $G(\widetilde{\beta}, \widetilde{\gamma})$.

For $\beta_i \in (0, \eta_i]$ and $\gamma_i > 0$, for $i = 1, 2$, let $\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})$ be the completion of the space $G(\widetilde{\eta}, \widetilde{\gamma})$ in $G(\widetilde{\beta}, \widetilde{\gamma})$ in the norm of $G(\widetilde{\beta}, \widetilde{\gamma})$. For $f \in \mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})$, we define $\|f\|_{\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta},\widetilde{\gamma})} := \|f\|_{G(\widetilde{\beta},\widetilde{\gamma})}$.

The distribution space $\big(\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})\big)'$ consists of all bounded linear functionals on $\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})$. We denote by $\langle f, h\rangle$ the natural pairing of elements $h \in \mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})$ and $f \in \big(\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})\big)'$.

Given Auscher-Hytönen wavelets $\{\psi_{\alpha_i}^{k_i}\}_{k_i\in\mathbb{Z},\alpha_i\in\mathscr{Y}^{k_i}}$ with Hölder regularity η_i on each space of homogeneous type (X_i, d_i, μ_i) for $i = 1, 2$, the corresponding normalized tensor product wavelets $\widetilde{\psi}_{\alpha_1}^{k_1}(x_1)\widetilde{\psi}_{\alpha_2}^{k_2}(x_2)$ belong to $\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}, \widetilde{\gamma})$ when $\beta_i \in (0, \eta_i]$ and $\gamma_i > 0$ for $i = 1, 2$. See [HLW, p.124]. The L^2-normalized wavelets are given by $\widetilde{\psi}_{\alpha_i}^{k_i}(x_i) := \psi_{\alpha_i}^{k_i}(x_i)/\sqrt{\mu_i\big(B_{X_i}(y_{\alpha_i}^{k_i}, \delta_i^{k_i})\big)}$ for $i = 1, 2$.

We are aware the tilde notation is being used to denote the product space and ordered pairs, now also to denote the L^2-normalized wavelets, and later on to denote enlargement of open domains in the product space and L^1-normalized functions. We expect the reader not to get too confused with the multiple purposes of this notation. We will write periodic reminders when a new tilde appears.

The following reproducing formula holds on the product space $\widetilde{X} = X_1 \times X_2$.

Theorem 3.10 ([HLW], Theorem 3.11). *For $i = 1, 2$, let $\{\psi_{\alpha_i}^{k_i}\}_{k_i\in\mathbb{Z},\alpha_i\in\mathscr{Y}^{k_i}}$ be Auscher-Hytönen wavelets with Hölder regularity $\eta_i > 0$ with reference dyadic grids $\mathscr{D}_i^W$ on the space of homogeneous type (X_i, d_i, μ_i), and fix constants β_i, $\gamma_i \in (0, \eta_i)$. Then the following hold:*

(a) *The reproducing formula*

$$f(x_1, x_2) = \sum_{k_1\in\mathbb{Z}} \sum_{\alpha_1\in\mathscr{Y}^{k_1}} \sum_{k_2\in\mathbb{Z}} \sum_{\alpha_2\in\mathscr{Y}^{k_2}} \langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle \psi_{\alpha_1}^{k_1}(x_1)\psi_{\alpha_2}^{k_2}(x_2) \tag{3.15}$$

holds in $\mathring{G}_{\widetilde{\eta}}(\widetilde{\beta}', \widetilde{\gamma}')$, for each $\beta_i' \in (0, \beta_i)$ and $\gamma_i' \in (0, \gamma_i)$, for $i = 1, 2$.

(b) *The reproducing formula* (3.15) *also holds in* $(\overset{\circ}{G}_{\widetilde{\eta}}(\widetilde{\beta},\widetilde{\gamma}))'$, *the space of distributions.*

Furthermore, when $f \in L^q(\widetilde{X})$ with $q > 1$, the series (3.15) converges unconditionally in the $L^q(\widetilde{X})$-norm. This is a consequence of the Auscher-Hytönen wavelets being an unconditional basis on $L^q(X_i)$ for $i = 1, 2$; see [AuH1, Corollary 10.4].

4. Product Hardy spaces, duals, predual, key auxiliary result and theorem

In this section we first recall the Hardy spaces $H^p(\widetilde{X})$, their duals the Carleson measure spaces $\mathrm{CMO}^p(\widetilde{X})$, and the spaces of bounded and vanishing mean oscillation, $\mathrm{BMO}(\widetilde{X})$ and $\mathrm{VMO}(\widetilde{X})$, respectively the dual and predual of $H^1(\widetilde{X})$. All these spaces, in the setting of product spaces of homogeneous type, were introduced in [HLW] in terms of a square function defined via the Auscher-Hytönen wavelet bases and their reference dyadic grids. We prove a key lemma that shows each of the Auscher-Hytönen wavelets can themselves be further decomposed into compactly supported building blocks with appropriate size, smoothness, and cancellation conditions inherited from the wavelets. Finally, we use the key lemma to prove a key auxiliary theorem stating that for $1 < q < \infty$ and $0 < p \le 1$ the set $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ is a subset of $L^p(\widetilde{X})$ with L^p-(semi)norm controlled by the H^p-(semi)norm. The key auxiliary results proved in this section will be needed in the proof of the Main Theorem in Section 5.

4.1. Biparameter Hardy spaces, CMO^p, BMO, and VMO. We focus on the bi-parameter setting $\widetilde{X} = X_1 \times X_2$, where each factor (X_i, d_i, μ_i) is a space of homogeneous type as defined in Section 1, with the constant ω_i being an upper dimension of X_i for $i = 1, 2$.

The family $\{\psi_{\alpha_i}^{k_i}\}_{k_i \in \mathbb{Z}, \alpha_i \in \mathscr{Y}^{k_i}}$ is an Auscher-Hytönen orthonormal wavelet basis on X_i with reference dyadic grid $\mathscr{D}_i^W$, exponential decay constants a_i and ν_i, and order of regularity $\eta_i \in (0, 1)$ for $i = 1, 2$, as in Theorem 3.3. All the dyadic rectangles in this section are of the form $R = Q_{\alpha_1}^{k_1} \times Q_{\alpha_2}^{k_2}$ where $Q_{\alpha_i}^{k_i} \in \mathscr{D}_i^W$ for $i = 1, 2$.

We denote by $\overset{\circ}{G}$ and $(\overset{\circ}{G})'$ for short the product test function spaces $\overset{\circ}{G}_{\widetilde{\eta}}(\widetilde{\beta'},\widetilde{\gamma'})$ and spaces of distributions $\left(\overset{\circ}{G}_{\widetilde{\eta}}(\widetilde{\beta'},\widetilde{\gamma'})\right)'$, respectively, where $\beta_i', \gamma_i' \in (0, \eta_i)$ for $i = 1, 2$. Note that we fix some β_i', γ_i' in $(0, \eta_i)$ and work with those test functions and the distributions in the dual space. At the end of the day it does not matter which β_i', γ_i' were chosen, as long as they belong to the interval $(0, \eta_i)$. The product wavelets $\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2} \in \overset{\circ}{G}$ and therefore if $f \in (\overset{\circ}{G})'$ the notation $\langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle$ means the action of the functional f on the product wavelet, which is an appropriate test function. We have used the prime $'$ on the parameters β_i' and γ_i' in the definition of $\overset{\circ}{G}$ and $(\overset{\circ}{G})'$ with a dash ($'$), so as not to confuse them with the parameters β_i and γ_i for which the wavelets $\psi^{k_i}_{\alpha_i}$ belong to $G(\beta_i, \gamma_i)$, namely all $\beta_i \in (0, \eta_i)$ and $\gamma_i > 0$ for $i = 1, 2$. In the proofs below, we will want to choose the wavelets' parameter γ_i as large as necessary. The space of distributions $(\overset{\circ}{G})'$ appears in the definitions of the product H^p, CMO^p, BMO, and VMO-spaces presented in this section as well as in the definition of atomic $H^{p,q}_{\mathrm{at}}$-spaces in Section 5.

In [HLW], the Hardy spaces $H^p(\widetilde{X})$ for $\widetilde{X} = X_1 \times X_2$ are defined as follows for $p_0 < p \le 1$, where we let $p_0 := \max\{\omega_i/(\omega_i + \eta_i) : i = 1, 2\}$.

Definition 4.1 ([HLW], Definition 5.1). Suppose $p_0 < p \le 1$. The *Hardy space* $H^p(\widetilde{X})$ is defined to be the collection of distributions in $(\overset{\circ}{G})'$ whose square function is in $L^p(\widetilde{X})$,

$$H^p(\widetilde{X}) := \{f \in (\overset{\circ}{G})' : S(f) \in L^p(\widetilde{X})\}.$$

Here the *product Littlewood-Paley square function* $S(f)$ *of* f *related to the given orthonormal basis* $\{\psi^k_\alpha\}_{k\in\mathbb{Z},\alpha\in\mathscr{Y}^k}$ *and reference dyadic grids* $\mathscr{D}^W_i$ *on* X_i *for* $i = 1, 2$, is defined by

$$S(f)(x_1, x_2) := \Big\{ \sum_{k_1\in\mathbb{Z}} \sum_{\alpha_1\in\mathscr{Y}^{k_1}} \sum_{k_2\in\mathbb{Z}} \sum_{\alpha_2\in\mathscr{Y}^{k_2}} \Big| \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle\, \widetilde{\chi}_{Q^{k_1}_{\alpha_1}}(x_1)\widetilde{\chi}_{Q^{k_2}_{\alpha_2}}(x_2) \Big|^2 \Big\}^{\frac{1}{2}} \tag{4.1}$$

with $Q^{k_i}_{\alpha_i} \in \mathscr{D}^W_i$ and $\widetilde{\chi}_{Q^{k_i}_{\alpha_i}}(x_i) := \chi_{Q^{k_i}_{\alpha_i}}(x_i)\, \mu_i(Q^{k_i}_{\alpha_i})^{-1/2}$ for $i = 1, 2$. For $f \in H^p(\widetilde{X})$, define the H^p-(semi)norm[3]

$$\|f\|_{H^p(\widetilde{X})} := \|S(f)\|_{L^p(\widetilde{X})}.$$

Definition (4.1) corresponds to [HLW, Definition 4.7, equation (4.10)], where the product square function is called $\widetilde{S}$ instead of S.

In [HLW] the Carleson measure spaces $\mathrm{CMO}^p(\widetilde{X})$ are defined as follows.

Definition 4.2 ([HLW], Definition 5.2). Suppose $p_0 < p \le 1$. The *Carleson measure space* $\mathrm{CMO}^p(\widetilde{X})$ is defined by

$$\mathrm{CMO}^p(\widetilde{X}) := \{f \in (\overset{\circ}{G})' : \mathcal{C}_p(f) < \infty\}.$$

Here the quantity $\mathcal{C}_p(f)$ is defined by

$$\mathcal{C}_p(f) := \sup_{\Omega} \Big\{ \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R=Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_1}\subset\Omega} |\langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle|^2 \Big\}^{1/2}, \tag{4.2}$$

where Ω runs over all open sets in $\widetilde{X}$ with finite measure, and it is understood, here and in the sequel, that the indices $k_i \in \mathbb{Z}$ and $\alpha_i \in \mathcal{Y}^{k_i}$ for $i = 1, 2$. The

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[3]For $p < 1$, the semi-norm $\|\cdot\|_{H^p(X_1\times X_2)}$ satisfies all the axioms of a norm except the triangle inequality, instead it satisfies $\|f + g\|^p_{H^p(\widetilde{X})} \le \|f\|^p_{H^p(\widetilde{X})} + \|g\|^p_{H^p(\widetilde{X})}$.

space BMO *of functions of bounded mean oscillation* is defined by

$$\mathrm{BMO}(\widetilde{X}) := \mathrm{CMO}^1(\widetilde{X}).$$

One of the main results in [HLW] establishes the duality between the Hardy spaces and the Carleson measure spaces.

Theorem 4.3 ([HLW], Theorem 5.3)**.** *Suppose $p_0 < p \le 1$. Then $(H^p(\widetilde{X}))' = \mathrm{CMO}^p(\widetilde{X})$. In particular, when $p = 1$ we have $\big(H^1(\widetilde{X})\big)' = \mathrm{BMO}(\widetilde{X})$.*

The vanishing mean oscillation space $\mathrm{VMO}(\widetilde{X})$ was introduced in [HLW], and it was shown in the same paper to be the predual of $H^1(\widetilde{X})$. For the convenience of the reader we record the definition and the duality theorem.

Definition 4.4 ([HLW], Definition 5.9)**.** The *space* $\mathrm{VMO}(\widetilde{X})$ *of functions of vanishing mean oscillation* is the subspace of $\mathrm{BMO}(\widetilde{X})$ whose elements satisfy the following three properties:

(a) $$\lim_{\delta\to 0^+} \sup_{\mu(\Omega)<\delta} \Big\{\frac{1}{\mu(\Omega)} \sum_{R=Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_1}\subset\Omega} |\langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle|^2\Big\}^{1/2} = 0;$$

(b) $$\lim_{N\to\infty} \sup_{\mathrm{diam}(\Omega)>N} \Big\{\frac{1}{\mu(\Omega)} \sum_{R=Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_1}\subset\Omega} |\langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle|^2\Big\}^{1/2} = 0; \text{ and}$$

(c) $$\lim_{N\to\infty} \sup_{\Omega:\ \Omega\subset\left(B(x_1,x_2,N)\right)^c} \Big\{\frac{1}{\mu(\Omega)} \sum_{R=Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_1}\subset\Omega} |\langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle|^2\Big\}^{1/2} = 0.$$

Here the suprema run over all open sets Ω in $\widetilde{X}$ with finite measure, and either with small measure in (a), with large diameter in (b), or living far away from an arbitrary fixed point $(x_1, x_2) \in \widetilde{X}$ in (c), where $B(x_1, x_2, N) := B(x_1, N) \times B(x_2, N)$.

Theorem 4.5 ([HLW], Theorem 5.10)**.** *The Hardy space $H^1(\widetilde{X})$ is the dual of the space of vanishing mean oscillation* $\mathrm{VMO}(\widetilde{X})$*. Namely, $\big(\mathrm{VMO}(\widetilde{X})\big)' = H^1(\widetilde{X})$.*

Note that the definitions for the H^p, CMO^p, BMO, and VMO spaces all use given Auscher-Hytönen wavelets and their underlying reference grids in X_i for $i = 1, 2$. Whether these definitions are independent of the chosen wavelets and reference grids is an important question, answered in the affirmative in the current paper.

4.2. Key decomposition lemma and $H^p \cap L^q \subset L^p$ theorem. We point out that $\mathring{G}$, and thus $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ for $q > 1$, are dense in $H^p(\widetilde{X})$ with respect to the $H^p(\widetilde{X})$-(semi)norm; see [HLW, p.40–41]. We now show that functions in the dense subset $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ also lie in $L^p(\widetilde{X})$, in other words for $q > 1$,

$$H^p(\widetilde{X}) \cap L^q(\widetilde{X}) \subset L^p(\widetilde{X}),$$

with L^p-(semi)norm controlled by the H^p-(semi)norm. As an aside recall that the L^p-(semi)norm is not a norm when $0 < p < 1$, satisfying instead of the triangle inequality the following inequality: $\|f+g\|^p_{L^p(\widetilde{X})} \le \|f\|^p_{L^p(\widetilde{X})} + \|g\|^p_{L^p(\widetilde{X})}$.

Our key auxiliary theorem in this section is the following.

Theorem 4.6. *Given spaces of homogeneous type (X_i, d_i, μ_i) with an upper dimension ω_i, with reference dyadic grids $\mathscr{D}_i^W$, and associated Auscher-Hytönen wavelet bases $\{\psi^{k_i}_{\alpha_i}\}_{k_i\in\mathbb{Z},\alpha_i\in\mathcal{Y}^{k_i}}$ with Hölder regularity $\eta_i \in (0,1)$, for $i = 1, 2$. Suppose $p_0 := \max\{\frac{\omega_1}{\omega_1+\eta_1}, \frac{\omega_2}{\omega_2+\eta_2}\} < p \le 1$, and take $q > 1$. If a function $f \in H^p(\widetilde{X}) \cap L^q(\widetilde{X})$, then $f \in L^p(\widetilde{X})$ and there exists a constant $C_p > 0$, independent of the L^q-norm of f, such that*

$$\|f\|_{L^p(\widetilde{X})} \le C_p \|f\|_{H^p(\widetilde{X})}.$$

As a consequence of Theorem 4.6, we have the following result.

Corollary 4.7. *Let $q > 1$. Then $H^1(\widetilde{X}) \cap L^q(\widetilde{X})$ is a subset of $L^1(\widetilde{X})$.*

To prove Theorem 4.6, we first establish an auxiliary result, Lemma 4.8, on the decomposition of the orthonormal basis functions ψ^k_α into building blocks with compact support and other convenient properties. These building blocks will inherit from the wavelets, appropriately scaled, size and smoothness conditions as well as cancellation.

We follow the approach of Nagel and Stein (see [NS, Section 3.5]).

Lemma 4.8. *Let (X, d, μ) be a space of homogeneous type with A_0 the quasi-triangle constant of the quasi-metric d, and ω an upper dimension of the Borel regular doubling measure μ. Fix parameters $\gamma > \omega$ and $\overline{C} > 1$. Suppose that ψ^k_α is a basis function (a wavelet) as in Theorem* 3.3*, with exponential decay exponents $\nu > 0$ and $a = (1 + 2\log_2 A_0)^{-1}$ and with Hölder-regularity exponent η. Then there exist functions $\varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}$ for each integer $\ell \ge 0$ such that for all $x \in X$ and for each $k \in \mathbb{Z}$, $\alpha \in \mathscr{Y}^k$, we have the following decomposition for the L^2-normalized wavelets $\widetilde{\psi}^k_\alpha := \psi^k_\alpha(x)/\sqrt{\mu(B(y^k_\alpha, \delta^k))}$:*

$$\widetilde{\psi}^k_\alpha(x) = \sum_{\ell=0}^{\infty} (2^\ell \overline{C})^{-\gamma} \varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}(x). \tag{4.3}$$

Here each $\varphi^\gamma_{\ell,k,\alpha}$ satisfies the following properties.

(i) (Compact support) $\operatorname{supp} \varphi^{\gamma,\overline{C}}_{\ell,k,\alpha} \subset B(y^k_\alpha, 2A_0^2\,\overline{C}2^\ell\,\delta^k)$.

(ii) (Boundedness) *There is a constant $C_\gamma > 0$ such that for all $x \in X$*

$$|\varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}(x)| \le C_\gamma (\overline{C}2^\ell)^\omega / \mu(B(y^k_\alpha, \overline{C}2^\ell \delta^k)).$$

(iii) (Local Hölder regularity) *There is a constant $C_\gamma > 0$ such that for all $x, y \in X$ with $d(x,y) \le \delta^k$,*

$$|\varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}(x) - \varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}(y)| \le C_\gamma\,(\overline{C}2^\ell\delta^k)^{-\eta}\,(\overline{C}2^\ell)^\omega d(x,y)^\eta / \mu(B(y^k_\alpha, \overline{C}2^\ell\delta^k)).$$

(iv) (Cancellation) $\int_X \varphi^{\gamma}_{\ell,k,\alpha}(x)\,d\mu(x) = 0$.

Here C_γ is a positive constant independent of y^k_α, δ^k, and ℓ. However C_γ will depend on the fixed $\gamma > 0$ and the geometric constants of the space X. The equality (4.3) *holds pointwise, as well as in $L^q(X)$ for $q \in (1,\infty)$.*

Lemma 4.8 allows for two parameters, a decay parameter $\gamma > \omega$ and a dilation parameter $\overline{C} > 1$. Later on we will pick γ large enough so that some geometric series converge and we will need $\overline{C}$ to match dilation parameters for the (p,q)-atoms which are independent of the wavelets, and based on possibly separate dyadic grids. When $\overline{C} = 1$ we simply write $\varphi^{\gamma}_{\ell,k,\alpha}$.

In the local Hölder regularity condition (iii) in Lemma 4.8, the range of validity, $d(x,y) \le \delta^k$, is inherited from the wavelets local regularity condition as in Theorem 3.3(iii). In the proof of Lemma 4.8 we will see that a type of Hölder regularity like the one test functions have, see Definition 3.5(ii), with range of validity $d(x,y) < \big(2A_0\big)^{-1}(\delta^k + d(x,y^k_\alpha))$ provided $x \in B(y^k_\alpha, A_0^2\,\overline{C}2^\ell\delta^k) \setminus B(y^k_\alpha, \overline{C}2^{\ell-1}\delta^k/4)$, will also hold because the wavelets are test functions by Theorem 3.6. We will need this estimate in the proof of the Main Theorem in Section 5.

What is gained in this decomposition is the compact support of the building blocks, as opposed to the exponential decay of the wavelets being decomposed. What is lost is the orthonormality of the wavelets, however the building blocks will have an appropriate "almost-orthogonality" property that will be needed in the proof of Theorem 4.6. This almost-orthogonality of the building blocks is captured in Lemma 4.9 stated on page 1208 and proved after the proof of Theorem 4.6 on page 1210.

Proof of Lemma 4.8. Fix $\gamma > \omega$, $k \in \mathbb{Z}$, and $\alpha \in \mathscr{Y}_k$. Let

$$\Lambda^{\overline{C}}_0(x) := h_0(x)\,\widetilde{\psi}^k_\alpha(x) \quad \text{and} \tag{4.4}$$

$$\Lambda^{\overline{C}}_\ell(x) := \big(h_\ell(x) - h_{\ell-1}(x)\big)\,\widetilde{\psi}^k_\alpha(x) \;\text{ for }\; \ell \ge 1. \tag{4.5}$$

The cut-off functions $h_\ell \in C^\eta(X)$ are given by Lemma 3.4 based on $x_0 = y^k_\alpha$ and with parameter $R_0 = \overline{C}2^\ell\delta^k$ for each $\ell \ge 0$. They have the following properties for $\ell > 0$: first $0 \le h_\ell(x) \le 1$; second

$$h_\ell(x) \equiv 1 \text{ when } x \in B(y^k_\alpha, \overline{C}2^\ell\delta^k/4),\; h_\ell(x) \equiv 0 \text{ when } x \in B(y^k_\alpha, A_0^2\,\overline{C}2^\ell\delta^k)^c; \tag{4.6}$$

and third, there exists a constant $C > 0$ independent of y^k_α and ℓ, depending only on the geometric constants of the space X, such that for all $x, y \in X$ the following global Hölder regularity holds:

$$|h_\ell(x) - h_\ell(y)| \le C\Big(\frac{d(x,y)}{\overline{C}2^\ell\delta^k}\Big)^\eta. \tag{4.7}$$

By definition, the function Λ_0 is supported on $B(y_\alpha^k, A_0^2\,\overline{C}\delta^k)$ and the function $\Lambda_\ell^{\overline{C}}$ for $\ell \geq 1$ is supported on $B(y_\alpha^k, A_0^2\,\overline{C}2^\ell\delta^k) \setminus B(y_\alpha^k, \overline{C}2^{\ell-1}\delta^k/4)$. By a telescoping sum argument we see that

$$\sum_{\ell=0}^{L} \Lambda_\ell^{\overline{C}}(x) = h_L(x)\,\widetilde{\psi}_\alpha^k(x) \text{ and is identical to } \widetilde{\psi}_\alpha^k(x) \text{ on } B(y_\alpha^k, \overline{C}2^L\delta^k/4).$$

It follows that $\widetilde{\psi}_\alpha^k(x) = \sum_{\ell\geq 0}\Lambda_\ell^{\overline{C}}(x)$ pointwise. Moreover, for all $x \in X$ and every $\gamma > 0$,

$$|\Lambda_\ell^{\overline{C}}(x)| \lesssim_\gamma \frac{(\overline{C}2^\ell)^{-\gamma}}{\mu\big(B(y_\alpha^k, \delta^k)\big)} \lesssim_\gamma \frac{(\overline{C}2^\ell)^{\omega-\gamma}}{\mu\big(B(y_\alpha^k, \overline{C}2^\ell\delta^k)\big)}. \tag{4.8}$$

The second inequality follows from the doubling property of the measure. The first inequality can be seen since $\psi_\alpha^k(x)$ has the exponential decay property (3.11), $|h_\ell(x) - h_{\ell-1}(x)| \in [0,1]$, and $\Lambda_\ell^{\overline{C}}$ is supported on the annulus $B(y_\alpha^k, A_0^2\,\overline{C}2^\ell\delta^k)\setminus B(y_\alpha^k, \overline{C}2^{\ell-1}\delta^k/4)$. Note that for $\nu, a > 0$ the function $e^{-\nu z^a}z^\gamma$ defined for $z \geq 0$ is a bounded function for each $\gamma > 0$, with an upper bound depending on $\gamma > 0$.

Following the argument in [NS, p.550–551], define $a_\ell := \int_X \Lambda_\ell^{\overline{C}}(x)\,d\mu(x)$. Using (4.8) it is clear that $a_\ell = O\big((\overline{C}2^\ell)^{\omega-\gamma}\big)$. Define $s_\ell := \sum_{0\leq j\leq\ell} a_j$. Note that by the Lebesgue domination theorem,

$$\sum_{\ell\geq 0} a_\ell = \int_X \widetilde{\psi}_\alpha^k(x)\,d\mu(x) = 0,$$

therefore we have $s_\ell = -\sum_{j>\ell} a_j$, which gives $s_\ell = O\big((\overline{C}2^\ell)^{\omega-\gamma}\big)$.

We now define the function $\widetilde{\Lambda}_\ell^{\overline{C}} : X \to \mathbb{R}$ by

$$\begin{aligned}\widetilde{\Lambda}_\ell^{\overline{C}}(x) &:= \Lambda_\ell^{\overline{C}}(x) - a_\ell\,\xi_\ell(x) + s_\ell\big(\xi_\ell(x) - \xi_{\ell+1}(x)\big)\\ &= \Lambda_\ell^{\overline{C}}(x) + s_{\ell-1}\,\xi_\ell(x) - s_\ell\,\xi_{\ell+1}(x).\end{aligned}$$

Here for each $\ell \geq 0$ the function ξ_ℓ is the L^1-normalization of the function h_ℓ supported on $B(y_\alpha^k, A_0^2\,\overline{C}2^\ell\delta^k)$ given by

$$\xi_\ell(x) := h_\ell(x)\Big[\int_X h_\ell(z)\,d\mu(z)\Big]^{-1}. \tag{4.9}$$

Finally we define the functions $\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}$ in the decomposition of the wavelets

$$\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}(x) := (\overline{C}2^\ell)^\gamma\,\widetilde{\Lambda}_\ell^{\overline{C}}(x). \tag{4.10}$$

Note that $\widetilde{\Lambda}_\ell^{\overline{C}}$ does not depend on γ, although it depends on the fixed k and α. It is easy to verify that the decomposition (4.3) holds. Namely

$$\begin{aligned}\sum_{\ell\ge 0}(\overline{C}2^\ell)^{-\gamma}\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}(x) &= \sum_{\ell\ge 0}\widetilde{\Lambda}_\ell^{\overline{C}}(x)\\ &= \sum_{\ell\ge 0}\Lambda_\ell^{\overline{C}}(x) - \sum_{\ell\ge 0} a_\ell\xi_\ell(x) + \sum_{\ell\ge 0} s_\ell\big(\xi_\ell(x)-\xi_{\ell+1}(x)\big)\\ &= \widetilde{\psi}_\alpha^k(x),\end{aligned}$$

where the last equality follows from the facts that $\widetilde{\psi}_\alpha^k(x) = \sum_{\ell\ge 0}\Lambda_\ell^{\overline{C}}(x)$ and $\sum_{\ell\ge 0} a_\ell\xi_\ell = \sum_{\ell\ge 0} s_\ell(\xi_\ell(x)-\xi_{\ell+1}(x))$, using summation by parts and noting that $a_\ell = s_\ell - s_{\ell-1}$.

Now we verify that $\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}$ satisfies properties (i), (ii), (iii), and (iv).

In fact, from the definition of $\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}$ it is easy to see that properties (i) and (iv) hold. We now turn to property (ii). From the size estimate (4.8) we have that

$$|\Lambda_\ell^{\overline{C}}(x)| \lesssim_\gamma \frac{(\overline{C}2^\ell)^{\omega-\gamma}}{\mu\big(B(y_\alpha^k,\overline{C}2^\ell\delta^k)\big)} \tag{4.11}$$

for each $\gamma > 0$, where ω is an upper dimension of the measure μ. Next, it follows from the definition of the function ξ_ℓ that

$$|\xi_\ell(x)| \lesssim \frac{1}{\mu\big(B(y_\alpha^k,\overline{C}2^\ell\delta^k)\big)}$$

because $0 \le h_\ell(x) \le 1$ and

$$\mu\big(B(y_\alpha^k,\overline{C}2^{\ell-1}\delta^k/4)\big) \le \int_X h_\ell(z)\,d\mu(z) \le \mu\big(B(y_\alpha^k,A_0^2\,\overline{C}2^\ell\delta^k)\big).$$

Furthermore, using the doubling property of μ, we conclude that

$$\int_X h_\ell(z)\,d\mu(z) \sim \mu\big(B(y_\alpha^k,\overline{C}2^\ell\delta^k)\big). \tag{4.12}$$

Consequently, recalling that $a_\ell = O\big((\overline{C}2^\ell)^{\omega-\gamma}\big)$ and $s_\ell = O\big((\overline{C}2^\ell)^{\omega-\gamma}\big)$, we conclude that property (ii) holds.

Similarly, from the Hölder regularity (3.12) of ψ_α^k and estimate (4.7) of the cut-off functions h_ℓ, together with the definition of the function ξ_ℓ, we obtain that property (iii) holds. More precisely, we need to verify that there is a constant $C_\gamma > 0$ depending only on the geometric constants of X and on γ, such that for all $x, y \in X$ with $d(x,y) \le \delta^k$, and for all ℓ, α, and k the following inequality holds:

$$|\varphi_{\ell,k,\alpha}^{\gamma\overline{C}}(x) - \varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}(y)| \le \frac{C_\gamma\,(\overline{C}2^\ell\delta^k)^{-\eta}\,(\overline{C}2^\ell)^\omega}{\mu\big(B(y_\alpha^k,\overline{C}2^\ell\delta^k)\big)}\,d(x,y)^\eta.$$

Without loss of generality we can assume that $d(x,y)>0$, in other words $x\neq y$. Using definition (4.10) of the atoms $\varphi^{\gamma}_{\ell,k,\alpha}$ and the triangle inequality we get that

$$|\varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}(x)-\varphi^{\gamma,\overline{C}}_{\ell,k,\alpha}(y)| \leq (\overline{C}2^{\ell})^{\gamma}\big(|\Lambda^{\overline{C}}_{\ell}(x)-\Lambda^{\overline{C}}_{\ell}(y)|+|s_{\ell-1}|\,|\xi_{\ell}(x)-\xi_{\ell}(y)| + |s_{\ell}|\,|\xi_{\ell+1}(x)-\xi_{\ell+1}(y)|\big).$$

Since $s_\ell = O\big((\overline{C}2^{\ell})^{-\gamma}\big)$, it suffices to show that there is a constant $C_\gamma>0$ such that for all $x,y\in X$ with $d(x,y)\leq\delta^k$ the following two inequalities hold:

$$(\overline{C}2^{\ell})^{\gamma}|\Lambda^{\overline{C}}_{\ell}(x)-\Lambda^{\overline{C}}_{\ell}(y)| \leq \frac{C_\gamma\,(\overline{C}2^{\ell}\delta^k)^{-\eta}\,(\overline{C}2^{\ell})^{\omega}}{\mu\big(B(y^k_\alpha,\overline{C}2^{\ell}\delta^k)\big)}\,d(x,y)^{\eta}, \tag{4.13}$$

$$|\xi_{\ell}(x)-\xi_{\ell}(y)| \leq \frac{C_\gamma\,(\overline{C}2^{\ell}\delta^k)^{-\eta}}{\mu\big(B(y^k_\alpha,\overline{C}2^{\ell}\delta^k)\big)}\,d(x,y)^{\eta}. \tag{4.14}$$

We first estimate (4.14). Using definition (4.9) of ξ_ℓ, estimate (4.12), and the fact that h_ℓ satisfies estimate (4.7) for all $x,y\in X$, we obtain

$$|\xi_{\ell}(x)-\xi_{\ell}(y)| \lesssim \frac{|h_\ell(x)-h_\ell(y)|}{\mu\big(B(y^k_\alpha,\overline{C}2^{\ell}\delta^k)\big)} \lesssim \frac{(\overline{C}2^{\ell}\delta^k)^{-\eta}}{\mu\big(B(y^k_\alpha,\overline{C}2^{\ell}\delta^k)\big)}\,d(x,y)^{\eta}.$$

This is more than what we wanted to show, since x and y are not required to be δ^k-close to each other, and the similarity constants are independent of γ.

We now estimate (4.13). We argue in the case when $\ell>0$ and note that when $\ell=0$ a similar calculation, somewhat simpler, yields the desired estimate. By definition (4.5) of $\Lambda^{\overline{C}}_{\ell}$ when $\ell>0$, we conclude that

$$|\Lambda^{\overline{C}}_{\ell}(x)-\Lambda^{\overline{C}}_{\ell}(y)| \leq |h_\ell(x)\widetilde{\psi}^k_\alpha(x)-h_\ell(y)\widetilde{\psi}^k_\alpha(y)| + |h_{\ell-1}(x)\widetilde{\psi}^k_\alpha(x)-h_{\ell-1}(y)\widetilde{\psi}^k_\alpha(y)|.$$

For all $\ell>0$ we estimate using the triangle inequality

$$|h_\ell(x)\,\widetilde{\psi}^k_\alpha(x)-h_\ell(y)\,\widetilde{\psi}^k_\alpha(y)| \leq \|h_\ell\|_{L^\infty(X)}|\widetilde{\psi}^k_\alpha(x)-\widetilde{\psi}^k_\alpha(y)| + \|\widetilde{\psi}^k_\alpha\|_{L^\infty(X)}|h_\ell(x)-h_\ell(y)|.$$

Using the exponential decay and Hölder regularity estimates (3.11) and (3.12) for the wavelet ψ^k_α, together with the fact that $\|h_\ell\|_{L^\infty(X)}\leq 1$ and the Hölder regularity estimate (4.7) of h_ℓ, we conclude that, when $d(x,y)\leq\delta^k$,

$$\begin{aligned}|h_\ell(x)\,\widetilde{\psi}^k_\alpha(x)-h_\ell(y)\,\widetilde{\psi}^k_\alpha(y)| &\lesssim \frac{\exp\Big[-\nu\Big(\frac{d(y^k_\alpha,x)}{\delta^k}\Big)^a\Big]}{\mu\big(B(y^k_\alpha,\delta^k)\big)}\Big[\frac{d(x,y)^\eta}{\delta^{k\eta}}+\frac{d(x,y)^\eta}{(\overline{C}2^{\ell}\delta^k)^\eta}\Big]\\ &\lesssim_\Gamma \frac{\delta^{-k\eta}(\overline{C}2^{\ell})^{\omega}\big(1+(\overline{C}2^{\ell})^{-\eta}\big)}{\mu\big(B(y^k_\alpha,\overline{C}2^{\ell}\delta^k)\big)}d(x,y)^\eta\Big[\frac{\delta^k}{\delta^k+d(y^k_\alpha,x)}\Big]^\Gamma,\end{aligned}$$

for all $\Gamma > 0$. We have used the doubling property (1.3) in the last inequality. When x is in the support of $\Lambda_\ell^{\overline{C}}$, namely in the annulus $B(y_\alpha^k, A_0^2\,\overline{C}2^\ell\delta^k) \setminus B(y_\alpha^k, \overline{C}2^{\ell-1}\delta^k/4)$, then $d(x, y_\alpha^k)\,\delta^{-k} \sim \overline{C}2^\ell$. We conclude that for all $\Gamma > 0$

$$\begin{aligned}(\overline{C}2^\ell)^\gamma|\Lambda_\ell^{\overline{C}}(x) - \Lambda_\ell^{\overline{C}}(y)| \quad &\lesssim_\Gamma \quad \frac{(\overline{C}2^\ell\delta^k)^{-\eta}\,(\overline{C}2^\ell)^\omega}{\mu\big(B(y_\alpha^k, \overline{C}2^\ell\delta^k)\big)} d(x,y)^\eta\,(\overline{C}2^\ell)^\gamma\big((\overline{C}2^\ell)^\eta + 1\big) \\ &\qquad\qquad \times \Big[\frac{1}{1 + d(y_\alpha^k, x)\,\delta^{-k}}\Big]^\Gamma \\ &\lesssim_\Gamma \quad \frac{(\overline{C}2^\ell\delta^k)^{-\eta}\,(\overline{C}2^\ell)^\omega}{\mu\big(B(y_\alpha^k, \overline{C}2^\ell\delta^k)\big)} d(x,y)^\eta\,(\overline{C}2^\ell)^{\gamma+\eta-\Gamma}.\end{aligned}$$

Picking $\Gamma = \gamma + \eta$ we get estimate (4.13) at least when x is in the support of $\Lambda_\ell^{\overline{C}}$ and $d(x,y) \le \delta^k$. Clearly when both x and y are not in the support of $\Lambda_\ell^{\overline{C}}$ then $\Lambda_\ell^{\overline{C}}(x) - \Lambda_\ell^{\overline{C}}(y) = 0$. The only remaining case is when y is in the support of $\Lambda_\ell^{\overline{C}}$ and x is not. The calculations above are symmetric in x and y; interchanging their roles we conclude that when $d(x,y) \le \delta^k$ then

$$(\overline{C}2^\ell)^\gamma|\Lambda_\ell^{\overline{C}}(x) - \Lambda_\ell^{\overline{C}}(y)| \lesssim_\gamma \frac{(\overline{C}2^\ell\delta^k)^{-\eta}\,(\overline{C}2^\ell)^\omega}{\mu\big(B(y_\alpha^k, \overline{C}2^\ell\delta^k)\big)} d(x,y)^\eta.$$

This proves estimate (4.13) and shows that condition (iii) in the lemma holds.

By Proposition 3.6, $\widetilde{\psi}_\alpha^k$ is a test function of type $(y_\alpha^k, \delta^k, \eta, \gamma + \eta)$. Using the test-function properties instead of the local Hölder regularity of the wavelets as we just did, one can verify in a similar manner that when $x \in \operatorname{supp}(\Lambda_\ell^{\overline{C}})$ and $d(x,y) \le (2A_0)^{-1}(\delta^k + d(x, y_\alpha^k))$ then

$$(\overline{C}2^\ell)^\gamma|\Lambda_\ell^{\overline{C}}(x) - \Lambda_\ell^{\overline{C}}(y)| \lesssim_\gamma \frac{(\overline{C}2^\ell\delta^k)^{-\eta}}{\mu\big(B(y_\alpha^k, \delta^k)\big) + \mu\big(B(x, d(x, y_\alpha^k))\big)} d(x,y)^\eta. \tag{4.15}$$

Finally we can verify that the convergence in equality (4.3) is not just pointwise, but also in $L^q(X)$ for $q \in (1, \infty)$. Indeed, let

$$\psi_\alpha^{k,N}(x) = \sqrt{\mu(B(y_\alpha^k, \delta^k))}\sum_{\ell=0}^N (\overline{C}2^\ell)^{-\gamma}\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}(x).$$

Then, using the already proven boundedness and support properties (i) and (ii) of $\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}$ in Lemma 4.8, we readily see that

$$\begin{aligned}||\psi_\alpha^k - \psi_\alpha^{k,N}||_{L^q(X)} &\le \sqrt{\mu\big(B(y_\alpha^k, \delta^k)\big)}\sum_{\ell=N+1}^\infty (\overline{C}2^\ell)^{-\gamma}||\varphi_{\ell,k,\alpha}^{\gamma,\overline{C}}||_{L^q(X)} \\ &\lesssim (\overline{C})^{\omega-\gamma}\mu\big(B(y_\alpha^k, \delta^k)\big)^{\frac{1}{2}}\sum_{\ell=N+1}^\infty 2^{(-\gamma+\omega)\ell}\mu\big(B(y_\alpha^k, \overline{C}2^\ell\delta^k)\big)^{-\frac{1}{q'}}\end{aligned}$$

$$\lesssim (\overline{C})^{\omega-\gamma}\mu\big(B(y_\alpha^k,\delta^k)\big)^{\frac{1}{2}-\frac{1}{q'}}\sum_{\ell=N+1}^{\infty}2^{(-\gamma+\omega)\ell}.$$

Since $\gamma>\omega$, as $N\to\infty$, the series on the right-hand-side converges to zero. In the last inequality we simply observed that

$$\mu\big(B(y_\alpha^k,\overline{C}2^\ell\delta^k)\big)^{-1/q'}\le\mu\big(B(y_\alpha^k,\delta^k)\big)^{-1/q'}$$

since the power is negative. □

We now present the proof of the key auxiliary theorem.

Proof of Theorem 4.6. Suppose that $f\in H^p(\widetilde{X})\cap L^q(\widetilde{X})$ and let μ denote the product measure $\mu_1\times\mu_2$. Then, by the reproducing formula (3.15), Lemma 4.8 with $\overline{C}_i=1$ for $i=1,2$, and Fubini for summations, we have

$$\begin{aligned}f(x_1,x_2)&=\sum_{k_1\in\mathbb{Z}}\sum_{\alpha_1\in\mathscr{Y}^{k_1}}\sum_{k_2\in\mathbb{Z}}\sum_{\alpha_2\in\mathscr{Y}^{k_2}}\langle f,\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle\,\psi_{\alpha_1}^{k_1}(x_1)\,\psi_{\alpha_2}^{k_2}(x_2)\\&=:\sum_{\ell_1,\ell_2\ge0}2^{-\ell_1\gamma_1}\,2^{-\ell_2\gamma_2}f_{\ell_1,\ell_2}(x_1,x_2),\end{aligned}\tag{4.16}$$

where f_{ℓ_1,ℓ_2} is defined by

$$f_{\ell_1,\ell_2}(x_1,x_2):=\sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}}\sum_{\substack{k_2\in\mathbb{Z}\\ \alpha_2\in\mathscr{Y}^{k_2}}}\langle f,\psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle\,\kappa_1\,\varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1}(x_1)\,\kappa_2\,\varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2}(x_2).\tag{4.17}$$

Here we are denoting $\varphi_{\ell_i,k_i,\alpha_i}^{\gamma_i}:=\varphi_{\ell_i,k_i,\alpha_i}^{\gamma_i,1}$ and $\kappa_i:=\sqrt{\mu_i\big(B(y_{\alpha_i}^{k_i},\delta^{k_i})\big)}$ for $i=1,2$ (we are abusing notation, to be more precise we should write $\kappa_{\alpha_i}^{k_i}$ instead of simply κ_i). The parameter γ_i is an arbitrary constant larger than the upper dimension of X_i, that is $\gamma_i>\omega_i$, for $i=1,2$, and to be determined later. All these series converge unconditionally in the $L^q(\widetilde{X})$-norm when $q>1$, allowing us to reorder the series at will.

Now for $j\in\mathbb{Z}$, we let Ω_j be a level set for $S(f)$, more precisely

$$\Omega_j:=\Big\{(x_1,x_2)\in\widetilde{X}\,:\,S(f)(x_1,x_2)>2^j\Big\}.\tag{4.18}$$

Notice that $\Omega_{j+1}\subset\Omega_j$ for all $j\in\mathbb{Z}$ and that by the well-known layer-cake[4] formula for the L^p-(semi)norm of $S(f)$ it holds that

$$\|S(f)\|_{L^p(\widetilde{X})}^p\sim_p\sum_{j\in\mathbb{Z}}2^{pj}\mu(\Omega_j).\tag{4.19}$$

Also, by Tchebichev's inequality, when $f\in L^p(X_1\times X_2)$,

$$\mu(\Omega_j)\le2^{-jp}\int_{\Omega_j}|S(f)(x_1,x_2)|^p\,d\mu(x_1,x_2).\tag{4.20}$$

[4]Assume $F\in L^p(X,\mu)$; then $\|F\|_{L^p(\mu)}^p=\int_0^\infty p\lambda^{p-1}\mu\{x\in X\,:\,|F(x)|>\lambda\}\,d\lambda$.

If $f = 0$ in $L^q(\widetilde{X})$ then $S(f) = 0$ in $L^q(\widetilde{X})$ and the theorem is trivially true. Assume $f \neq 0$ in $L^q(\widetilde{X})$, notice that this implies that $S(f) \neq 0$ in $L^q(X_1 \times X_2)$, and it ensures that there is $j_0 \in \mathbb{Z}$ such that $\mu(\Omega_j) > 0$ for all $j \leq j_0$.

Recall that the reference dyadic grids underlying the wavelets on X_i are denoted $\mathcal{D}_i^W$ for $i = 1, 2$. Given dyadic cubes $Q_{\alpha_i}^{k_i} \in \mathscr{D}_i^W$ for $i = 1, 2$, let $R = R_{\alpha_1,\alpha_2}^{k_1,k_2}$ denote the dyadic rectangle in $X_1 \times X_2$ they determine, that is, $R_{\alpha_1,\alpha_2}^{k_1,k_2} := Q_{\alpha_1}^{k_1} \times Q_{\alpha_2}^{k_2}$. Let

$$\mathcal{B}_j := \Big\{R \text{ dyadic rectangle} : \mu(R \cap \Omega_j) > \frac{1}{2}\mu(R),\ \mu(R \cap \Omega_{j+1}) \leq \frac{1}{2}\mu(R)\Big\}. \tag{4.21}$$

In particular, since $S(f) \neq 0$ in $L^q(\widetilde{X})$, each dyadic rectangle $R_{\alpha_1,\alpha_2}^{k_1,k_2}$ belongs to exactly one set $\mathcal{B}_j$. We can reorder the quadruple sum in (4.17) over

$$(k_1, k_2, \alpha_1, \alpha_2) \in \mathbb{Z}^2 \times \mathscr{Y}^{k_1} \times \mathscr{Y}^{k_2}$$

by first adding over $j \in \mathbb{Z}$ and second adding over those $(k_1, k_2, \alpha_1, \alpha_2)$ such that $R_{\alpha_1,\alpha_2}^{k_1,k_2} \in \mathcal{B}_j$, obtaining

$$f_{\ell_1,\ell_2}(x_1, x_2) = \sum_{j\in\mathbb{Z}} \sum_{R_{\alpha_1,\alpha_2}^{k_1,k_2} \in \mathcal{B}_j} \langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle \kappa_1 \varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1}(x_1)\, \kappa_2\, \varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2}(x_2). \tag{4.22}$$

Next, we will show below that for each $j \in \mathbb{Z}$,

$$\Big\| \sum_{R_{\alpha_1,\alpha_2}^{k_1,k_2} \in \mathcal{B}_j} \langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle \kappa_1 \varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1} \kappa_2 \varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2}\Big\|_{L^p(\widetilde{X})}^p$$
$$\lesssim (\ell_1\omega_1 + \ell_2\omega_2)^{1-\frac{p}{q}} 2^{\ell_1\omega_1(1+\frac{p}{q'})} 2^{\ell_2\omega_2(1+\frac{p}{q'})} 2^{jp}\mu(\Omega_j). \tag{4.23}$$

Together with the special reproducing formula (4.16) and estimate (4.19), inequality (4.23) yields the conclusion of Theorem 4.6. More precisely, since $0 < p \leq 1$,

$$\begin{aligned}
\|f\|_{L^p(\widetilde{X})}^p &\leq \sum_{\ell_1,\ell_2\geq 0} 2^{-\ell_1\gamma_1 p} 2^{-\ell_2\gamma_2 p} \|f_{\ell_1,\ell_2}\|_{L^p(\widetilde{X})}^p \\
&\leq \sum_{\ell_1,\ell_2\geq 0} 2^{-\ell_1\gamma_1 p} 2^{-\ell_2\gamma_2 p} \sum_{j\in\mathbb{Z}} \Big\| \sum_{R_{\alpha_1,\alpha_2}^{k_1,k_2} \in \mathcal{B}_j} \langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle \kappa_1\varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1} \kappa_2\varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2}\Big\|_{L^p(\widetilde{X})}^p \\
&\lesssim \sum_{\ell_1,\ell_2\geq 0} 2^{-\ell_1\gamma_1 p} 2^{-\ell_2\gamma_2 p} (\ell_1\omega_1 + \ell_2\omega_2)^{1-\frac{p}{q}} 2^{\ell_1\omega_1(1+\frac{p}{q'})} 2^{\ell_2\omega_2(1+\frac{p}{q'})} \sum_{j\in\mathbb{Z}} 2^{jp}\mu(\Omega_j) \\
&\lesssim \|S(f)\|_{L^p(\widetilde{X})}^p = \|f\|_{H^p(\widetilde{X})}^p.
\end{aligned}$$

Where we have chosen $\gamma_i > \omega_i(1/p + 1/q')$ for $i = 1, 2$, to ensure convergence of the relevant series over ℓ_1 and ℓ_2. Note that since $1/p \geq 1$, this constraint implies that $\gamma_i > \omega_i$ for $i = 1, 2$, a constraint needed in Lemma 4.8.

Thus, it suffices to verify the claim (4.23). To this end, we define the ϵ_0-*enlargement* $\widetilde{\Omega}_j := \widetilde{\Omega}_j^{\epsilon_0}$ of the open set Ω_j by

$$\widetilde{\Omega}_j := \Big\{(x_1, x_2) \in \widetilde{X} : M_s(\chi_{\Omega_j})(x_1, x_2) > \epsilon_0 := \frac{1}{2C_{\mu_1}C_{\mu_2}}\Big(\frac{c_1^1}{C_1^1}\Big)^{\omega_1}\Big(\frac{c_1^2}{C_1^2}\Big)^{\omega_2}\Big\}. \tag{4.24}$$

Here c_1^i, C_1^i are the dilation constants of the grids $\mathscr{D}_i^W$ and M_s is the *strong maximal function*

$$M_s g(x_1, x_2) := \sup_{B_1\times B_2 \ni (x_1,x_2)} \frac{1}{\mu_1(B_1)\mu_2(B_2)} \int_{B_1\times B_2} |g(y_1, y_2)|\, d\mu(y_1, y_2),$$

defined for functions $g \in L^1_{\rm loc}(\widetilde{X})$, and where B_i are balls in X_i for $i = 1, 2$.

The constant ϵ_0 in (4.24) is determined by the doubling constants of the measures μ_i, the upper dimensions ω_i, and the ratio of the dilation constants $c_1^i = (A_0^{(i)})^{-5}/6$ and $C_1^i = 6(A_0^{(i)})^4$ involved in the radius of the inner and outer balls sandwiching the reference dyadic cubes for the wavelets, as in property (3.6), for $i = 1, 2$. More precisely, ϵ_0 is a constant depending only on the geometric constants of X_i for $i = 1, 2$,

$$\epsilon_0 = \Big(2C_{\mu_1}C_{\mu_2}\big(36(A_0^{(1)})^9\big)^{\omega_1}\big(36(A_0^{(2)})^9\big)^{\omega_2}\Big)^{-1}. \tag{4.25}$$

Furthermore $\epsilon_0 \in (0, 1)$ and is chosen so that if $R \in \mathcal{B}_j$ then $R \subset \widetilde{\Omega}_j$. More precisely, if $R \in \mathcal{B}_j$ then by definition $\mu(R\cap\Omega_j)/\mu(R) > 1/2$. The dyadic rectangle $R = Q_1 \times Q_2$ and, for $i = 1, 2$, each dyadic cube $Q_i \in \mathscr{D}_i^W$ contains B_i', its inner ball, and is contained in B_i'', its outer ball, that is $B_i' \subset Q_i \subset B_i''$. Moreover, $\mu_i(B_i'') \le C_{\mu_i}\Big(\frac{C_1^i}{c_1^i}\Big)^{\omega_i}\mu_i(B_i')$ by the doubling property (1.3) of the measure μ_i for $i = 1, 2$. Hence

$$\begin{aligned}\frac{1}{2} &< \frac{\mu(R\cap\Omega_j)}{\mu(R)} \le \frac{\mu\big((B_1''\times B_2'')\cap\Omega_j\big)}{\mu_1(B_1')\,\mu_2(B_2')}\\ &\le C_{\mu_1}C_{\mu_2}\Big(\frac{C_1^1}{c_1^1}\Big)^{\omega_1}\Big(\frac{C_1^2}{c_1^2}\Big)^{\omega_2}\frac{\mu\big((B_1''\times B_2'')\cap\Omega_j\big)}{\mu_1(B_1'')\,\mu_2(B_2'')}.\end{aligned}$$

We conclude that $B_1''\times B_2'' \subset \widetilde{\Omega}_j$ and therefore $R = Q_1 \times Q_2 \subset \widetilde{\Omega}_j$.

By definition every open set Ω is contained in its ϵ-enlargement

$$\widetilde{\Omega}^\epsilon := \big\{(x_1, x_2) \in X_1 \times X_2 : M_s(\chi_\Omega)(x_1, x_2) > \epsilon\big\} \tag{4.26}$$

for $\epsilon \in (0, 1)$, that is $\Omega \subset \widetilde{\Omega}^\epsilon$. In particular $\Omega_j \subset \widetilde{\Omega}_j$ and hence $\mu(\Omega_j) \le \mu(\widetilde{\Omega}_j)$ for all $j \ge 0$. More interestingly, by weak-L^2 properties of the strong maximal function we get

$$\mu(\widetilde{\Omega}_j) \le C\bigg(\frac{\|\chi_{\Omega_j}\|_{L^2(X_1\times X_2)}}{\epsilon_0}\bigg)^2 = \frac{C}{\epsilon_0^2}\,\mu(\Omega_j). \tag{4.27}$$

We also define the (ℓ_1, ℓ_2)-*enlargement* $\widetilde{\Omega}_{j,\ell_1,\ell_2}$ of $\widetilde{\Omega}_j$. Recall that $2^{\ell_i}Q_i := B(y^{k_i}_{\alpha_i}, 2^{\ell_i}C^i_1\delta^{k_i})$, where C^i_1 is the dilation constant determining the radius of the outer ball of the dyadic cube $Q_i \in \mathcal{D}^W_i$ for each $i = 1, 2$. Let

$$\widetilde{\Omega}_{j,\ell_1,\ell_2} := \bigcup_{R=Q_1\times Q_2\subset\widetilde{\Omega}_j} 2^{\ell_1}Q_1 \times 2^{\ell_2}Q_2. \tag{4.28}$$

It is clear from this definition that $\widetilde{\Omega}_j \subset \widetilde{\Omega}_{j,\ell_1,\ell_2}$ for all $\ell_1, \ell_2 \geq 0$. Note that $\widetilde{\Omega}_{j,\ell_1,\ell_2}$ is a subset of $\{(x_1, x_2) \in X_1 \times X_2 : M_s(\chi_{\widetilde{\Omega}_j})(x_1, x_2) \geq 2^{-\ell_1\omega_1-\ell_2\omega_2}\}$. Indeed, for every $(x_1, x_2) \in \widetilde{\Omega}_{j,\ell_1,\ell_2}$ there must be a dyadic rectangle $R = Q_1 \times Q_2 \in \widetilde{\Omega}_j$ such that $(x_1, x_2) \in 2^{\ell_1}Q_1 \times 2^{\ell_2}Q_2$. Also for $2^{\ell_1}Q_1 \times 2^{\ell_2}Q_2$ we get

$$\frac{\mu\big(\widetilde{\Omega}_j \cap (2^{\ell_1}Q_1 \times 2^{\ell_2}Q_2)\big)}{\mu(2^{\ell_1}Q_1 \times 2^{\ell_2}Q_2)} \geq \frac{\mu\big(\widetilde{\Omega}_j \cap (Q_1 \times Q_2)\big)}{2^{\ell_1\omega_1+\ell_2\omega_2}\mu(Q_1 \times Q_2)} = \frac{1}{2^{\ell_1\omega_1+\ell_2\omega_2}}.$$

Hence $M_s(\chi_{X_{\widetilde{\Omega}_j}})(x_1, x_2) \geq 2^{-\ell_1\omega_1-\ell_2\omega_2}$. We conclude that

$$\mu(\widetilde{\Omega}_{j,\ell_1,\ell_2}) \lesssim (\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1}2^{\ell_2\omega_2}\mu(\widetilde{\Omega}_j), \tag{4.29}$$

by an argument similar to [CF, p.191, line 17], denoting $\widetilde{x} = (x_1, x_2)$ and using the $L\log_+ L$ to weak L^1 estimate for the strong maximal function applied to $f = \chi_{\widetilde{\Omega}_j}$, namely

$$\mu\{(\widetilde{x}) \in \widetilde{X} : M_s(f)(\widetilde{x}) > \lambda\} \lesssim \int_{\widetilde{X}} \frac{|f(\widetilde{x})|}{\lambda} \log\left(1 + \frac{|f(\widetilde{x})|}{\lambda}\right) d\mu(\widetilde{x}). \tag{4.30}$$

The $L\log_+ L$ to weak L^1 estimate (4.30) for the strong maximal function can be deduced for the strong dyadic maximal function (defined as M_s but instead of product of balls we consider products of dyadic cubes in X_1 and X_2) from the weak (1, 1) estimates on each individual dyadic maximal function on X_i for $i = 1, 2$, see [Fa, Theorem 1] and also [Fe2]. By [KLPW, Theorem 3.1(ii)] we can control pointwise the strong maximal function M_s (with respect to balls) by a finite sum of strong dyadic maximal functions (with respect to adjacent systems of dyadic cubes [KLPW, Section 2.4], the equivalent to the 1/3 trick in $\mathbb{R}$ for spaces of homogeneous type). Hence we obtain the desired estimate (4.30).

For each set $\mathcal{B}_j$ of dyadic rectangles, we define the function $f_{\mathcal{B}_j} : \widetilde{X} \to \mathbb{R}$ to be

$$f_{\mathcal{B}_j}(x_1, x_2) := \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle\, \psi^{k_1}_{\alpha_1}(x_1)\, \psi^{k_2}_{\alpha_2}(x_2), \tag{4.31}$$

and hence by definition of the square function

$$S(f_{\mathcal{B}_j})(x_1, x_2) = \Big(\sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \big|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\big|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}(x_1, x_2)\Big)^{\frac{1}{2}}, \tag{4.32}$$

where $\widetilde{\psi}^{k_i}_{\alpha_i} = \psi^{k_i}_{\alpha_i}/\kappa_i$ denotes the normalized wavelets for $i = 1, 2$.

Note that by construction, the function $\varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}$ has compact support on

$$B(y^{k_i}_{\alpha_i}, 2(A^{(i)}_0)^2\, 2^{\ell_i}\delta^{k_i})$$

which is contained in $B(y^{k_i}_{\alpha_i}, 2^{\ell_i}C^i_1\delta^{k_i})$ for $i = 1, 2$. The last statement holds since in the Auscher-Hytönen construction the dilation constant C^i_1 determining the radius of the outer balls is $C^i_1 = 6(A^{(i)}_0)^4 > 2(A^{(i)}_0)^2$ for each $i = 1, 2$ [AuH1, Theorem 2.11]. As explained on page 1205, if $R^{k_1,k_2}_{\alpha_1,\alpha_2} \in \mathcal{B}_j$, then $R^{k_1,k_2}_{\alpha_1,\alpha_2} \in \widetilde{\Omega}_j$, and thus the support of $\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}(x_1)\,\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}(x_2)$ is contained in $\widetilde{\Omega}_{j,\ell_1,\ell_2}$.

Therefore, by Hölder's inequality with exponents $s = q/p > 1$ and $s' = q/(q-p)$,

$$\Big\| \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\kappa_2\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}\Big\|^p_{L^p(\widetilde{X})} \tag{4.33}$$
$$\le \mu(\widetilde{\Omega}_{j,\ell_1,\ell_2})^{1-\frac{p}{q}} \Big\| \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\kappa_2\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}\Big\|^p_{L^q(\widetilde{X})}.$$

To estimate the L^q-norm of the sum in the right-hand-side of (4.33) we use a duality argument. Hence, for all $g \in L^{q'}(\widetilde{X})$ with $\|g\|_{L^{q'}(\widetilde{X})} \le 1$, we estimate the inner product

$$\begin{aligned}
&\Big|\Big\langle \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\kappa_2\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2},\ g\Big\rangle\Big| \\
&\quad = \Big| \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \kappa_1^2\kappa_2^2 \big\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\big\rangle \langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\Big| \\
&\quad \le \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \mu_1(Q^{k_1}_{\alpha_1})\mu_2(Q^{k_2}_{\alpha_2}) \Big|\big\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\big\rangle\Big|\, \Big|\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\Big| \\
&\quad \le \int_{X_1\times X_2} \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\big\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\big\rangle\Big|\, \Big|\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\Big|\, \chi_{R^{k_1k_2}_{\alpha_1,\alpha_2}}(\widetilde{x})\, d\mu(\widetilde{x}).
\end{aligned}$$

In the last inequality $\widetilde{x} = (x_1, x_2) \in X_1 \times X_2$ and we used that

$$\mu_1(Q^{k_1}_{\alpha_1})\mu_2(Q^{k_2}_{\alpha_2}) = \int_{X_1\times X_2} \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}(\widetilde{x})\, d\mu(\widetilde{x}).$$

We continue estimating, first applying the Cauchy-Schwarz inequality on the sum, second applying Hölder's inequality, with exponents $q > 1$ and q', to the integral, and third using the notation introduced in (4.31) and (4.32):

$$\Big|\Big\langle \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\kappa_2\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2},\ g\Big\rangle\Big|$$

$$\leq \Big(\int_{X_1\times X_2} \Big(\sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\Big|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}(\widetilde{x})\Big)^{\frac{q}{2}} d\mu(\widetilde{x})\Big)^{\frac{1}{q}}$$

$$\times\Big(\int_{X_1\times X_2} \Big(\sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\Big|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}(\widetilde{x})\Big)^{\frac{q'}{2}} d\mu(\widetilde{x})\Big)^{\frac{1}{q'}}$$

$$\lesssim_q 2^{\ell_1\omega_1}2^{\ell_2\omega_2}\|S(f_{\mathcal{B}_j})\|_{L^q(\widetilde{X})}. \tag{4.34}$$

The last inequality is deduced from the fact that $\|g\|_{L^{q'}(\widetilde{X})} \leq 1$ and the following Littlewood–Paley estimate, whose proof will be provided after finishing the proof of Theorem 4.6.

Lemma 4.9. *There is a constant $C > 0$ (depending on the geometric constants and on $q > 1$) such that for all functions $g \in L^{q'}(\widetilde{X})$ and all positive integers ℓ_1 and ℓ_2,*

$$\Big\|\Big[\sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\Big|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}\Big]^{\frac{1}{2}}\Big\|_{L^{q'}(\widetilde{X})}$$
$$\lesssim_q 2^{\ell_1\omega_1}2^{\ell_2\omega_2}\|g\|_{L^{q'}(\widetilde{X})}. \tag{4.35}$$

The dual estimate (4.34) implies that

$$\Big\| \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\kappa_2\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}\Big\|_{L^q(\widetilde{X})}$$
$$\lesssim_q 2^{\ell_1\omega_1}2^{\ell_2\omega_2}\|S(f_{\mathcal{B}_j})\|_{L^q(\widetilde{X})}$$
$$= 2^{\ell_1\omega_1}2^{\ell_2\omega_2}\Big(\int_{X_1\times X_2} \Big\{ \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\Big|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}(\widetilde{x})\Big\}^{\frac{q}{2}} d\mu(\widetilde{x})\Big)^{\frac{1}{q}}$$
$$\lesssim_q 2^{\ell_1\omega_1}2^{\ell_2\omega_2}\Big(\int_{X_1\times X_2} \Big\{ \sum_{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\Big|^2 \tag{4.36}$$
$$\times \Big|M_s\big(\chi_{R\cap(\widetilde{\Omega}_j\setminus\Omega_{j+1})}\big)(\widetilde{x})\Big|^2\Big\}^{\frac{q}{2}} d\mu(\widetilde{x})\Big)^{\frac{1}{q}}.$$

In the last inequality we have used the definitions (4.21), of the set $\mathcal{B}_j$, and (4.24), of the enlargement set $\widetilde{\Omega}_j$ via the strong maximal function, to deduce that

$$\chi_R(x_1,x_2) \lesssim \Big|M_s\big(\chi_{R\cap(\widetilde{\Omega}_j\setminus\Omega_{j+1})}\big)(x_1,x_2)\Big|^2.$$

More precisely, recall that if $R = Q_1 \times Q_2$ belongs to $\mathcal{B}_j$ then it is a subset of $\widetilde{\Omega}_j$. Hence $R \cap (\widetilde{\Omega}_j \setminus \Omega_{j+1}) = R \setminus \Omega_{j+1}$, and since $R \in \mathcal{B}_j$ it is also true that $\mu(R\cap\Omega_{j+1}) \leq \frac{1}{2}\mu(R)$. Therefore $\mu(R\setminus\Omega_{j+1}) \geq \frac{1}{2}\mu(R)$. As before, denote by B_i' and B_i'' the inner and outer balls of the dyadic cubes Q_i for $i = 1, 2$. Recall that

$B'_i \subset Q_i \subset B''_i$, therefore $R \setminus \Omega_{j+1} \subset B''_1 \times B''_2$. Using the doubling property (1.3) we get for $R \in \mathcal{B}_j$

$$\begin{aligned}\frac{1}{\mu(B''_1 \times B''_2)} \int_{B''_1 \times B''_2} \chi_{R\cap(\widetilde{\Omega}_j \setminus \Omega_{j+1})}(z_1, z_2)\, d\mu(z_1, z_2) &= \frac{\mu(R \setminus \Omega_{j+1})}{\mu(B''_1 \times B''_2)} \\ &\geq \frac{1}{2} \frac{\mu(R)}{\mu(B''_1 \times B''_2)} \geq \frac{1}{2} \frac{\mu(B'_1 \times B'_2)}{\mu(B''_1 \times B''_2)} = \frac{1}{2} \frac{\mu_1(B'_1)}{\mu_1(B''_1)} \frac{\mu_2(B'_2)}{\mu_2(B''_2)} \\ &\geq \frac{1}{2C_{\mu_1} C_{\mu_2}} \Big[\frac{c_1^1}{C_1^1}\Big]^{\omega_1} \Big[\frac{c_1^2}{C_1^2}\Big]^{\omega_2} = \epsilon_0.\end{aligned}$$

Therefore, for all $R \in \mathcal{B}_j$ and for all $\widetilde{x} = (x_1, x_2) \in R$, we get

$$M_s\big(\chi_{R\cap(\widetilde{\Omega}_j \setminus \Omega_{j+1})}\big)(\widetilde{x}) \geq \epsilon_0 > 0.$$

Hence we obtain $\chi_R(\widetilde{x}) = \chi_R^2(\widetilde{x}) \lesssim \left|M_s\big(\chi_{R\cap(\widetilde{\Omega}_j \setminus \Omega_{j+1})}\big)(\widetilde{x})\right|^2$, as claimed. Note that the similarity constant is ϵ_0^{-2}, which only depends on the geometric constants of X_i for $i = 1, 2$, by definition (4.25).

Recall the Fefferman-Stein vector-valued strong maximal function estimate in [FS]: given $q, r > 1$, there is a constant $C_q > 0$ such that for appropriate sequences of functions $\{f_k\}_{k\geq 1}$

$$\Big\|\Big\{\sum_{k=1}^{\infty} M_s(f_k)^r\Big\}^{1/r}\Big\|_{L^q(\widetilde{X})} \leq C_q \Big\|\Big\{\sum_{k=1}^{\infty} |f_k|^r\Big\}^{1/r}\Big\|_{L^q(\widetilde{X})}. \tag{4.37}$$

We use estimate (4.37) with $r = 2$ and $q > 1$, to conclude that

$$\begin{aligned}\|S(f_{\mathcal{B}_j})\|_{L^q(\widetilde{X})} &\lesssim_q \bigg(\int_{\widetilde{X}} \bigg(\sum_{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\Big|^2 \chi_{R\cap(\widetilde{\Omega}_j \setminus \Omega_{j+1})}(\widetilde{x})\bigg)^{\frac{q}{2}} d\mu(\widetilde{x})\bigg)^{\frac{1}{q}} \\ &= \bigg(\int_{\widetilde{\Omega}_j \setminus \Omega_{j+1}} \bigg(\sum_{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \Big|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\Big|^2 \chi_R(\widetilde{x})\bigg)^{\frac{q}{2}} d\mu(\widetilde{x})\bigg)^{\frac{1}{q}} \\ &= \bigg(\int_{\widetilde{\Omega}_j \setminus \Omega_{j+1}} |S(f_{\mathcal{B}_j})(x_1, x_2)|^q\, d\mu(\widetilde{x})\bigg)^{\frac{1}{q}}.\end{aligned} \tag{4.38}$$

The function $f_{\mathcal{B}_j}$ was defined in (4.31), and its square function $S(f_{\mathcal{B}_j})$ in (4.32). Note that pointwise $S(f_{\mathcal{B}_j}) \leq S(f)$. Moreover when $(x_1, x_2) \notin \Omega_{j+1}$ by definition $S(f)(x_1, x_2) \leq 2^{j+1}$. Therefore,

$$\|S(f_{\mathcal{B}_j})\|_{L^q(\widetilde{X})} \lesssim_q 2^j \mu(\widetilde{\Omega}_j)^{1/q}. \tag{4.39}$$

All together we conclude that

$$\Big\| \sum_{R_{\alpha_1,\alpha_2}^{k_1,k_2}\in\mathcal{B}_j} \langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle \kappa_1\varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1}\kappa_2\varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2}\Big\|_{L^q(\widetilde{X})} \lesssim_q 2^{\ell_1\omega_1}2^{\ell_2\omega_2}2^j\mu(\widetilde{\Omega}_j)^{\frac{1}{q}}. \quad (4.40)$$

Finally, first using estimates (4.33) and (4.40), and second using estimate (4.29), we get the L^p-estimate claimed in (4.23):

$$\begin{aligned}
&\Big\| \sum_{R_{\alpha_1,\alpha_2}^{k_1,k_2}\in\mathcal{B}_j} \langle f, \psi_{\alpha_1}^{k_1}\psi_{\alpha_2}^{k_2}\rangle \kappa_1\varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1}\kappa_2\varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2}\Big\|_{L^p(\widetilde{X})}^p \\
&\lesssim_q \mu(\widetilde{\Omega}_{j,\ell_1,\ell_2})^{1-\frac{p}{q}} 2^{\ell_1\omega_1 p}2^{\ell_2\omega_2 p}2^{jp}\mu(\widetilde{\Omega}_j)^{\frac{p}{q}} \\
&\lesssim_q (\ell_1\omega_1+\ell_2\omega_2)^{1-\frac{p}{q}} 2^{\ell_1\omega_1(1-\frac{p}{q})}\, 2^{\ell_2\omega_2(1-\frac{p}{q})}\, \mu(\widetilde{\Omega}_j)^{1-\frac{p}{q}}\, 2^{\ell_1\omega_1 p}\, 2^{\ell_2\omega_2 p}\, 2^{jp}\, \mu(\widetilde{\Omega}_j)^{\frac{p}{q}} \\
&\lesssim_q (\ell_1\omega_1+\ell_2\omega_2)^{1-\frac{p}{q}} 2^{\ell_1\omega_1(1+\frac{p}{q'})} 2^{\ell_2\omega_2(1+\frac{p}{q'})} 2^{jp}\mu(\Omega_j).
\end{aligned}$$

Here the last estimate follows from $\mu(\widetilde{\Omega}_j) \lesssim \mu(\Omega_j)$ by (4.27). Note that all constants depend only on the geometric constants of X_i for $i = 1, 2$, sometimes via the parameter ϵ_0 defined in (4.25). This estimate finishes the proof of the claim (4.23), and hence Theorem 4.6 is proved. □

Proof of Lemma 4.9. Estimate (4.35) can be established using an argument similar to the one made when proving the second inequality in the product Plancherel–Pólya inequalities from [HLW, Theorem 4.9, equation (4.13)]. More specifically, there are sufficiently large integers $N_i > 0$ for $i = 1, 2$, and a constant $C_q > 0$ (depending only on the geometric constants of X_i for $i = 1, 2$ and $q > 1$) such that for all $g \in L^{q'}(\widetilde{X})$ the following inequality holds:

$$\begin{aligned}
&\bigg\|\bigg\{\sum_{k_1,k_2}\sum_{\alpha_1\in\mathscr{Y}^{k_1},\alpha_2\in\mathscr{Y}^{k_2}} \Big|\langle\varphi_{\ell_1,k_1,\alpha_1}^{\gamma_1}\varphi_{\ell_2,k_2,\alpha_2}^{\gamma_2},\ g\rangle\Big|^2 \chi_{R_{\alpha_1,\alpha_2}^{k_1,k_2}}\bigg\}^{\frac{1}{2}}\bigg\|_{L^{q'}(\widetilde{X})} \\
&\quad\le C_q 2^{\ell_1 w_1}2^{\ell_2 w_2}\bigg\|\bigg\{\sum_{k_1,k_2}\sum_{\substack{\alpha_1\in\mathscr{X}^{k_1+N_1}\\ \alpha_2\in\mathscr{X}^{k_2+N_2}}} \inf_{\substack{z_1\in Q_{\alpha_1}^{k_1+N_1}\\ z_2\in Q_{\alpha_2}^{k_2+N_2}}} |D_{k_1}^{(1)}D_{k_2}^{(2)}(g)(z_1,z_2)|^2 \\
&\qquad\qquad \times \chi_{Q_{\alpha_1}^{k_1+N_1}}\chi_{Q_{\alpha_2}^{k_2+N_2}}\bigg\}^{\frac{1}{2}}\bigg\|_{L^{q'}(\widetilde{X})}, \qquad (4.41)
\end{aligned}$$

where $D_{k_1}^{(1)}$ is the integral operator in X_1 with kernel

$$D_{k_1}^{(1)}(x,y) = \sum_{\beta_1\in\mathscr{Y}^{k_1}} \psi_{\beta_1}^{k_1}(x)\psi_{\beta_1}^{k_1}(y),$$

and similarly for $D_{k_2}^{(2)}$. The statement in [HLW, Theorem 4.9] refers to Plancherel-Pólya inequalities with the wavelets $\psi_{\alpha_i}^{k_i}$ instead of the functions $\varphi_{\ell_i,k_i,\alpha_i}^{\gamma_i}$ on the left-hand-side of equation (4.41). However, carefully tracing the proof of [HLW, Equation (4.13)], one realizes that all that is required are the size, smoothness, and cancellation conditions of the functions $\varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}$ (proved in Lemma 4.8) and of the kernels $D^{(i)}_{k_i}(x,y)$ for $i=1,2$ (proved in [HLW, Lemma 3.6]). The key observations are first, for every $(y_1,y_2)\in\widetilde{X}$

$$\begin{aligned}\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, D^{(1)}_{k_1}D^{(2)}_{k_2}(\cdot,y_1,\cdot,y_2)\rangle_{\widetilde{X}} \\ = \langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}, D^{(1)}_{k_1}(\cdot,y_1)\rangle_{X_1}\,\langle \varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, D^{(2)}_{k_2}(\cdot,y_2)\rangle_{X_2}.\end{aligned}$$

Second, the following almost-orthogonality estimate is valid for $i=1,2$: for all integers k_i and k_i' let $\delta_i' := \delta_i^{\min\{k_i,k_i'\}}$, where δ_i is the base side length for the reference dyadic cubes in X_i. Then for each positive integer N_i, each $\gamma>0$, each point $z\in Q^{k_i'+N_i}_{\alpha_i'}\subset X_i$ and each center point $x^{k_i'+N_i}_{\alpha_i'}\in Q^{k_i'+N_i}_{\alpha_i'}$

$$\begin{aligned}|\langle \varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}(\cdot), D^{(i)}_{k_i'}(\cdot,z)\rangle| \;\lesssim\; & \frac{2^{\ell_i\omega_i}\delta_i^{|k_i-k_i'|\eta}}{V_{\delta_i'}(x^{k_i}_{\alpha_i})+V_{\delta_i'}(x^{k_i'+N_i}_{\alpha_i'})+V(x^{k_i}_{\alpha_i},x^{k_i'+N_i}_{\alpha_i'})} \\ & \times\Big(\frac{\delta_i'}{\delta_i'+d_i(x^{k_i}_{\alpha_i},x^{k_i'+N_i}_{\alpha_i'})}\Big)^{\gamma}.\end{aligned}\tag{4.42}$$

Here $V_{r_i}(x_i)=\mu_i(B_{X_i}(x_i,r_i))$, $V(x_i,y_i)=\mu_i(B_{X_i}(x_i,d_i(x_i,y_i)))$, and the similarity constants depend only on the geometric constants of X_i for $i=1,2$. This estimate is the analogue of estimate [HLW, Equation (4.4)] with the functions φ instead of the wavelets on the left-hand side of the inner product. It is in proving estimate (4.42) that the size, smoothness, and cancellation properties of the functions $\varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}$ are needed. Also needed are the corresponding properties for the kernels of the operators $D^{(i)}_{k_i'}$ established in [HLW, Lemma 3.6]. The right-hand side of (4.41) is pointwise bounded by the same expression where the infimum in the sum is replaced by the supremum. Another application of Plancherel-Pólya as stated in [HLW, equation (4.12)] shows that for all positive integers N_1 and N_2 there is a constant $C_q>0$ (depending only on geometric constants and $q>1$) such that

$$\begin{aligned}\Big\|\Big\{\sum_{k_1,k_2}\sum_{\substack{\alpha_1\in\mathscr{X}^{k_1+N_1}\\ \alpha_2\in\mathscr{X}^{k_2+N_2}}}\sup_{\substack{z_1\in Q^{k_1+N_1}_{\alpha_1}\\ z_2\in Q^{k_2+N_2}_{\alpha_2}}}|D^{(1)}_{k_1}D^{(2)}_{k_2}(g)(z_1,z_2)|^2\chi_{Q^{k_1+N_1}_{\alpha_1}}\chi_{Q^{k_2+N_2}_{\alpha_2}}\Big\}^{\frac12}\Big\|_{L^{q'}(\widetilde{X})} \\ \le\; C_q\|S(g)\|_{L^{q'}(\widetilde{X})}.\end{aligned}\tag{4.43}$$

Here $S(g)$ is the product Littlewood-Paley square function of g as in Definition 4.1. This time there are wavelets on both sides of (4.43) exactly as in [HLW, equation (4.12)].

From (4.41) and the product Plancherel–Pólya inequality (4.43) we see that the left-hand side of (4.35) is bounded by the $L^{q'}$-norm of $S(g)$. From Theorem 4.8 in [HLW], since we are in the case $q' > 1$, we obtain that

$$\|S(g)\|_{L^{q'}(\widetilde{X})} \leq C_{q'}\|g\|_{L^{q'}(\widetilde{X})}.$$

Putting all the pieces together we get estimate (4.35), with a constant $C > 0$ that depends only on the geometric constants of X_i for $i = 1, 2$ and on $q > 1$. This finishes the proof of Lemma 4.9. □

5. Atomic product Hardy spaces

We now provide an atomic decomposition for $H^p(\widetilde{X})$. More precisely, we will find an atomic decomposition for each function $f \in L^q(\widetilde{X}) \cap H^p(\widetilde{X})$ with $1 < q < \infty$ and $p_0 < p \leq 1$, where the decomposition converges both in the $L^q(\widetilde{X})$-norm and in the $H^p(\widetilde{X})$-(semi)norm. Recall that $p_0 := \max\{\omega_i/(\omega_i + \eta_i) : i = 1, 2\}$. To achieve this decomposition we will need a Journé-type covering lemma and a suitable definition of product (p, q)-atoms on $\widetilde{X} = X_1 \times X_2$, valid for (X_i, d_i, μ_i) spaces of homogeneous type in the sense of Coifman and Weiss for $i = 1, 2$. We will also define atomic product Hardy spaces $H^{p,q}_{\rm at}(\widetilde{X})$, and as a consequence of the main theorem we will show these spaces coincide with $H^p(\widetilde{X})$ for all $q > 1$.

The definition of the product Hardy spaces $H^p(\widetilde{X})$ uses Auscher-Hytönen wavelet bases on each space of homogeneous type X_i, with Hölder regularity $\eta_i \in (0, 1]$, and corresponding reference dyadic grids $\mathscr{D}_i^W$, for $i = 1, 2$, provided $p > p_0$. In this section we will show that functions in $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ can be decomposed into product (p, q)-atoms based on the wavelets' reference dyadic grids $\mathscr{D}_i^W$ for $i = 1, 2$. Product (p, q)-atoms do not require wavelets in their definition, but there is an underlying dyadic grid associated to each atom. We will show that product (p, q)-atoms, based on regular families of dyadic grids, are in $H^p(\widetilde{X})$ with uniform bounds on their H^p-(semi)norm dependent only on the geometric constants of the spaces X_i for $i = 1, 2$. These observations allow us to deduce that the product H^p, CMO^p, BMO, and VMO spaces, defined *a priori* using Auscher-Hytönen wavelets, are independent of the wavelets and the reference dyadic grids chosen (and indeed of the reference dyadic points $\{x_\alpha^k\}$ chosen), yielding Corollary B and Corollary C stated in the introduction.

We would like to point out that the convergence in both the $L^2(\widetilde{X})$-norm and $H^p(\widetilde{X})$-(semi)norm is crucial for proving the boundedness of Calderón-Zygmund operators from $H^p(\widetilde{X})$ to $L^p(\widetilde{X})$ as described in [HLLin].

5.1. Journé-type covering lemma. In the product theory the Journé-type covering lemmas play a fundamental role. The Journé covering lemma was established by Journé [J] on $\mathbb{R} \times \mathbb{R}$, and by Pipher [P] on $\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_k}$. Recently, following the same ideas and techniques as in [P], a Journé-type covering lemma was developed for $\widetilde{X} = X_1 \times X_2$ by the first two authors and Lin [HLLin] for certain spaces of homogeneous type.

In this section, for $i = 1, 2$, (X_i, d_i, μ_i) denotes a space of homogeneous type in the sense of Coifman and Weiss with ω_i an upper dimension, $A_0^{(i)}$ the quasi-triangle constant, C_{μ_i} the doubling constant, and with an underlying dyadic grid $\mathscr{D}_i$ whose structural constants are c_0^i, C_0^i, c_1^i, C_1^i, and δ_i, as in Theorem 3.1.

Let $\Omega \subset \widetilde{X}$ be an open set of finite measure and for $i = 1, 2$, let $m_i(\Omega)$ denote the family of dyadic rectangles $R = Q_1 \times Q_2$ in Ω which are maximal in the ith "direction", here $Q_i \in \mathscr{D}_i$. Also denote by $m(\Omega)$ the set of all maximal dyadic rectangles contained in Ω. Note that neither $m(\Omega)$ nor $m_1(\Omega)$ nor $m_2(\Omega)$ are disjoint collections of rectangles; this is one of the main difficulties when dealing with the product and multi-parameter settings.

Given a dyadic rectangle $R = Q_1 \times Q_2 \in m_1(\Omega)$, let $\widehat{Q}_2 = \widehat{Q}_2(Q_1)$ be the largest dyadic cube in $\mathscr{D}_2$ containing Q_2 such that

$$\mu\big((Q_1 \times \widehat{Q}_2) \cap \Omega\big) > \frac{1}{2}\mu(Q_1 \times \widehat{Q}_2), \tag{5.1}$$

where $\mu = \mu_1 \times \mu_2$ is the measure on $\widetilde{X}$. Similarly, given a dyadic rectangle $R = Q_1 \times Q_2 \in m_2(\Omega)$, let $\widehat{Q}_1 = \widehat{Q}_1(Q_2)$ be the largest dyadic cube in $\mathscr{D}_1$ containing Q_1 such that

$$\mu\big((\widehat{Q}_1 \times Q_2) \cap \Omega\big) > \frac{1}{2}\mu(\widehat{Q}_1 \times Q_2).$$

We now state the Journé-type covering lemma on $X_1 \times X_2$.

Lemma 5.1 ([HLLin], Lemma 2.2). *For $i = 1, 2$, let (X_i, d_i, μ_i) be spaces of homogeneous type in the sense of Coifman and Weiss as described in the Introduction, with quasi-metrics d_i and Borel-regular doubling measures μ_i, each space with an underlying dyadic grid $\mathscr{D}_i$. Let Ω be an open subset in $\widetilde{X}$ with finite measure. Let $w : [0,\infty) \to [0,\infty)$ be any fixed increasing function such that $\sum_{j=0}^{\infty} jw(C_0 2^{-j}) < \infty$, where C_0 is any given positive constant. Then there exists a positive constant C (dependent on the fixed increasing function w, the geometric constants of the spaces X_i, and the structural constants of the underlying dyadic grids via the ratios of the dilation constants C_1^i/c_1^i, for $i = 1, 2$) such that*

$$\sum_{R=Q_1\times Q_2\in m_1(\Omega)} \mu(R)\, w\Big(\frac{\ell(Q_2)}{\ell(\widehat{Q}_2)}\Big) \le C\mu(\Omega) \tag{5.2}$$

and

$$\sum_{R=Q_1\times Q_2\in m_2(\Omega)} \mu(R)\, w\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big) \le C\mu(\Omega). \tag{5.3}$$

In applications, we may take $w(t) = t^{\delta}$ for any $\delta > 0$ and the underlying dyadic grids may be reference dyadic grids for the wavelets, or may belong to a regular family of dyadic grids that contains them. In these cases the constant $C = C_\delta$ depends only on δ and the geometric constants of the spaces X_i for $i = 1, 2$.

In [HLLin] the setting is the product of two spaces of homogeneous type with a regularity condition on the metrics and a reverse doubling condition on the measures. However the proof of the Journé-type lemma uses only the doubling property of the measures, and goes through in the present setting. In the same paper the authors introduced (p, q)-atoms in their setting, similar to those we define in this paper. Our (p, q)-atoms will have additional enlargement parameters $(\ell_1, \ell_2) \in \mathbb{Z}_+^2$ that were not present in [HLLin].

5.2. Product (p, q)-atoms and atomic Hardy spaces. First we define product (p, q)-atoms for all $p \in (0, 1]$ and $q > 1$. Second we define product atomic Hardy spaces, $H^{p,q}_{\mathrm{at}}(\widetilde{X})$, for all $q > 1$ and for all p with $p_0 < p \le 1$, where $p_0 := \max\{\omega_i/(\omega_i + \eta_i) : i = 1, 2\}$.

Definition 5.2 (Product (p, q)-atoms)**.** Suppose that $0 < p \le 1$ and $1 < q < \infty$. For $i = 1, 2$, let (X_i, d_i, μ_i) be spaces of homogeneous type in the sense of Coifman and Weiss, with upper dimension ω_i. A function $a(x_1, x_2)$ defined on $\widetilde{X}$ is a *product* (p, q)*-atom* if it satisfies the following conditions.

(1) (Support condition on open set) There are an open set Ω of $\widetilde{X}$ with finite measure and integers $\ell_1, \ell_2 \ge 0$, such that $\operatorname{supp} a \subset \widetilde{\Omega}_{\ell_1,\ell_2}$, where $\widetilde{\Omega}_{\ell_1,\ell_2}$ is the (ℓ_1, ℓ_2)-enlargement of $\widetilde{\Omega}$, the ϵ_0-enlargement of Ω, defined respectively in (4.28) and in (4.26), with ϵ_0 as defined in (4.25).

(2) (Size condition) There is a constant $C_q > 0$ such that

$$\|a\|_{L^q(X_1\times X_2)} \le C_q \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega})\right)^{1/q-1/p}.$$

(3) (Further decomposition into rectangle atoms with cancellation) There are underlying dyadic grids $\mathscr{D}_i^a$ on X_i for $i = 1, 2$, such that the function a can be decomposed into *rectangle* (p, q)*-atoms* a_R associated to a dyadic rectangle $R = Q_1\times Q_2$, with $Q_i \in \mathscr{D}_i^a$ and satisfying the following conditions.

   (i) (Support condition) Let $C_i = 2(A_0^{(i)})^2 > 0$ for $i = 1, 2$. For all rectangle atoms a_R, we have that

$$\operatorname{supp} a_R \subset C_1 2^{\ell_1} Q_1 \times C_2 2^{\ell_2} Q_2 \subset \widetilde{\Omega}_{\ell_1,\ell_2}.$$

   (ii) (Cancellation condition on each variable)

$$\int_{X_i} a_R(x_1, x_2)\, d\mu_i(x_i) = 0 \text{ for a.e. } x_j \in X_j \text{ and } (i, j) \in \{(1, 2), (2, 1)\}.$$

   (iii-a) (Decomposition and size condition for $2 \le q < \infty$) If $q \ge 2$ then

$$a = \sum_{R\in m(\Omega)} a_R$$

   and there is a constant $C_q > 0$ such that

$$\Big[\sum_{R\in m(\Omega)} \|a_R\|^q_{L^q(\widetilde{X})}\Big]^{1/q} \le C_q \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega})\right)^{1/q-1/p}.$$

(iii-b) (Decomposition and size condition for $1 < q < 2$) If $q \in (1,2)$ then

$$a = \sum_{R\in m_1(\Omega)} a_R + \sum_{R\in m_2(\Omega)} a_R,$$

and for all $\delta > 0$, there exists a constant $C_{q,\delta} > 0$ such that we have, for each $(i,j) \in \{(1,2),(2,1)\}$,

$$\Big[\sum_{R\in m_i(\Omega)} \Big(\frac{\ell(Q_j)}{\ell(\widehat{Q}_j)}\Big)^{\delta} \|a_R\|^q_{L^q(\widetilde{X})}\Big]^{\frac{1}{q}} \le C_{q,\delta}\left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega})\right)^{\frac{1}{q}-\frac{1}{p}}.$$

The constants ϵ_0, C_q, $C_{q,\delta}$ depend only on the geometric constants of X_i for $i = 1,2$ and as indicated on q and δ. The families of rectangles $m(\Omega)$, $m_i(\Omega)$ for $i = 1,2$ were defined on page 1213. We will call the integers $\ell_i \ge 0$ *enlargement parameters* of the atom.

We remark that, when $\widetilde{X} = \mathbb{R}^n\times\mathbb{R}^m$, $(p,2)$-atoms with conditions (i), (ii) and (iii-a) (with $q = 2$, and $\ell_1 = \ell_2 = 0$) were introduced by R. Fefferman [Fe1]. When (X_i, d_i, μ_i) are spaces of homogeneous type with the quasi-metric d_i satisfying the regularity condition (2.3) and the doubling measure μ_i satisfying a reverse doubling condition (1.2), for $i = 1,2$, the (p,q)-atoms with $\ell_1 = \ell_2 = 0$ were defined in [HLLin, Definition 2.3]. In [KLPW, Definition 5.3] the product $(1,2)$-atoms as in Definition 5.2 were used when $\ell_1 = \ell_2 = 0$.

Note that there are no wavelets and no regularity parameters η_i involved in the definition of the (p,q)-atoms. In item (3) of Definition 5.2 any pair of underlying dyadic grids is acceptable, as long as properties (i)–(iii) are met. However we will be interested in the situation when the underlying dyadic grids $\mathscr{D}_i^a$ belong to a regular family of dyadic grids on X_i that contains all possible reference dyadic grids $\mathscr{D}_i^W$ for all possible wavelets on X_i for $i = 1,2$.

The open set Ω is a placeholder and the maximal rectangles in item (3) do refer to Ω. The positive constants $C_i = 2(A_0^{(i)})^2$ for $i = 1,2$ in item (3)(i) are the same for all (p,q)-atoms. However the enlargement parameters, ℓ_i for $i = 1,2$, in item (1) may change from (p,q)-atom to (p,q)-atom. We will see, in the proof of the atomic decomposition for $H^p(\widetilde{X})$, that the (p,q)-atoms will be indexed by a parameter $j \in \mathbb{Z}$ and by the enlargement parameters $\ell_i \ge 0$ for $i = 1,2$.

We can now define atomic product Hardy spaces $H^{p,q}_{\rm at}(\widetilde{X})$.

Definition 5.3 (Atomic product Hardy spaces)**.** For $i = 1$, 2, let (X_i, d_i, μ_i) be spaces of homogeneous type in the sense of Coifman and Weiss as described in the Introduction, with quasi-metrics d_i and Borel-regular doubling measures μ_i. Let ω_i be an upper dimension for X_i, and let η_i be the exponent of regularity of a family of Auscher-Hytönen wavelets on X_i. Let $p_0 := \max\{\omega_i/(\omega_i+\eta_i) : i = 1,2\}$. Suppose that $p_0 < p \le 1$ and $1 < q < \infty$. Then

$$H^{p,q}_{\rm at}(\widetilde{X}) := \{f \in (\overset{\circ}{G})' : \ f = \sum_{j=-\infty}^{\infty} \lambda_j a_j, \quad \sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty\},$$

where for each $j \in \mathbb{Z}$, the function a_j is a (p,q)-atom with underlying dyadic grids $\mathscr{D}_i^{a_j}$ for $i = 1, 2$, belonging to a regular family of dyadic grids on X_i that contains the reference dyadic grids of all possible Auscher-Hytönen wavelets on X_i. Furthermore, the convergence of the series is in $(\overset{\circ}{G})'$. We define a semi-norm on $H^{p,q}_{\rm at}(\widetilde{X})$ as follows

$$\|f\|_{H^{p,q}_{\rm at}(\widetilde{X})} := \inf\Big\{\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{\frac{1}{p}} : f = \sum_{j=-\infty}^{\infty} \lambda_j a_j\Big\},$$

where the infimum is taken over all possible atomic decompositions of f.

Recall that $(\overset{\circ}{G})'$ is short for the spaces of distributions $\big(\overset{\circ}{G}_{\widetilde{\eta}}(\widetilde{\beta}', \widetilde{\gamma}')\big)'$, where we have fixed $\beta_i', \gamma_i' \in (0, \eta_i)$ and η_i is the regularity exponent of the Auscher-Hytönen wavelets on X_i for $i = 1, 2$. In the one parameter theory, in the corresponding definition of atomic Hardy space $H^p_{\rm at}(X)$, it is required that $f \in (\mathcal{C}_{\frac{1}{p}-1}(X))'$ the dual of the Campanato space, see [HHL1, discussion surrounding Lemma 2.6 on p.3448].

The underlying dyadic grids can change from atom to atom. The underlying dyadic grids $\mathscr{D}_i^a$ for $i = 1, 2$, for a given atom a, can be any dyadic grids belonging to a regular family of dyadic grids on X_i that contains all the reference dyadic grids associated to all possible wavelets on X_i for $i = 1, 2$. In particular they may not coincide with the reference dyadic grids $\mathscr{D}_i^W$ associated to the wavelet basis on X_i for $i = 1, 2$, used in the definition of the product Hardy space $H^p(\widetilde{X})$. This ensures that by definition, the product atomic Hardy spaces $H^{p,q}_{\rm at}(\widetilde{X})$ are independent of the reference dyadic grids and wavelets used in the definition of $H^p(\widetilde{X})$. We may as well restrict the regular family of dyadic grids on each X_i in the definition of atomic Hardy spaces to be the collection of reference dyadic grids for all possible wavelets on X_i for $i = 1, 2$.

We will show in Section 5.3 that $H^{p,q}_{\rm at}(\widetilde{X})$ is the same space for all $q > 1$, hence we can safely write $H^p_{\rm at}(\widetilde{X})$. Moreover we will show that $H^p_{\rm at}(\widetilde{X}) = H^p(\widetilde{X})$. In [HHL1] they work with $(p, 2)$-atoms only, and therefore their $H^p_{\rm at}(\widetilde{X})$ is by definition what we denote $H^{p,2}_{\rm at}(\widetilde{X})$. Note that if $f \in H^{p,q}_{\rm at}(\widetilde{X}) \cap L^q(\widetilde{X})$ the convergence of the atomic series also holds in $L^q(\widetilde{X})$ and that $H^{p,q}_{\rm at}(\widetilde{X}) \cap L^q(\widetilde{X})$ is dense in $H^{p,q}_{\rm at}(\widetilde{X})$ in the atom (semi)norm.

5.3. Main theorem on atomic decomposition, and corollaries. The main result in this section, Theorem 5.4, is to show that $L^q(\widetilde{X}) \cap H^p(\widetilde{X})$ has an atomic decomposition. This theorem was cited and used in [KLPW, Theorem 5.4], in the case $p = 1$ and $q = 2$, to establish dyadic structure theorems for $H^1(\widetilde{X})$ and $\mathrm{BMO}(\widetilde{X})$.

Theorem 5.4 was stated in the introduction and called Main Theorem. For the convenience of the reader we restate the theorem here.

Theorem 5.4 (Main Theorem). *For $i = 1, 2$, let (X_i, d_i, μ_i) be spaces of homogeneous type in the sense of Coifman and Weiss as described in the Introduction, with quasi-metrics d_i and Borel-regular doubling measures μ_i. Let ω_i be an upper dimension for X_i, let η_i be the exponent of regularity of the Auscher-Hytönen wavelets used in the construction of the Hardy space $H^p(\widetilde{X})$, let $p_0 := \max\{\omega_i/(\omega_i + \eta_i) : i = 1, 2\}$, and let $\mathscr{D}_i^W$ be the reference dyadic grids for the wavelets in X_i. Suppose that $p_0 < p \le 1$, $1 < q < \infty$, and $f \in L^q(\widetilde{X})$. Then $f \in H^p(\widetilde{X})$ if and only if f has an atomic decomposition, that is,*

$$f = \sum_{j=-\infty}^{\infty} \lambda_j a_j. \tag{5.4}$$

Here, first the functions a_j are (p, q)-atoms with respect to an underlying dyadic grid $\mathscr{D}_i^{a_j}$ belonging to a regular family of dyadic grids on X_i that contains all possible reference grids for wavelets on X_i for $i = 1, 2$, second $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$, and third the series converges in $L^q(\widetilde{X})$. Moreover, the series also converges in $H^p(\widetilde{X})$ and

$$\|f\|_{H^p(\widetilde{X})} \sim \inf\Big\{\Big(\sum_{j=-\infty}^{\infty} |\lambda_j|^p\Big)^{\frac{1}{p}} : f = \sum_{j=-\infty}^{\infty} \lambda_j a_j\Big\},$$

where the infimum is taken over all decompositions as in (5.4) *and the implicit constants are independent of the $L^q(\widetilde{X})$ and $H^p(\widetilde{X})$-(semi)norms of f, and depend only on the geometric constants of X_i for $i = 1, 2$.*

We repeat, the underlying dyadic grid $\mathscr{D}_i^a$ needed for each atom may or not coincide with the reference dyadic grid $\mathscr{D}_i^W$ associated to the underlying Auscher-Hytönen wavelets on X_i for $i = 1, 2$, used in the definition of $H^p(\widetilde{X})$.

As corollaries of the Main Theorem 5.4 we conclude first that $H^{p,q}_{\rm at}(\widetilde{X})$ coincides with $H^p(\widetilde{X})$ for all $q > 1$, and second that the Hardy spaces $H^p(\widetilde{X})$ defined via specific Auscher-Hytönen wavelet bases based on specific reference dyadic grids on X_i for $i = 1, 2$, are indeed independent of the choices of both wavelet bases and reference dyadic grids.

Corollary 5.5 (Corollary A in the Introduction). *For all $1 < q < \infty$ and $p_0 < p \le 1$ then*

$$H^{p,q}_{\rm at}(\widetilde{X}) = H^p(\widetilde{X}).$$

Proof. By Theorem 5.4 for each $q > 1$,

$$H^{p,q}_{\rm at}(\widetilde{X}) \cap L^q(\widetilde{X}) = L^q(\widetilde{X}) \cap H^p(\widetilde{X}),$$

the closure of the right-hand-side in the H^p-(semi)norm is $H^p(\widetilde{X})$, and the closure of the left-hand-side in the atom (semi)norm is $H^{p,q}_{\rm at}(\widetilde{X})$. Both (semi)norms are equivalent by Theorem 5.4, therefore we conclude that $H^p(\widetilde{X}) = H^{p,q}_{\rm at}(\widetilde{X})$. This is precisely what we wanted to prove. □

For any p with $p_0 < p \leq 1$ we now define $H^p_{\rm at}(\widetilde{X})$, the *atomic product* H^p-*space*, by

$$H^p_{\rm at}(\widetilde{X}) := H^{p,q}_{\rm at}(\widetilde{X}), \tag{5.5}$$

for any given $q > 1$. The atomic product H^p-space is well-defined by Corollary 5.5.

Corollary 5.6 (Corollary B in the Introduction)**.** *Let $p > p_0$. Then the Hardy spaces $H^p(\widetilde{X})$ as defined in* [HLW] *are independent of the particular choices of the Auscher-Hytönen wavelets and of the reference dyadic grids used in their construction.*

Proof. Given $p > p_0$, define $H^p(\widetilde{X})$ as in [HLW], using a particular choice of reference dyadic grids, $\mathscr{D}^W_i$ for $i = 1, 2$, and a particular choice of basis of Auscher-Hytönen wavelets defined on those grids. For $p > 1$ we already know that $H^p(\widetilde{X}) = L^p(\widetilde{X})$; see [HLW]. For $p_0 < p \leq 1$, choose $q > 1$. By the Main Theorem, the set $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ coincides with the set of functions in $L^q(\widetilde{X})$ that have atomic decompositions in terms of (p, q)-atoms. Each (p, q)-atom a in a decomposition, has underlying dyadic grids $\mathscr{D}^a_i$ for $i = 1, 2$, possibly different from $\mathscr{D}^W_i$, but belonging to a regular families of dyadic grids on X_i that contain all possible reference dyadic grids on X_i. The atomic decompositions are *a priori* unrelated to the Auscher-Hytönen wavelets and their reference dyadic grids. Further, $H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ is dense in $H^p(\widetilde{X})$ in the H^p-(semi)norm. Note that the closure is independent of the choice of square function (which depends on the choice of wavelets and hence of reference dyadic grids) in the H^p-(semi)norm, because we can instead use the equivalent atom (semi)norm. Thus $H^p(\widetilde{X})$ is independent of the particular choice of reference dyadic grids and the particular choice of basis of Auscher-Hytönen wavelets defined on these grids, as required. □

As a further corollary of these results and the duality theorems, Theorem 4.3 and Theorem 4.5, we conclude that the Carleson measure spaces $\mathrm{CMO}^p(\widetilde{X})$, the space of bounded mean oscillation $\mathrm{BMO}(\widetilde{X})$, and the space of vanishing mean oscillation $\mathrm{VMO}(\widetilde{X})$ are all independent of the chosen wavelets and reference dyadic grids.

Corollary 5.7 (Corollary C in the Introduction)**.** *Let $p_0 < p \leq 1$. Then the Carleson measure spaces* $\mathrm{CMO}^p(\widetilde{X})$*, the space of bounded mean oscillation* $\mathrm{BMO}(\widetilde{X})$*, and the space of vanishing mean oscillation* $\mathrm{VMO}(\widetilde{X})$*, as defined in* [HLW]*, are independent of the particular choices of the Auscher-Hytönen wavelets and of the reference dyadic grids used in their construction.*

Proof. By Theorem 4.3, if $p_0 < p \leq 1$ then $\mathrm{CMO}^p(\widetilde{X})$ is the dual of $H^p(\widetilde{X})$. By Corollary 5.6, the Hardy space $H^p(\widetilde{X})$ is independent of the particular choice of reference dyadic grids and the particular choice of basis of Auscher-Hytönen wavelets defined on these grids, therefore so will be its dual $\mathrm{CMO}^p(\widetilde{X})$. Also by Definition 4.2 we know that $\mathrm{BMO}(\widetilde{X}) = \mathrm{CMO}^1(\widetilde{X})$, and by Theorem 4.5 we know that $\big(\mathrm{VMO}(\widetilde{X})\big)' = H^1(\widetilde{X})$, hence since $H^1(\widetilde{X})$ is independent of chosen reference dyadic grids and wavelets so will be $\mathrm{BMO}(\widetilde{X})$ and $\mathrm{VMO}(\widetilde{X})$. □

5.4. Proof of the main theorem. In the proof of the Main Theorem 5.4, given a function $f \in H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ we will show it can be decomposed into (p,q)-atoms based upon the reference dyadic grids, $\mathscr{D}_i^W$ for $i = 1,2$, corresponding to the underlying wavelets. For the converse, it will suffice to verify that a given (p,q)-atom a, based on possibly different dyadic grids $\mathscr{D}_i^a$ belonging to a regular family of dyadic grids that contains all possible reference dyadic grids for wavelets on X_i for $i = 1,2$, must belong to $H^p(\widetilde{X})$ with uniform control on its H^p-(semi)norm. We will have to carefully balance the geometry on both sets of dyadic grids with the size, support, and cancellation properties of the functions $\varphi^{\gamma,\overline{C}_i}_{\ell,k_i,\alpha_i}$ for $i = 1,2$ (building blocks for the wavelet $\psi^{k_i}_{\alpha_i}$ found in Lemma 4.3) and the rectangular (p,q)-atoms a_R; for example, when estimating the inner product $\langle \varphi^{\gamma,\overline{C}_1}_{\ell,k,\alpha_1}(\cdot), a_R(\cdot,x_2)\rangle_{L^2(X_1)}$ for μ_2-a.e. $x_2 \in X_2$, as we do on page 1230. To achieve this balance we will choose $\overline{C}_i = C_i 2^{\ell_i}$ where $C_i = 2(A_0^{(i)})^2$ and ℓ_i for $i = 1,2$ are the enlargement parameters appearing in the definition of the (p,q)-atom.

Proof of Theorem 5.4. $(\Rightarrow)$ Following the proof of Theorem 4.6, for any function $f \in H^p(\widetilde{X}) \cap L^q(\widetilde{X})$, we have by (4.16) and (4.22), that for some sufficiently large $\gamma_i > 0$ (in fact for $\gamma_i > \omega_i(1/p + 1/q')$), letting $\overline{C}_i = 1$, and denoting $\varphi^{\gamma_i,1}_{\ell_i,k_i,\alpha_i} = \varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}$, for $i = 1,2$,

$$\begin{aligned} f(x_1,x_2) &= \sum_{\ell_1,\ell_2\ge 0} 2^{-\ell_1\gamma_1-\ell_2\gamma_2} f_{\ell_1,\ell_2}(x_1,x_2) \\ &= \sum_{\ell_1,\ell_2\ge 0} 2^{-\ell_1\gamma_1-\ell_2\gamma_2} \sum_{j\in\mathbb{Z}} \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\, \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}(x_1)\, \kappa_2\, \varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}(x_2). \end{aligned}$$

Here the series converges unconditionally in the $L^q(\widetilde{X})$-norm. As before, the constants $\kappa_i := \sqrt{\mu_i\big(B(y^{k_i}_{\alpha_i},\delta^{k_i})\big)}$ for $i = 1,2$, the dyadic rectangle $R^{k_1,k_2}_{\alpha_1,\alpha_2} := Q^{k_1}_{\alpha_1}\times Q^{k_2}_{\alpha_2}$, with $Q^{k_i}_{\alpha_i} \in \mathscr{D}^W_i$ for $i = 1,2$, and the set $\mathcal{B}_j$ was defined by (4.21). We now set

$$f(x_1,x_2) \;:=\; \sum_{\ell_1,\ell_2\ge 0}\sum_{j\in\mathbb{Z}} 2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}\, a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}(x_1,x_2), \tag{5.6}$$

where the functions $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ will be (p,q)-atoms with respect to the reference dyadic grids $\mathscr{D}^W_i$ for $i = 1,2$ associated to the wavelets (as shown below), provided γ_1 and γ_2 are sufficiently large, and are defined by

$$a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}(x_1,x_2) := \frac{1}{\lambda_{j,\ell_1,\ell_2}} \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle \kappa_1\, \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}(x_1)\, \kappa_2\, \varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}(x_2),$$

and the coefficients $\lambda_{j,\ell_1,\ell_2}$ are defined differently according to whether $q < 2$ or not.

First, when $2 \le q < \infty$, define the coefficient $\lambda_{j,\ell_1,\ell_2}$ as follows:

$$\lambda_{j,\ell_1,\ell_2} := 2^{\ell_1\omega_1+\ell_2\omega_2} \|S(f_{\mathcal{B}_j})\|_{L^q(\widetilde{X})} \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{5.7}$$

Second, when $1 < q < 2$, define the coefficient $\lambda_{j,\ell_1,\ell_2}$ as follows:

$$\lambda_{j,\ell_1,\ell_2} := 2^{\ell_1\omega_1+\ell_2\omega_2} \|S(f_{\mathcal{B}_j})\|_{L^2(\widetilde{X})} \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{\frac{1}{p}-\frac{1}{2}}. \tag{5.8}$$

Here $f_{\mathcal{B}_j}$ was defined in (4.31), and hence

$$S(f_{\mathcal{B}_j}) = \Big(\sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j} \left|\left\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\right\rangle\right|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}\Big)^{\frac{1}{2}},$$

where $\widetilde{\psi}^{k_i}_{\alpha_i} = \psi^{k_i}_{\alpha_i}/\kappa_i$ denotes the normalized wavelets for $i = 1, 2$. The open set $\widetilde{\Omega}_{j,\ell_1,\ell_2}$ is the (ℓ_1, ℓ_2)-enlargement of $\widetilde{\Omega}_j$ defined in (4.28), the open set $\widetilde{\Omega}_j$ is the ϵ_0-enlargement of Ω_j defined in (4.24), and the level set Ω_j is defined in (4.18). The constant $\epsilon_0 > 0$ was defined in (4.25) and is purely dependent on the geometric constants of the spaces X_i for $i = 1, 2$.

Notice that when $1 < q < \infty$ estimate (4.36) provides

$$\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}\|^q_{L^q(\widetilde{X})} \lesssim_q \lambda^{-q}_{j,\ell_1,\ell_2} 2^{q(\ell_1\omega_1+\ell_2\omega_2)}\|S(f_{\mathcal{B}_j})\|^q_{L^q(\widetilde{X})}, \tag{5.9}$$

where the similarity depends only on the geometric constants of X_i for $i = 1, 2$ and on $q > 1$.

When $2 \le q < \infty$, using (5.7), the definition of the coefficient $\lambda_{j,\ell_1,\ell_2}$ provides the following L^q-estimate for the atom:

$$\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}\|^q_{L^q(\widetilde{X})} \lesssim_q \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{1-\frac{q}{p}}. \tag{5.10}$$

In particular when $q = 2$ we obtain the following L^2-estimate for the atom:

$$\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}\|^2_{L^2(\widetilde{X})} \lesssim \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{1-\frac{2}{p}}. \tag{5.11}$$

We now verify that the functions $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ are (p, q)-atoms with respect to the reference dyadic grids $\mathscr{D}^W_i$ for $i = 1, 2$ associated to the underlying wavelets, with the open set Ω_j playing the role of Ω in Definition 5.2, and with enlargement parameters $\ell_1, \ell_2 \ge 0$.

First we check that $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ satisfies condition (1) of Definition 5.2. Recall that $\varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}(x_i)$ is supported on the ball $B(y^{k_i}_{\alpha_i}, 2(A^{(i)}_0)^2\, 2^{\ell_i}\delta^{k_i}) \subset X_i$ for each $i = 1, 2$. Hence, if $R \in \mathcal{B}_j$, then the support of $\varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}(x_1)\,\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}(x_2)$ is contained in the open set $\widetilde{\Omega}_{j,\ell_1,\ell_2} = (\widetilde{\Omega}_j)_{\ell_1,\ell_2}$, as explained on page 1205. Note that since

$f \in L^q(\widetilde{X})$ for $1 < q < \infty$, then Ω_j and $\widetilde{\Omega}_{j,\ell_1,\ell_2}$ have finite measure. More precisely, by estimates (4.29) and (4.27) and by Tchebichev's inequality (4.20),

$$\begin{aligned}\mu(\widetilde{\Omega}_{j,\ell_1,\ell_2}) &\lesssim (\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\Omega_j)\\ &\leq (\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}2^{-jq}\|S(f)\|^q_{L^q(\widetilde{X})}\\ &\lesssim_q (\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}2^{-jq}\|f\|^q_{L^q(\widetilde{X})} < \infty.\end{aligned}$$

Thus condition (1) of Definition 5.2 holds.

Second we verify that $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ satisfies condition (2) of Definition 5.2. For $2 \leq q < \infty$ this is estimate (5.10). For $1 < q < 2$, since $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ is supported in $\widetilde{\Omega}_{j,\ell_1,\ell_2}$, applying Hölder's inequality with exponent $s = 2/q > 1$, and using (4.29) and the L^2-estimate (5.11), yields

$$\begin{aligned}\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}\|_{L^q(\widetilde{X})} \quad &\leq \quad \|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}\|_{L^2(\widetilde{X})}\,\mu(\widetilde{\Omega}_{j,\ell_1,\ell_2})^{\frac{1}{q}-\frac{1}{2}}\\ \lesssim\ &\big((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\big)^{\frac{1}{2}-\frac{1}{p}}\big((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\big)^{\frac{1}{q}-\frac{1}{2}}\\ \lesssim\ &\big((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\big)^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

As a consequence, we get that $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ satisfies condition (2) of Definition 5.2.

Third, it remains to check that $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ satisfies condition (3) of Definition 5.2. To see this, we can further decompose $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ into rectangular atoms $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}$ defined for each $\widetilde{x} = (x_1, x_2)$ by

$$a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}(\widetilde{x}) := \frac{1}{\lambda_{j,\ell_1,\ell_2}} \sum_{\substack{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j,\\ \tau(R)=\overline{R}}} \langle f, \psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle\, \kappa_1\, \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}(x_1)\, \kappa_2\, \varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}(x_2),$$

where $\overline{R} = \overline{Q}_1 \times \overline{Q}_2$ with $\overline{Q}_i \in \mathscr{D}^W_i$, a dyadic cube associated to the wavelets on X_i for $i = 1, 2$. Here $\tau : \mathcal{B}_j \to m(\Omega_j)$ denotes a function that assigns to each $R \in \mathcal{B}_j$ a rectangle $\tau(R) = \overline{R} \in m(\Omega_j)$, so that $R \subset \overline{R}$. This will be important when verifying condition (3)(iii-a) in Definition 5.2. Likewise when verifying condition (3)(iii-b) in Definition 5.2 we will assign each $R \in \mathcal{B}_j$ to only one $\overline{R} \in m_1(\Omega_j) \cup m_2(\Omega_j)$ with $R \subset \overline{R}$.

We can verify that supp $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}} \subset 2(A^{(1)}_0)^2\, 2^{\ell_1}\overline{Q}_1 \times 2(A^{(2)}_0)^2\, 2^{\ell_2}\overline{Q}_2$, by definition of the rectangle atoms and the support conditions of the functions $\varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}$, for $i = 1, 2$. We deduce that $\int_{X_i} a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}(x_1, x_2)\, d\mu_i(x_i) = 0$ for a.e. $x_j \in X_j$, by the cancellation conditions of the functions $\varphi^{\gamma_i}_{\ell_i,k_i,\alpha_i}$ for $(i, j) \in \{(1, 2), (2, 1)\}$, and the facts that the integrand $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}} \in L^q(\widetilde{X})$ for $q > 1$ and has compact support. These show that the support and cancellation conditions (3)(i) and (3)(ii) of Definition 5.2 hold, with support constants $C_i = 2(A_0^{(i)})^2$, as required, and enlargement constants $\ell_i \geq 0$, for $i = 1, 2$.

We now show that $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}$ satisfies the decomposition and size conditions in (3)(iii-a) when $2 \leq q < \infty$, and (3)(iii-b) when $1 < q < 2$, of Definition 5.2.

For $2 \leq q < \infty$, first observe that $a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2} = \sum_{\overline{R}\in m(\Omega_j)} a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}$, this is true because each $R \in \mathcal{B}_j$ is assigned to exactly one $\overline{R} \in m(\Omega)$, namely to $\overline{R} = \tau(R)$. Second, we have by definition of the rectangular atom and the triangle inequality

$$\begin{aligned}\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|_{L^q(\widetilde{X})} &= \sup_{g:\ \|g\|_{L^{q'}(\widetilde{X})}\leq 1} \left|\langle a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}, g\rangle\right| \\ &\leq \sup_{g:\ \|g\|_{L^{q'}(\widetilde{X})}\leq 1} \lambda^{-1}_{j,\ell_1,\ell_2} \sum_{\substack{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j,\\ \tau(R)=\overline{R}}} \left|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\right| \kappa_1^2\kappa_2^2 \left|\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\right|.\end{aligned}$$

Therefore, first raising to the q power, and second using the Cauchy-Schwarz inequality on the sum together with Lemma 4.9 as we did when estimating (4.34), we conclude that

$$\begin{aligned}&\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|^q_{L^q(\widetilde{X})} \\ &\lesssim_q \sup_{g:\ \|g\|_{L^{q'}(\widetilde{X})}\leq 1} \lambda^{-q}_{j,\ell_1,\ell_2} \Bigg| \sum_{\substack{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j,\\ \tau(R)=\overline{R}}} \left|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\right| \kappa_1^2\kappa_2^2 \left|\langle \varphi^{\gamma_1}_{\ell_1,k_1,\alpha_1}\varphi^{\gamma_2}_{\ell_2,k_2,\alpha_2}, g\rangle\right| \Bigg|^q \\ &\lesssim_q 2^{(\ell_1\omega_1+\ell_2\omega_2)q} \lambda^{-q}_{j,\ell_1,\ell_2} \int_{\widetilde{X}} \Bigg| \sum_{\substack{R=R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j,\\ \tau(R)=\overline{R}}} \left|\langle f, \widetilde{\psi}^{k_1}_{\alpha_1}\widetilde{\psi}^{k_2}_{\alpha_2}\rangle\right|^2 \chi_{R^{k_1,k_2}_{\alpha_1,\alpha_2}}(\widetilde{x}) \Bigg|^{\frac{q}{2}} d\mu(\widetilde{x}).\end{aligned}$$

We now add this estimate over all $\overline{R} \in m(\Omega_j)$, note that the power $q/2 \geq 1$ can be pulled out of the sum (namely $\sum_k |a_k|^{q/2} \leq (\sum_k |a_k|)^{q/2}$), and remember that each $R \in \mathcal{B}_j$ is assigned to exactly one $\overline{R} \in m(\Omega_j)$ that contains it, and get

$$\begin{aligned}\sum_{\overline{R}\in m(\Omega_j)} \left\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\right\|^q_{L^q(\widetilde{X})} &\lesssim_q 2^{(\ell_1\omega_1+\ell_2\omega_2)q}\lambda^{-q}_{j,\ell_1,\ell_2}\|S(f_{\mathcal{B}_j})\|^q_{L^q(\widetilde{X})} \\ &\lesssim_q \left((\ell_1\omega_1+\ell_2\omega_2)\,2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{1-\frac{q}{p}}, \qquad (5.12)\end{aligned}$$

where in the last inequality we used the definition (5.7) of $\lambda_{j,\ell_1,\ell_2}$. This proves condition (3)(iii-a) of Definition 5.2.

For $1 < q < 2$, applying Hölder's inequality and the Journé-type covering lemma, we will show that condition (3)(iii-b) of Definition 5.2 holds. First we observe that in this case the decomposition

$$a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2} = \sum_{\overline{R}\in m_1(\Omega_j)} a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}} + \sum_{\overline{R}\in m_2'(\Omega_j)} a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}$$

holds. Here the second sum is over $m_2'(\Omega_j) := m_2(\Omega_j) \setminus m_1(\Omega_j)$ to avoid duplicates. The decomposition is true because this time we assign each $R \in \mathcal{B}_j$ to exactly one $\overline{R} \in m_1(\Omega_j) \cup m_2(\Omega_j)$, namely $\overline{R} = \tau(R)$ where the function $\tau : \mathcal{B}_j \to m_1(\Omega_j) \cup m_2(\Omega_j)$. Second, let us show that given $\delta > 0$ there is a constant $C_{q,\delta} > 0$ such that

$$\sum_{\overline{R}\in m_1(\Omega_j)} \Big(\frac{\ell(Q_2)}{\ell(\widehat{Q}_2)}\Big)^{\delta} \|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|^q_{L^q(\widetilde{X})} \le C_{q,\delta} \left((\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{1-\frac{q}{p}}.$$

A similar argument will take care of the sum over $\overline{R} \in m_2(\Omega_j)$, and hence over $\overline{R} \in m_2'(\Omega_j)$. First, using Hölder's inequality with exponent $s = 2/q > 1$, the support property of the rectangular atoms, and the doubling condition of the measures (as in (3.9)), we get that

$$\|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|^q_{L^q(\widetilde{X})} \lesssim \|a_{j,\ell_1,\ell_2,\overline{R}}\|^q_{L^2(\widetilde{X})}\big(2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\overline{R})\big)^{\frac{2-q}{2}}.$$

Second, substituting this estimate and using Hölder's inequality in the sum with exponents $s = 2/q$ and $s' = 2/(2-q)$, we get

$$\begin{aligned}
&\sum_{\overline{R}\in m_1(\Omega_j)} \Big(\frac{\ell(Q_2)}{\ell(\widehat{Q}_2)}\Big)^{\delta} \|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|^q_{L^q(\widetilde{X})} \\
&\lesssim \sum_{\overline{R}\in m_1(\Omega_j)} \Big(\frac{\ell(Q_2)}{\ell(\widehat{Q}_2)}\Big)^{\delta} \|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|^q_{L^2(\widetilde{X})}\big(2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\overline{R})\big)^{\frac{2-q}{2}} \\
&\lesssim \Big(2^{\ell_1\omega_1+\ell_2\omega_2}\sum_{\overline{R}\in m_1(\Omega_j)} \Big(\frac{\ell(Q_2)}{\ell(\widehat{Q}_2)}\Big)^{\frac{2\delta}{2-q}}\mu(\overline{R})\Big)^{\frac{2-q}{2}} \Big(\sum_{\overline{R}\in m_1(\Omega_j)} \|a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2,\overline{R}}\|^2_{L^2(\widetilde{X})}\Big)^{\frac{q}{2}} \\
&\lesssim_{q,\delta} \big(2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\Omega_j)\big)^{1-\frac{q}{2}}\big((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\big)^{\frac{q}{2}-\frac{q}{p}} \\
&\lesssim_{q,\delta} \big((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\big)^{1-\frac{q}{p}}.
\end{aligned}$$

We used the Journé-type covering lemma with $\delta' = \frac{2\delta}{2-q} > 0$, and estimate (5.12) (for $q = 2$), in the third inequality. In the last inequality we used the fact that $\mu(\widetilde{\Omega}_j) \sim \mu(\Omega_j)$. Altogether we obtain the desired atomic decomposition for f.

Finally by computations similar to those in the proof of Theorem 4.6 we conclude that when $f \in H^p(\widetilde{X}) \cap L^q(\widetilde{X})$ then $\inf \sum_{j\in\mathbb{Z}} |\lambda_j|^p \le C\,\|f\|^p_{H^p(\widetilde{X})}$, where the infimum is taken over all decompositions of the form $f = \sum_{j\in\mathbb{Z}} \lambda_j a_j$, the functions a_j are (p,q)-atoms, and $\sum_{j\in\mathbb{Z}}|\lambda_j|^p < \infty$. More precisely, it suffices to show that for the decomposition we just proved, namely $f(x_1,x_2) = \sum_{j\in\mathbb{Z};\,\ell_1,\ell_2\ge 0} 2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}a^{\gamma_1,\gamma_2}_{j,\ell_1,\ell_2}(x_1,x_2)$, the following inequality holds:

$$\sum_{j\in\mathbb{Z};\,\ell_1,\ell_2\ge 0}|2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}|^p \lesssim_q \|S(f)\|^p_{L^p(\widetilde{X})}. \tag{5.13}$$

When $1<q<2$, according to definition (5.8) we get, using that the square function is bounded on $L^2(\widetilde{X})$, that

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\ \ell_1,\ell_2\ge 0}}|2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}|^p &= \sum_{\substack{j\in\mathbb{Z}\\ \ell_1,\ell_2\ge 0}}\|S(f_{\mathcal{B}_j})\|^p_{L^2(\widetilde{X})}2^{\ell_1 p(\omega_1-\gamma_1)}2^{\ell_2 p(\omega_2-\gamma_2)}\\ &\qquad\times\Big((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\Big)^{1-\frac{p}{2}}\\ &\lesssim \sum_{j\in\mathbb{Z}}\|f_{\mathcal{B}_j}\|^p_{L^2(\widetilde{X})}\mu(\widetilde{\Omega}_j)^{1-\frac{p}{2}}\sum_{\ell_1,\ell_2\ge 0}2^{\ell_1 p(\omega_1-\gamma_1)}2^{\ell_2 p(\omega_2-\gamma_2)}\\ &\qquad\times 2^{\ell_1\omega_1(1-\frac{p}{2})}2^{\ell_2\omega_2(1-\frac{p}{2})}(\ell_1\omega_1+\ell_2\omega_2)^{1-\frac{p}{2}}.\end{aligned}$$

The series over ℓ_1,ℓ_2 converges if we choose $\gamma_i>\omega_i\big(\frac{1}{p}+\frac{1}{2}\big)$ for $i=1,2$. Therefore,

$$\sum_{\substack{j\in\mathbb{Z}\\ \ell_1,\ell_2\ge 0}}|2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}|^p \lesssim \sum_{j\in\mathbb{Z}}2^{pj}\mu(\widetilde{\Omega}_j\setminus\Omega_{j+1})^{\frac{p}{2}}\mu(\widetilde{\Omega}_j)^{1-\frac{p}{2}} \lesssim \sum_{j\in\mathbb{Z}}2^{pj}\mu(\widetilde{\Omega}_j).$$

In the first inequality we have used the following estimate for the L^2-norm of $f_{\mathcal{B}_j}$:

$$\begin{aligned}\|f_{\mathcal{B}_j}\|^2_{L^2(\widetilde{X})} &= \sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j}\Big|\langle f,\psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle\Big|^2\\ &\le 2\sum_{R^{k_1,k_2}_{\alpha_1,\alpha_2}\in\mathcal{B}_j}\frac{\mu\big(R^{k_1,k_2}_{\alpha_1,\alpha_2}\cap(\widetilde{\Omega}_j\backslash\Omega_{j+1})\big)}{\mu_1(Q^{k_1}_{\alpha_1})\mu_2(Q^{k_2}_{\alpha_2})}\Big|\langle f,\psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}\rangle\Big|^2\\ &= 2\|S(f_{\mathcal{B}_j})\|^2_{L^2(\widetilde{\Omega}_j\backslash\Omega_{j+1})} \le 2\|S(f)\|^2_{L^2(\widetilde{\Omega}_j\backslash\Omega_{j+1})} \lesssim 2^{2j}\mu(\widetilde{\Omega}_j\setminus\Omega_{j+1}).\end{aligned}$$

In the above calculation we used Plancherel in the first line, and we used the fact that when $R\in\mathcal{B}_j$ then $2\mu\big(R\cap(\widetilde{\Omega}_j\backslash\Omega_{j+1})\big)>\mu(R)$ in the second line (as shown on page 1208). In the third line, the last inequality holds because if $(x_1,x_2)\notin\Omega_{j+1}$ then $|S(f)(x_1,x_2)|\le 2^{j+1}$.

Finally, recalling that $\mu(\widetilde{\Omega}_j)\lesssim\mu(\Omega_j)$, and using (4.19) we conclude that

$$\sum_{j\in\mathbb{Z};\,\ell_1,\ell_2\ge 0}|2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}|^p \lesssim \sum_{j\in\mathbb{Z}}2^{pj}\mu(\Omega_j)\lesssim\|S(f)\|^p_{L^p(\widetilde{X})}.$$

Therefore inequality (5.13) holds when $1 < q < 2$ whenever the parameters γ_i satisfy the constraint $\gamma_i > \omega_i(\frac{1}{p} + \frac{1}{2})$ for $i = 1, 2$. Notice that in this range $q' > 2$ and $(\frac{1}{p} + \frac{1}{2}) > (\frac{1}{p} + \frac{1}{q'})$, therefore the constraint needed in the proof of Theorem 4.6 on page 1204 is satisfied.

When $q \geq 2$, according to definition (5.7), by a similar argument to that in the proof of Theorem 4.6, specifically using (4.39) and provided that $\gamma_i > \omega_i(\frac{1}{p}+\frac{1}{q'})$ for $i = 1, 2$, we get that

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\ \ell_1,\ell_2\geq 0}} |2^{-\ell_1\gamma_1-\ell_2\gamma_2}\lambda_{j,\ell_1,\ell_2}|^p &= \sum_{\substack{j\in\mathbb{Z}\\ \ell_1,\ell_2\geq 0}} \|S(f_{\mathcal{B}_j})\|^p_{L^q(\widetilde{X})} 2^{\ell_1 p(\omega_1-\gamma_1)} 2^{\ell_2 p(\omega_2-\gamma_2)} \\ &\quad \times \left((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}_j)\right)^{1-\frac{p}{q}} \\ &\lesssim_q \sum_{j\in\mathbb{Z}} 2^{pj}\mu(\widetilde{\Omega}_j)^{\frac{p}{q}}\,\mu(\widetilde{\Omega}_j)^{1-\frac{p}{q}} \\ \times \sum_{\ell_1,\ell_2\geq 0} 2^{\ell_1 p(\omega_1-\gamma_1)} & 2^{\ell_2 p(\omega_2-\gamma_2)} 2^{\ell_1\omega_1(1-\frac{p}{q})} 2^{\ell_2\omega_2(1-\frac{p}{q})} (\ell_1\omega_1+\ell_2\omega_2)^{1-\frac{p}{q}} \\ &\lesssim_q \sum_{j\in\mathbb{Z}} 2^{pj}\mu(\Omega_j) \ \lesssim_q \ \|S(f)\|^p_{L^p(\widetilde{X})}.\end{aligned}$$

We conclude that (5.13) holds when $q \geq 2$ whenever the parameters γ_i satisfy the constraint $\gamma_i > \omega_i(\frac{1}{p}+\frac{1}{q'})$ for $i = 1, 2$. Notice that this is the same constraint needed in the proof of Theorem 4.6 on page 1204. All the constants appearing in the inequalities $\leq$ and similarities $\sim$ depend on the geometric constants of the spaces X_i for $i = 1, 2$, and possibly on the parameters $q > 1$ or $\delta > 0$ as indicated.

($\Leftarrow$) Given an atomic decomposition $f = \sum_{j\in\mathbb{Z}} \lambda_j a_j$ for a function $f \in L^q(\widetilde{X}) \cap H^{p,q}_{\text{at}}(\widetilde{X})$, with $\sum_{j\in\mathbb{Z}} |\lambda_j|^p < \infty$. By definition each product (p, q)-atom a_j has underlying dyadic grids $\mathscr{D}_i^{a_j}$ on X_i for $i = 1, 2$ belonging to regular families of dyadic grids on X_i that contain all the reference dyadic grids for wavelets on X_i. The series is assumed to converge in $L^q(\widetilde{X})$, hence it suffices to verify that there is a constant $C > 0$ such that for all such (p, q)-atoms a

$$\|S(a)\|_{L^p(\widetilde{X})} \leq C. \tag{5.14}$$

The constant $C > 0$ will depend only on the geometric constants of the spaces X_i for $i = 1, 2$ and on p and q, but not on the enlargement parameters $\ell_1, \ell_2 \geq 0$ of Definition 5.2 of the (p, q)-atom. The constant will depend on the structural constants of the atom's underlying dyadic grids, $\mathcal{D}_i^a$ for $i = 1, 2$, via the outer balls dilation constants C_1^i and the ratio of the outer and inner balls dilation constants C_1^i/c_1^i. These quantities will appear when using the doubling property for dilates of cubes as in (3.8). Both quantities are uniformly bounded by a constant depending only on the quasi-triangle constants of the quasi-metric d_i, since the grids $\mathcal{D}_i^a$ are assumed to belong to a regular family of dyadic grids on X_i for $i = 1, 2$. See Definition 3.2 and inequality (3.9).

Once we prove estimate (5.14) for the atoms, if $f \in L^q(\widetilde{X})$ has an atomic decomposition $f = \sum_i \lambda_i a_i$, where the series converges in both L^q-norm and H^p-(semi)norm, then by subadditivity of the square function, and since $p \le 1$, together with (5.14), we conclude that

$$\|f\|^p_{H^p(\widetilde{X})} = \|S(f)\|^p_{L^p(\widetilde{X})} \le \sum_{i\in\mathbb{Z}} |\lambda_i|^p \|S(a_i)\|^p_{L^p(\widetilde{X})} \le C^p \sum_{i\in\mathbb{Z}} |\lambda_i|^p < \infty,$$

which immediately proves the norm estimate $\|f\|_{H^p(\widetilde{X})} \lesssim \inf\{\big(\sum_{i\in\mathbb{Z}} |\lambda_i|^p\big)^{1/p}\}$.

To this end, fix a (p, q)-atom a with $\operatorname{supp} a \subset \Omega_*$, where Ω_* is an appropriate enlargement of the open set Ω in Definition 5.2, more precisely $\Omega_* = \widetilde{\Omega}^{\epsilon_0}_{\ell_1,\ell_2}$ for some enlargement parameters $\ell_1, \ell_2 > 0$. Recall that $\mu(\Omega) \sim \mu(\widetilde{\Omega}^{\epsilon_0}) \le \mu(\widetilde{\Omega}^{\epsilon_0}_{\ell_1,\ell_2}) \lesssim (\ell_1\omega_1 + \ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega}^{\epsilon_0})$, where the last inequality holds by (4.29). Assume the (p, q)-atom has a decomposition $a = \sum_{R\in m(\Omega)} a_R$ when $q \ge 2$, and a decomposition $a = \sum_{R\in m_1(\Omega)} a_R + \sum_{R\in m_2'(\Omega)} a_R$ when $1 < q < 2$. We will work in detail the first case when $q \ge 2$. A similar argument will take care of the second case, $1 < q < 2$; we only need to start with dyadic rectangles R in $m_1(\Omega)$ or in $m_2(\Omega)$.

Let $\widetilde{\Omega}$ be the ϵ-enlargement of Ω and let $\widetilde{\widetilde{\Omega}}$ be the ϵ-enlargement of $\widetilde{\Omega}$, as defined in (4.26) for $\epsilon = 1/2$, that is,

$$\begin{aligned}\widetilde{\Omega} &= \{(x_1, x_2) \in \widetilde{X} : M_s(\chi_\Omega)(x_1, x_2) > 1/2\},\\ \widetilde{\widetilde{\Omega}} &= \{(x_1, x_2) \in \widetilde{X} : M_s(\chi_{\widetilde{\Omega}})(x_1, x_2) > 1/2\}.\end{aligned}$$

It will be useful to keep in mind that $\Omega \subset \widetilde{\Omega} \subset \widetilde{\widetilde{\Omega}}$ and that $\mu(\Omega) \sim \mu(\widetilde{\Omega}) \sim \mu(\widetilde{\widetilde{\Omega}})$ by (4.27).

Moreover, recall that $m_i(\Omega)$ denotes the family of dyadic rectangles $R \subset \Omega$, $R = Q_1 \times Q_2$, with $Q_i \in \mathscr{D}_i^a$, which are maximal in the ith "direction", $i = 1, 2$. We define $m_i(\widetilde{\Omega})$ similarly. Also recall that $m(\Omega)$ is the set of all maximal dyadic rectangles contained in Ω. Then for any $R = Q_1 \times Q_2 \in m(\Omega)$, set $\widehat{R} := \widehat{Q}_1 \times Q_2$. By definition of $\widehat{Q}_1$ in page 1213, one has that $Q_1 \subset \widehat{Q}_1$, $\mu(\widehat{R} \cap \Omega) > \mu(\widehat{R})/2$, and that $\widehat{Q}_1 \in \mathscr{D}_1^a$ is maximal with respect to these properties, hence $\widehat{R} \in m_1(\widetilde{\Omega})$. Similarly, set $\widehat{\widehat{R}} := \widehat{Q}_1 \times \widehat{Q}_2 \in m_2(\widetilde{\widetilde{\Omega}})$, since $Q_2 \subset \widehat{Q}_2$, $\mu(\widehat{\widehat{R}} \cap \widetilde{\Omega}) > \mu(\widehat{R})/2$, and $\widehat{Q}_2 \in \mathscr{D}_2^a$ is maximal with respect to these properties.

The set Ω is a placeholder, rectangles R refer back to Ω, rectangles $\widehat{R}$ to $\widetilde{\Omega}$, and rectangles $\widehat{\widehat{R}}$ to $\widetilde{\widetilde{\Omega}}$. However we want to relate to the support of the (p, q)-atom a for the estimates, hence we will consider the (ℓ_1, ℓ_2)-enlargement of these sets. Specifically echoing the $*$ notation we are using for the support Ω_* of a, we write

$\widetilde{\Omega}_* := (\widetilde{\Omega})_{\ell_1,\ell_2}$ and $\widetilde{\widetilde{\Omega}}_* := (\widetilde{\widetilde{\Omega}})_{\ell_1,\ell_2}$. We will also consider appropriate (ℓ_1,ℓ_2)-enlargements of the rectangles, specifically $\widehat{\widehat{R}}_* := 2^{\ell_1}\widehat{Q}_1 \times 2^{\ell_2}\widehat{Q}_2$ and $R_* = 2^{\ell_1}Q_1 \times 2^{\ell_2}Q_2$.

Decompose $\|S(a)\|^p_{L^p(\widetilde{X})}$ into pieces that are near or far from Ω_* (the support of a):

$$\|S(a)\|^p_{L^p(\widetilde{X})} = A + B,$$

where

$$\begin{aligned} A &:= \int_{\cup_{R\in m(\Omega)}100\overline{C}\widehat{\widehat{R}}_*} |S(a)(x_1,x_2)|^p \, d\mu(x_1,x_2) \qquad \text{(near } \Omega_*\text{)},\\ B &:= \int_{(\cup_{R\in m(\Omega)}100\overline{C}\widehat{\widehat{R}}_*)^c} |S(a)(x_1,x_2)|^p \, d\mu(x_1,x_2) \quad \text{(far from } \Omega_*\text{)}. \end{aligned}$$

Here $\overline{C}\widehat{\widehat{R}}_* := C_1 2^{\ell_1}\widehat{Q}_1 \times C_2 2^{\ell_2}\widehat{Q}_2$. The constants $C_i = 2(A_0^{(i)})^2$, for $i = 1,2$, are the dilation constants appearing in the support of the rectangular atoms property (3)(i) of Definition 5.2, and the parameters ℓ_i, for $i = 1,2$, are the enlargement parameters in the support of the (p,q)-atom in property (1) of Definition 5.2. To ease notation, we will denote $\overline{C}_i = C_i 2^{\ell_i}$ for $i = 1,2$. This ensures that $\overline{C}_1 Q_1 \times \overline{C}_2 Q_2 \subset \overline{C}\widehat{\widehat{R}}_*$, and $\operatorname{supp}(a) \subset \cup_{R\in m(\Omega)}\overline{C}\widehat{\widehat{R}}_*$.

Applying Hölder's inequality with exponent $s = q/p > 1$, the desired estimate $A \lesssim 1$ for the integral A follows from the L^q-boundedness of S and the L^q-norm estimate of the atom a as in (2) of Definition 5.2. More precisely,

$$\begin{aligned} A &\lesssim \|a\|^p_{L^q(\widetilde{X})}\big(\mu(\cup_{R\in m(\Omega)}100\overline{C}\widehat{\widehat{R}}_*)\big)^{1-\frac{p}{q}} \lesssim_q \big((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\Omega)\big)^{\frac{p}{q}-1} \\ &\qquad \times \big((\ell_1\omega_1+\ell_2\omega_2)(100\overline{C}_1)^{\omega_1}(100\overline{C}_2)^{\omega_2}\mu(\widetilde{\widetilde{\Omega}})\big)^{1-\frac{p}{q}} \\ &\lesssim_q \big(\mu(\Omega)\big)^{\frac{p}{q}-1}\big(\mu(\widetilde{\widetilde{\Omega}})\big)^{1-\frac{p}{q}} \lesssim_q 1. \end{aligned}$$

In the second inequality, similar to (4.29), we again used the $L\log_+ L$ to weak L^1 estimate of the strong maximal function to estimate the upper bound of $\mu(\cup_{R\in m(\Omega)}100\overline{C}\widehat{\widehat{R}}_*)$. In the last inequality we used the fact that $\mu(\Omega) \sim \mu(\widetilde{\widetilde{\Omega}})$.

Using the decomposition of the atom a as in (3)(ii-a) of Definition 5.2, the sublinearity of the product square function S, and that $p \le 1$, the integral B can be estimated as follows:

$$B \le \sum_{R\in m(\Omega)} \int_{(100\overline{C}\widehat{\widehat{R}}_*)^c} |S(a_R)(x_1,x_2)|^p \, d\mu(x_1,x_2).$$

We split the integral over $(100\overline{C}\widehat{\widehat{R}}_*)^c$ into two parts, one over $(100\overline{C}_1\widehat{Q}_1)^c \times X_2$ and the other over $X_1 \times (100\overline{C}_2\widehat{Q}_2)^c$. Denote the sum over $R \in m(\Omega)$ of the first integrals by B_1 and of the second integrals by B_2, respectively, so that $B \le B_1 + B_2$. It suffices to estimate B_1 since the estimate for B_2 is similar by symmetry.

To estimate the sum B_1, we further split each integral into two pieces, one over $(100\overline{C}_1\widehat{Q}_1)^c\times 100\overline{C}_2Q_2$ and the other over $(100\overline{C}_1\widehat{Q}_1)^c\times(100\overline{C}_2Q_2)^c$. Denote the sum over $R\in m(\Omega)$ of the first integrals by B_{11} and of the second integrals B_{12} respectively, so that $B_1 = B_{11} + B_{12}$.

Estimate for B_{11}. Applying Fubini for the integrals, then Hölder's inequality on the second variable with exponent $s = q/p > 1$, using the doubling property of μ_2, and writing $\widetilde{x} = (x_1, x_2)$, we estimate

$$\begin{aligned} B_{11} &= \sum_{R\in m(\Omega)} \int_{(100\overline{C}_1\widehat{Q}_1)^c\times(100\overline{C}_2Q_2)} |S(a_R)(\widetilde{x})|^p \, d\mu(\widetilde{x}) \\ &\lesssim \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \Big[\int_{X_2} |S(a_R)(\widetilde{x})|^q \, d\mu(x_2)\Big]^{\frac{p}{q}} d\mu(x_1). \end{aligned}$$

We estimate the inner integral on the right-hand side using an L^q-vector-valued one-parameter square function estimate with respect to the variable x_2 for μ_1-a.e. x_1, where we consider x_1 a fixed parameter. More precisely, let $F : X_2 \to L^q_{\ell^2(\mathbb{S})}(X_2, \mu_2) =: L^q_{\ell^2}(X_2)$ where $\mathbb{S}$ is a countable set, meaning that for each $x_2 \in X_2$, $F(x_2) = \{F_k(x_2)\}_{k\in\mathbb{S}} \in \ell^2(\mathbb{S})$ where $\|F(\cdot)\|_{\ell_2(\mathbb{S})} \in L^q(X_2)$. Furthermore we let $\|F\|_{L^q_{\ell_2}(X_2)} := \big\|\|F(\cdot)\|_{\ell_2(\mathbb{S})}\big\|_{L^q(X_2)}$. Then, using the notation $\widetilde{\chi}_{Q^{k_i}_{\alpha_i}} = \chi_{Q^{k_i}_{\alpha_i}}/\mu_i(Q^{k_i}_{\alpha_i})$ (denoting an L^1-normalization instead of denoting an L^2-normalization) and where $Q^{k_i}_{\alpha_i} \in \mathscr{D}^W_i$, we define

$$S_2(F)(x_2) := \Big(\sum_{k_2\in\mathbb{Z}} \sum_{\alpha_2\in\mathscr{Y}^{k_2}} \Big\|\langle\psi^{k_2}_{\alpha_2}, F\rangle_{L^2(X_2)}\Big\|^2_{\ell^2(\mathbb{S})} \widetilde{\chi}_{Q^{k_2}_{\alpha_2}}(x_2)\Big)^{\frac{1}{2}}.$$

Here $\langle\psi^{k_2}_{\alpha_2}, F\rangle_{L^2(X_2)}$ denotes the sequence $\{\langle\psi^{k_2}_{\alpha_2}, F_k\rangle_{L^2(X_2)}\}_{k\in\mathbb{S}}$. For all $q > 1$ the following vector-valued inequality holds: $\big\|S_2(F)\big\|_{L^q(X_2)} \leq C_q\big\|F\big\|_{L^q_{\ell^2}(X_2)}$. We recall that $\{\psi^{k_2}_{\alpha_2}\}_{k_2\in\mathbb{Z},\mathscr{Y}^{k_2}}$ is an orthogonal wavelet basis in X_2 satisfying suitable size, smoothness, and cancellation conditions. Hence by following the proof of the L^q-boundedness of the Littlewood-Paley square function as in [HLW] for $q > 1$, we obtain the L^q-boundedness of the vector-valued Littlewood-Paley operator S_2. For the Euclidean version, we refer to [Gra, Section 5.1.2].

With these preliminaries in mind, we can now estimate for μ_1-a.e. $x_1 \in X_1$ the $L^q(X_2)$-norm of $S(a_R)(x_1, \cdot)$, more precisely,

$$\begin{aligned} &\int_{X_2} |S(a_R)(x_1, x_2)|^q \, d\mu_2(x_2) \\ &= \int_{X_2} \Big[\sum_{\substack{k_2\in\mathbb{Z}\\ \alpha_2\in\mathscr{Y}^{k_2}}} \sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}} \big|\langle\psi^{k_1}_{\alpha_1}\psi^{k_2}_{\alpha_2}, a_R\rangle_{L^2(X_1\times X_2)}\big|^2 \widetilde{\chi}_{Q^{k_1}_{\alpha_1}}(x_1)\widetilde{\chi}_{Q^{k_2}_{\alpha_2}}(x_2)\Big]^{\frac{q}{2}} d\mu_2(x_2) \end{aligned}$$

$$= \int_{X_2} \Big[\sum_{\substack{k_2\in\mathbb{Z}\\ \alpha_2\in\mathscr{Y}^{k_2}}} \Big(\sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}} \big|\langle \psi^{k_2}_{\alpha_2}, F^{(x_1)}\rangle_{L^2(X_2)}\big|^2 \widetilde{\chi}_{Q^{k_1}_{\alpha_1}}(x_1)\Big)\widetilde{\chi}_{Q^{k_2}_{\alpha_2}}(x_2)\Big]^{\frac{q}{2}} d\mu_2(x_2)$$

$$= \int_{X_2} \Big[\sum_{k_2\in\mathbb{Z}} \sum_{\alpha_2\in\mathscr{Y}^{k_2}} \Big\|\langle \psi^{k_2}_{\alpha_2}, F^{(x_1)}\rangle_{L^2(X_2)}\Big\|^2_{\ell^2(\mathbb{S})} \widetilde{\chi}_{Q^{k_2}_{\alpha_2}}(x_2)\Big]^{\frac{q}{2}} d\mu_2(x_2)$$

$$= \int_{X_2} \big|S_2(F^{(x_1)})(x_2)\big|^q d\mu_2(x_2) \ \le\ C \int_{X_2} \big\|F^{(x_1)}(x_2)\big\|^q_{\ell^2(\mathbb{S})}\, d\mu_2(x_2).$$

Here $F^{(x_1)}(x_2) = \{F^{x_2,x_1}_{k_1,\alpha_1}\}_{k_1\in\mathbb{Z},\alpha_1\in\mathscr{Y}^{k_1}}$, where

$$F^{x_2,x_1}_{k_1,\alpha_1} := \langle a_R(\cdot, x_2), \psi^{k_1}_{\alpha_1}\rangle_{L^2(X_1)} \big(\widetilde{\chi}_{Q^{k_1}_{\alpha_1}}(x_1)\big)^{1/2}$$

and $\mathbb{S} = \{(k_1, \alpha_1) :\ k_1 \in \mathbb{Z},\ \alpha_1 \in \mathcal{Y}^{k_1}\}$ is a countable set.

Altogether we now estimate the term B_{11} as follows:

$$B_{11} \ \lesssim \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \times \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \Big[\int_{X_2} \big\|F^{(x_1)}(x_2)\big\|^q_{\ell^2(\mathbb{S})}\, d\mu_2(x_2)\Big]^{\frac{p}{q}} d\mu_1(x_1)$$

$$= \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \Big[\int_{X_2}\Big[\sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}} \Big|\int_{X_1} \psi^{k_1}_{\alpha_1}(y_1) a_R(y_1, x_2)\, d\mu_1(y_1)\Big|^2 \widetilde{\chi}_{Q^{k_1}_{\alpha_1}}(x_1)\Big]^{\frac{q}{2}} d\mu_2(x_2)\Big]^{\frac{p}{q}} d\mu_1(x_1).$$

Applying the decomposition (4.3) in Lemma 4.8 to $\psi^{k_1}_{\alpha_1}$, we get that for $\gamma > \omega_1$ (where γ is to be determined later) and for $\overline{C}_1 = C_1\, 2^{\ell_1}$ playing the role of $\overline{C}$,

$$\psi^{k_1}_{\alpha_1}(y_1) = \sqrt{\mu\big(B(y^{k_1}_{\alpha_1}, \delta^{k_1})\big)} \sum_{\ell=0}^{\infty} (2^\ell \overline{C}_1)^{-\gamma} \varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}(y_1).$$

Substituting and noting that $\mu\big(B(y^{k_1}_{\alpha_1}, \delta^{k_1})\big)\, \widetilde{\chi}_{Q^{k_1}_{\alpha_1}}(x_1) = \chi_{Q^{k_1}_{\alpha_1}}(x_1)$, we continue estimating:

$$B_{11} \ \lesssim \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \Big[\int_{X_2} \Big[\sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}} \Big|\sum_{\ell=0}^{\infty} (2^\ell\overline{C}_1)^{-\gamma} \big\langle \varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}, a_R(\cdot, x_2)\big\rangle_{L^2(X_1)}\Big|^2 \chi_{Q^{k_1}_{\alpha_1}}(x_1)\Big]^{\frac{q}{2}} d\mu_2(x_2)\Big]^{\frac{p}{q}} d\mu_1(x_1).$$

First apply the Cauchy-Schwarz inequality to the sum over $\ell \geq 0$ after factoring out the constant $(\overline{C}_1)^{-\gamma}$ and considering the decaying exponential factor as a weight so that $\sum_{\ell\geq 0} 2^{-\ell\gamma} < \infty$ is a harmless finite constant. Second, interchange sums over ℓ and over $(k_1, \alpha_1) \in \mathbb{S}$, applying Hölder's inequality with exponent $s = q/2 > 1$ (we are in the case $q \geq 2$ and when $q = 2$ this step is unnecessary. When $1 < q < 2$ the power $q/2 < 1$ and it will travel into the sum over ℓ, the only difference being that the exponential factor will be $2^{-\frac{\ell\gamma q}{2}}$ instead of $2^{-\ell\gamma}$) to the sum over ℓ and considering the decaying exponential factor as a weight as before. Third, interchange the sum over ℓ and the integral over X_2, using the fact that $p/q < 1$ and the exponent can travel inside the sum over ℓ. Finally, interchange the sum over ℓ with the outer integral over $(100\overline{C}_1\widehat{Q}_1)^c$ and then with the sum over $R \in m(\Omega)$. We find that

$$B_{11} \lesssim (\overline{C}_1)^{-\gamma p} \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \int\limits_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}}$$

$$\Big[\int\limits_{X_2}\Big[\sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}} \Big|\big\langle\varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}, a_R(\cdot,x_2)\big\rangle_{L^2(X_1)}\Big|^2 \chi_{Q^{k_1}_{\alpha_1}}(x_1)\Big]^{\frac{q}{2}} d\mu_2(x_2)\Big]^{\frac{p}{q}} d\mu_1(x_1)$$

$$\lesssim (\overline{C}_1)^{-\gamma p} \sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}} \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \int\limits_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \Big[\int\limits_{X_2}$$

$$\Big[\sum_{\substack{k_1\in\mathbb{Z}\\ \alpha_1\in\mathscr{Y}^{k_1}}} \Big|\big\langle\varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}, a_R(\cdot,x_2)\big\rangle_{L^2(X_1)}\Big|^2 \chi_{Q^{k_1}_{\alpha_1}}(x_1)\Big]^{\frac{q}{2}} d\mu_2(x_2)\Big]^{\frac{p}{q}} d\mu_1(x_1).$$

(In the case $1 < q < 2$ the only difference in the estimate is that instead of $2^{-\frac{\ell\gamma p}{q}}$ one gets the exponential $2^{-\frac{\ell\gamma p}{2}}$, where q has been replaced by 2 in the exponent's denominator.)

The support of a_R is $\overline{C}_1Q_1 \times \overline{C}_2Q_2$. Note that if $y_1 \in \overline{C}_1Q_1$ then $d_1(y_1, z_1) \leq C_1^1\overline{C}_1\,\ell(Q_1)$, where z_1 is the center of Q_1 and C_1^1 is the dilation constant for the outer balls in the fixed dyadic grid $\mathscr{D}_1^a$ on X_1. Recall that $R = Q_1 \times Q_2$. If $C_1^1\overline{C}_1\,\ell(Q_1) \leq \delta_1^{k_1}$, then $d_1(y_1, z_1) \leq \delta_1^{k_1}$ and using the smoothness property (iii) in Lemma 4.8 of $\varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}$, the cancellation condition (3)(ii) in the first variable of a_R in Definition 5.2, together with the geometric considerations and Hölder's inequality, we conclude that when both x_1 and $y^{k_1}_{\alpha_1}$ are in $Q^{k_1}_{\alpha_1}$,

$$\Big|\big\langle\varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}(\cdot), a_R(\cdot,x_2)\big\rangle_{L^2(X_1)}\Big|$$

$$
\begin{aligned}
&\le \int_{\overline{C}_1 Q_1} \left|\varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}(y_1) - \varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}(z_1)\right|\left|a_R(y_1,x_2)\right| d\mu_1(y_1)\\
&\lesssim \frac{\big(\overline{C}_1\ell(Q_1)\big)^{\eta_1}(\overline{C}_1 2^\ell\delta_1^{k_1})^{-\eta_1}(\overline{C}_1 2^\ell)^{\omega_1}}{\mu_1\big(B_{X_1}(y^{k_1}_{\alpha_1},\overline{C}_1 2^\ell\delta_1^{k_1})\big)}\int_{\overline{C}_1 Q_1}\left|a_R(y_1,x_2)\right| d\mu_1(y_1)\\
&\lesssim \frac{\ell(Q_1)^{\eta_1}(2^\ell\delta_1^{k_1})^{-\eta_1}(\overline{C}_1 2^\ell)^{\omega_1}}{\mu_1\big(B_{X_1}(x_1,\overline{C}_1 2^\ell\delta_1^{k_1})\big)}\big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{\frac{q-1}{q}}\|a_R(\cdot,x_2)\|_{L^q(X_1)}.
\end{aligned}
$$

Here the doubling condition on the measure allows us to compare nearby balls with the same radius; specifically, $\frac{\mu_1(B_{X_1}(x_1,2^\ell\delta_1^{k_1}))}{\mu_1(B_{X_1}(y^{k_1}_{\alpha_1},2^\ell\delta_1^{k_1}))} \sim 1$, since we are assuming that x_1 and $y^{k_1}_{\alpha_1}$ are in $Q^{k_1}_{\alpha_1}$.

Assume now that $C_1^1\overline{C}_1\ell(Q_1) > \delta_1^{k_1}$. Recall that to get the desired estimate for the inner product it suffices to obtain the estimate for the inner product with differences of the functions $(2^\ell\overline{C}_1)^\gamma\Lambda_\ell^{\overline{C}_1}$ instead of differences of the functions $\varphi^{\gamma,\overline{C}_1}_{\ell,k_1,\alpha_1}$. The other piece can be estimated as above. Therefore we can assume that $y_1 \in \overline{C_1}Q_1 \cap \operatorname{supp}(\Lambda_\ell^{\overline{C}_1})$. This means

$$2^{\ell-3}\overline{C}_1\delta_1^{k_1} \le d_1(y_1, y^{k_1}_{\alpha_1}) \le (A_0^{(1)})^2 2^\ell\overline{C}_1\delta_1^{k_1}$$

and $d_1(y_1,z_1) \le C_1^1\overline{C}_1\ell(Q_1)$. We also know that $x_1 \in (100(2A_0^{(1)})\overline{C}_1\widehat{Q}_1)^c$, hence $d_1(z_1,x_1) \ge 100(2A_0^{(1)})C_1^1\overline{C}_1\ell(\widehat{Q}_1) \ge 100(2A_0^{(1)})C_1^1\overline{C}_1\ell(Q_1)$ and $x_1 \in Q^{k_1}_{\alpha_1}$ therefore

$$d_1(z_1,y^{k_1}_{\alpha_1}) \sim d_1(y_1,y^{k_1}_{\alpha_1}) \ge 100(2A_0^{(1)})C_1^1\overline{C}_1\ell(Q_1).$$

From the proof of Lemma 4.8, we can use a test-function-like smoothness property for the function $\Lambda_\ell^{\overline{C}_1}$ encoded in (4.15) and valid when $y_1 \in \operatorname{supp}(\Lambda_\ell^{\overline{C}_1})$ and $d(y_1,z_1) \le (2A_0^{(1)})^{-1}\big(\delta_1^{k_1} + d(y_1,y^{k_1}_{\alpha_1})\big)$, which both hold by the assumptions made in this case, namely:

$$(2^\ell\overline{C}_1)^\gamma|\Lambda_\ell^{\overline{C}_1}(y_1) - \Lambda_\ell^{\overline{C}_1}(z_1)| \lesssim \frac{(\overline{C}_1 2^\ell\delta_1^{k_1})^{-\eta_1}d(y_1,z_1)^{\eta_1}}{\mu\big(B(y^{k_1}_{\alpha_1},\delta_1^{k_1})\big) + \mu\big(B(y_1,d(y_1,y^{k_1}_{\alpha_1}))\big)}.$$

Furthermore since nearby balls with same radius have comparable measure by the doubling property, $\mu\big(B(y_1,d(y_1,y^{k_1}_{\alpha_1}))\big) \sim \mu\big(B(y^{k_1}_{\alpha_1},\overline{C}_1 2^\ell\delta_1^{k_1})\big)$; we get that in our case

$$(2^\ell\overline{C}_1)^\gamma|\Lambda_\ell^{\overline{C}_1}(x) - \Lambda_\ell^{\overline{C}_1}(y)| \lesssim \frac{(\overline{C}_1 2^\ell\delta_1^{k_1})^{-\eta_1}}{\mu\big(B(y^{k_1}_{\alpha_1},\overline{C}_1 2^\ell\delta_1^{k_1})\big)}\big(C_1^1\overline{C}_1\ell(Q_1)\big)^{\eta_1}. \tag{5.15}$$

Inequality (5.15) together with the geometric considerations and Hölder's inequality, and given that both x_1 and $y_{\alpha_1}^{k_1}$ are in $Q_{\alpha_1}^{k_1}$, yield

$$\begin{aligned}\int_{\overline{C}_1 Q_1 \cap \operatorname{supp}(\Lambda_\ell^{\overline{C}_1})} &(\overline{C}_1 2^\ell)^\gamma \left|\Lambda_\ell^{\overline{C}_1}(y_1) - \Lambda_\ell^{\overline{C}_1}(z_1)\right| \left|a_R(y_1, x_2)\right| d\mu_1(y_1) \\ &\lesssim \frac{\ell(Q_1)^{\eta_1}(2^\ell \delta_1^{k_1})^{-\eta_1}}{\mu_1\big(B_{X_1}(y_{\alpha_1}^{k_1}, \overline{C}_1 2^\ell \delta_1^{k_1})\big)} \int_{\overline{C}_1 Q_1} \left|a_R(y_1, x_2)\right| d\mu_1(y_1) \\ &\lesssim \frac{\ell(Q_1)^{\eta_1}(2^\ell \delta_1^{k_1})^{-\eta_1}}{\mu_1\big(B_{X_1}(x_1, \overline{C}_1 2^\ell \delta_1^{k_1})\big)} \big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{\frac{q-1}{q}} \|a_R(\cdot, x_2)\|_{L^q(X_1)}.\end{aligned}$$

Note that in the first $\lesssim$ the constant $(C_1^1)^{\eta_1} \le C_1^1$ has been absorbed since it is bounded above by a constant depending only on the geometric constants of the space X_1.

Therefore we conclude that in all cases, when both x_1 and $y_{\alpha_1}^{k_1}$ are in $Q_{\alpha_1}^{k_1}$,

$$\begin{aligned}&\left|\big\langle \varphi_{\ell,k_1,\alpha_1}^{\gamma,\overline{C}_1}, a_R(\cdot, x_2)\big\rangle_{L^2(X_1)}\right| \\ &\quad\lesssim \frac{\ell(Q_1)^{\eta_1}(2^\ell \delta_1^{k_1})^{-\eta_1}(\overline{C}_1 2^\ell)^{\omega_1}}{\mu_1\big(B_{X_1}(x_1, \overline{C}_1 2^\ell \delta_1^{k_1})\big)} \big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{\frac{q-1}{q}} \|a_R(\cdot, x_2)\|_{L^q(X_1)}.\end{aligned}$$

Notice that in the above calculation $\ell(Q_1)$ refers to the underlying dyadic grid $\mathscr{D}_1^a$ for the atom, possibly different than the reference dyadic grid $\mathscr{D}_1^W$ for the wavelets on X_1. Also note that the inequalities $\lesssim$ and the similarities $\sim$ introduce constants depending only on the geometric constants of the space of homogeneous type, in this case X_1.

Now recall that $\operatorname{supp}(\varphi_{\ell,k_1,\alpha_1}^{\gamma,\overline{C}_1}) \subset B_{X_1}(y_{\alpha_1}^{k_1}, 2(A_0^{(1)})^2\, 2^\ell \overline{C}_1 \delta_1^{k_1})$, so the inner product we just estimated will be nonzero only when

$$(\overline{C}_1\, Q_1) \cap B_{X_1}(y_{\alpha_1}^{k_1}, 2(A_0^{(1)})^2\, 2^\ell \overline{C}_1 \delta_1^{k_1}) \neq \emptyset,$$

where $y_{\alpha_1}^{k_1}$ is the center of the cube $Q_{\alpha_1}^{k_1} \in \mathscr{D}_1^W$ that contains $x_1 \notin 100\overline{C}_1\widehat{Q}_1$. Therefore, when estimating B_{11}, in the sum over k_1 and α_1 the only scales that intervene are those integers $\ell \ge 0$ such that $2^\ell \overline{C}_1 \delta_1^{k_1} \sim d_1(x_1, z_1)$, where z_1 is the center of Q_1 (it helps to draw a picture to understand the geometry). With this in mind, applying the above estimate on the inner product we conclude that

$$\begin{aligned}B_{11} \quad&\lesssim\quad (\overline{C}_1)^{-\gamma p} \sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}} \sum_{R \in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \\ &\times \int_{x_1 \notin 100\overline{C}_1\widehat{Q}_1} \Big[\int_{X_2} \Big[\sum_{\substack{k_1,\alpha_1: \\ 2^\ell \overline{C}_1 \delta_1^{k_1} \sim d_1(x_1, z_1)}} \Big| \frac{\ell(Q_1)^{\eta_1}(2^\ell \delta_1^{k_1})^{-\eta_1}(\overline{C}_1 2^\ell)^{\omega_1}}{\mu_1\big(B_{X_1}(x_1, 2^\ell \delta_1^{k_1})\big)} \big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{1-\frac{1}{q}}\end{aligned}$$

$$\times \|a_R(\cdot,x_2)\|_{L^q(X_1)}\Big|^2 \chi_{Q^{k_1}_{\alpha_1}}(x_1)\Big]^{\frac{q}{2}} d\mu_2(x_2)\Big]^{\frac{p}{q}} d\mu_1(x_1).$$

There is exactly one dyadic cube $Q^{k_1}_{\alpha_1} \in \mathscr{D}^W_1$ in generation k_1 containing x_1, so the double sum over k_1, α_1 reduces to a single sum over k_1. Furthermore, note that when $2^\ell \overline{C}_1\delta_1^{k_1} \sim d_1(x_1,z_1)$ then

$$\mu_1\big(B_{X_1}(x_1, 2^\ell\overline{C}_1\delta_1^{k_1})\big) \sim \mu_1\big(B_{X_1}(x_1, d_1(x_1,z_1))\big) \lesssim (\overline{C}_1)^{\omega_1}\mu_1\big(B_{X_1}(x_1, 2^\ell\delta_1^{k_1})\big).$$

Therefore

$$\begin{aligned} B_{11} &\lesssim (\overline{C}_1)^{-\gamma p}\sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}} \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \|a_R\|^p_{L^q(\widetilde{X})} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \\ &\Big[\sum_{\substack{k_1:\\ 2^\ell\overline{C}_1\delta_1^{k_1}\sim d_1(x_1,z_1)}} \Big|\frac{\ell(Q_1)^{\eta_1}(2^\ell\delta_1^{k_1})^{-\eta_1}(2^\ell\overline{C}_1)^{\omega_1}}{\mu_1\big(B_{X_1}(x_1,d_1(x_1,z_1))\big)}\big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{1-\frac{1}{q}}\Big|^2\Big]^{\frac{p}{2}} d\mu_1(x_1) \\ &\lesssim (\overline{C}_1)^{-\gamma p}\sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}} \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \|a_R\|^p_{L^q(X_1\times X_2)}\, 2^{\ell\omega_1 p} \\ &\quad\times \big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{p-\frac{p}{q}} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \Big[\sum_{\substack{k_1:\\ 2^\ell\overline{C}_1\delta^{k_1}\sim d_1(x_1,z_1)}} (2^\ell\delta^{k_1})^{-2\eta_1}\Big]^{\frac{p}{2}} \\ &\quad\times \frac{\ell(Q_1)^{\eta_1 p}(\overline{C}_1)^{\omega_1 p}}{\mu_1\big(B_{X_1}(x_1,d_1(x_1,z_1))\big)^p}\, d\mu_1(x_1). \end{aligned}$$

Notice that the sum over k_1 is a geometric sum comparable to $\big(\frac{\overline{C}_1}{d_1(x_1,z_1)}\big)^{2\eta_1}$. Therefore

$$\begin{aligned} B_{11} &\lesssim (\overline{C}_1)^{(\omega_1-\gamma)p}\sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}} \sum_{R\in m(\Omega)} \big((\overline{C}_2)^{\omega_2}\mu_2(Q_2)\big)^{1-\frac{p}{q}} \|a_R\|^p_{L^q(X_1\times X_2)} \\ &\times 2^{\ell\omega_1 p}\big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{p-\frac{p}{q}} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \frac{d_1(x_1,z_1)^{-\eta_1 p}\ell(Q_1)^{\eta_1 p}(\overline{C}_1)^{\eta_1 p}}{\mu_1\big(B_{X_1}(x_1,d_1(x_1,z_1))\big)^p}\, d\mu_1(x_1). \end{aligned}$$

The integral over $(100\overline{C}_1\widehat{Q}_1)^c$ can be further decomposed into integrals over disjoint annuli $D_{j+1}\backslash D_j$. Here $D_j := 2^j 100\overline{C}_1\widehat{Q}_1$, so that for all $j\ge 0$, $(100\overline{C}_1\widehat{Q}_1)^c = \cup_{j\ge 0}(D_{j+1}\setminus D_j)$. For $x_1 \in D_j\setminus D_{j-1}$ we have that $(\overline{C}_1)^{\eta_1 p} d_1(x_1,z_1)^{-\eta_1 p} \sim 2^{-j\eta_1 p}\,\ell(\widehat{Q}_1)^{-\eta_1 p}$. Note that nearby balls with the same radius have comparable mass by the doubling property of the measure. In particular,

$$\mu_1\big(B_{X_1}(z_1, d_1(x_1,z_1))\big) \sim \mu_1\big(B_{X_1}(x_1, d_1(x_1,z_1)\big),$$

and certainly $\widehat{Q}_1 \subset D_j \subset B_{X_1}(z_1, d_1(x_1, z_1)) \subset D_{j+1}$. All together, we obtain the following estimate

$$\begin{aligned}\int_{x_1\in(100\overline{C}_1\widehat{Q}_1)^c} &\frac{d_1(x_1,z_1)^{-\eta_1 p}\ell(Q_1)^{\eta_1 p}(\overline{C}_1)^{\eta_1 p}}{\mu_1\big(B_{X_1}(x_1,d_1(x_1,z_1))\big)^p}\,d\mu_1(x_1)\\ &\lesssim \frac{\ell(Q_1)^{\eta_1 p}\ell(\widehat{Q}_1)^{-\eta_1 p}}{\mu_1(\widehat{Q}_1)^{p-1}}\sum_{j\geq 0}\Big(\frac{\mu_1(D_{j+1})}{\mu_1(D_j)}2^{-j\eta_1 p}\Big),\end{aligned}$$

where the sum over j is comparable to 1 by the doubling condition of μ_1. Note that $\gamma > \omega_1$ hence $(\overline{C}_1)^{(\omega_1-\gamma)p} < 1$, that $p-1 < 0$ hence $(\overline{C}_1)^{\omega_1(p-1)} < 1$, and recall that $(\overline{C}_i)^{\omega_i} \sim 2^{\ell_i\omega_i}$ for $i = 1, 2$. Substituting we obtain

$$\begin{aligned}B_{11} \;&\lesssim\; \sum_{\ell=0}^{\infty} 2^{-\frac{\ell\gamma p}{q}} \sum_{R\in m(\Omega)} \big(2^{\ell_1\omega_1+\ell_2\omega_2}\mu(R)\big)^{1-\frac{p}{q}} \|a_R\|^p_{L^q(\widetilde{X})}\ell(Q_1)^{\eta_1 p}\,2^{\ell\omega_1 p}\\ &\qquad\times\big((\overline{C}_1)^{\omega_1}\mu_1(Q_1)\big)^{p-1}\ell(\widehat{Q}_1)^{-\eta_1 p}/\mu_1(\widehat{Q}_1)^{p-1}\\ &\lesssim \sum_{R\in m(\Omega)} \big(2^{\ell_1\omega_1+\ell_2\omega_2}\mu(R)\big)^{1-\frac{p}{q}}\|a_R\|^p_{L^q(\widetilde{X})}\Big[\ell(Q_1)/\ell(\widehat{Q}_1)\Big]^{\eta_1 p}\\ &\qquad\times\Big[\mu_1(Q_1)/\mu_1(\widehat{Q}_1)\Big]^{p-1}\sum_{\ell=0}^{\infty}2^{-\frac{\ell\gamma p}{q}+\ell\omega_1 p}.\end{aligned}$$

For the geometric sum to converge we need to choose $\gamma > q\,\omega_1$ when $q \geq 2$, and when $1 < q < 2$ we choose $\gamma > 2\omega_1$. With this choice and using the doubling property (1.3) once more since $p-1 < 0$, we estimate for $2 \leq q < \infty$,

$$B_{11} \;\lesssim\; 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\sum_{R\in m(\Omega)}\|a_R\|^p_{L^q(\widetilde{X})}\,\mu(R)^{1-\frac{p}{q}}\,w\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big), \quad (5.16)$$

where $w(x) = x^\alpha$ with $\alpha = p\eta_1 + (p-1)\omega_1 > 0$. This is where we explicitly used the choice of $p > p_0$ where $p_0 = \max\big(\omega_i/(\omega_i+\eta_i)\,:\, i = 1,2\}$. It was also used in the definition of $H^p(\widetilde{X})$ in [HLW].

To be more precise on how the doubling condition was used in (5.16), let $\widehat{z}_1$ be the center of $\widehat{Q}_1$ and z_1 the center of Q_1. Recall that $Q_1 \subset \widehat{Q}_1$, then

$$\begin{aligned}\frac{\mu_1(\widehat{Q}_1)}{\mu_1(Q_1)} \;&\leq\; \frac{\mu_1\big(B_{X_1}(\widehat{z}_1, C_1^1\ell(\widehat{Q}_1))\big)}{\mu_1\big(B_{X_1}(z_1, c_1^1\ell(Q_1))\big)}\\ &\leq\; \frac{\mu_1\big(B_{X_1}(z_1, A_0^{(1)}(d_1(\widehat{z}_1,z_1)+C_1^1\ell(\widehat{Q}_1)))\big)}{\mu_1\big(B_{X_1}(z_1, c_1^1\ell(Q_1))\big)}\\ &\leq\; \frac{\mu_1\big(B_{X_1}(z_1, 2A_0^{(1)}C_1^1\ell(\widehat{Q}_1))\big)}{\mu_1\big(B_{X_1}(z_1, c_1^1\ell(Q_1))\big)}\end{aligned}$$

$$\lesssim \Big(\frac{2A_0^{(1)}C_1^1\ell(\widehat{Q}_1)}{c_1^1\ell(Q_1)}\Big)^{\omega_1} \lesssim \Big(\frac{\ell(\widehat{Q}_1)}{\ell(Q_1)}\Big)^{\omega_1}.$$

We continue estimating B_{11}. Applying Hölder's inequality to the right-hand side of (5.16), with exponent $s = q/p > 1$, setting $\widetilde{w} = w^{\frac{q}{q-p}}$, using property (iii-a) in the definition of (p,q)-atoms, and applying the Journé-type covering lemma gives

$$\begin{aligned} B_{11} &\lesssim 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\Big|\sum_{R\in m(\Omega)}\|a_R\|^q_{L^q(\widetilde{X})}\Big|^{\frac{p}{q}}\Big|\sum_{R\in m(\Omega)}\mu(R)\,\widetilde{w}\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big)\Big|^{1-\frac{p}{q}} \\ &\lesssim 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\big((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega})\big)^{\frac{p}{q}-1}\mu(\Omega)^{1-\frac{p}{q}} \lesssim 1. \end{aligned}$$

The last inequality holds because $\mu(\widetilde{\Omega}) \sim \mu(\Omega)$.

For $1 < q < 2$, setting $\overline{w} = w^{\frac{1}{2}}$, $\widetilde{w} = \overline{w}^{\frac{q}{q-p}}$ and $\widetilde{\widetilde{w}} = \overline{w}^{\frac{q}{p}}$ and applying the same estimate as above, we obtain

$$\begin{aligned} B_{11} &\lesssim 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\sum_{R\in m(\Omega)}\|a_R\|^p_{L^q(\widetilde{X})}\,\mu(R)^{1-\frac{p}{q}}\,w\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big) \\ &\lesssim 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\sum_{R\in m(\Omega)}\|a_R\|^p_{L^q(\widetilde{X})}\,\overline{w}\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big)\,\mu(R)^{1-\frac{p}{q}}\,\overline{w}\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big). \end{aligned}$$

Applying Hölder's inequality with exponent $s = q/p > 1$, and then using property (iii-b), with $\delta = q/(2p) > 0$, from the atoms and the Journé-type covering lemma implies

$$\begin{aligned} B_{11} &\lesssim 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\Big(\sum_{R\in m(\Omega)}\|a_R\|^q_{L^q(\widetilde{X})}\,\widetilde{\widetilde{w}}\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big)\Big)^{\frac{p}{q}} \\ &\qquad\qquad \times\Big(\sum_{R\in m(\Omega)}\mu(R)\,\widetilde{w}\Big(\frac{\ell(Q_1)}{\ell(\widehat{Q}_1)}\Big)\Big)^{1-\frac{p}{q}} \\ &\lesssim 2^{(\ell_1\omega_1+\ell_2\omega_2)(1-\frac{p}{q})}\big((\ell_1\omega_1+\ell_2\omega_2)2^{\ell_1\omega_1+\ell_2\omega_2}\mu(\widetilde{\Omega})\big)^{\frac{p}{q}-1}\mu(\Omega)^{1-\frac{p}{q}} \lesssim 1. \end{aligned}$$

The last inequality holds because $\frac{p}{q} - 1 < 0$ so $(\ell_1\omega_1 + \ell_2\omega_2)^{\frac{p}{q}-1} < 1$. The constants involved in the similarities depend only on p and q, the geometric constants of the spaces directly (quasi-triangle constants $A_0^{(i)}$, doubling constants, and upper dimensions ω_i for $i = 1,2$) or indirectly via the absolute constants C_i appearing in the definition of the (p,q)-atoms, or the constants appearing in the Journé Lemma, or the dilation constants of the underlying dyadic grids or their ratios, themselves depending only on the geometric constants.

Estimate for B_{12}. Using the cancellation condition of the atoms a_R, and denoting $\widetilde{y} = (y_1, y_2)$ and $\widetilde{x} = (x_1, x_2)$, we write B_{12} as

$$B_{12} = \sum_{R\in m(\Omega)} \int_{x_1\notin 100\overline{C}_1\widehat{Q}_1} \int_{x_2\notin 100\overline{C}_2 Q_2} \bigg| \sum_{k_1=-\infty}^{\widehat{k}_1} \sum_{k_2=-\infty}^{\widehat{k}_2} \bigg| \int_{\widetilde{X}} [\psi^{k_1}_{\alpha_1}(y_1) - \psi^{k_1}_{\alpha_1}(z_1)] \\ \times [\psi^{k_1}_{\alpha_1}(y_2) - \psi^{k_1}_{\alpha_1}(z_2)]\, a_R(\widetilde{y})\, d\mu(\widetilde{y}) \frac{\chi_{Q^{k_1}_{\alpha_1}}(x_1)}{\mu_1(Q^{k_1}_{\alpha_1})} \frac{\chi_{Q^{k_2}_{\alpha_2}}(x_2)}{\mu_2(Q^{k_2}_{\alpha_2})} \bigg|^q \bigg|^{\frac{p}{q}} d\mu(\widetilde{x}).$$

Here the constants $\widehat{k}_1$ and $\widehat{k}_2$ satisfy $\delta_1^{\widehat{k}_1} \approx \ell(\widehat{Q}_1)$ and $\delta_2^{\widehat{k}_2} \approx \ell(Q_2)$, respectively. Applying the smoothness properties of $\psi^{k_1}_{\alpha_1}(x_1, y_1)$ and $\psi^{k_1}_{\alpha_1}(x_2, y_2)$ yields that B_{12} satisfies the same estimate as B_{11} does, as in (5.16). This concludes the proof of Theorem 5.4. $\square$

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(Yongsheng Han) Department of Mathematics, Auburn University, AL 36849-5310, USA
hanyong@auburn.edu

(Ji Li) Department of Mathematics and Statistics, Macquarie University, NSW 2019, Australia
ji.li@mq.edu.au

(M. Cristina Pereyra) Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, USA
crisp@math.unm.edu

(Lesley A. Ward) UniSA STEM, University of South Australia, Mawson Lakes SA 5095, Australia
lesley.ward@unisa.edu.au

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