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Solutions of diophantine equations as periodic points of p-adic algebraic functions, III

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Abstract. All the periodic points of a certain algebraic function related to the Rogers-Ramanujan continued fraction $r(\tau)$ are determined. They turn out to be 0, $\frac{-1\pm\sqrt{5}}{2}$, and the conjugates over $\mathbb{Q}$ of the values $r(w_d/5)$, where w_d is one of a specific set of algebraic integers, divisible by the square of a prime divisor of 5, in the field $K_d = \mathbb{Q}(\sqrt{-d})$, as $-d$ ranges over all negative quadratic discriminants for which $\left(\frac{-d}{5}\right) = +1$. This yields a new class number formula for orders in the fields K_d. Conjecture 1 of Part I is proved for the prime $p = 5$, showing that the ring class fields over fields of type K_d whose conductors are relatively prime to 5 coincide with the fields generated over $\mathbb{Q}$ by the periodic points (excluding -1) of a fixed 5-adic algebraic function.

Contents

1. Introduction

In Part I a periodic point of an algebraic function $w = \mathfrak{g}(z)$, with minimal polynomial $g(z,w)$ over $F(z)$, F a given field (often algebraically closed), was defined to be an element a of F, for which numbers $a_i \in F$ exist satisfying the simultaneous equations

$$g(a,a_1) = g(a_1,a_2) = \cdots = g(a_{n-1},a) = 0,$$

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for some $n \geq 1$. The numbers $a_i = \mathfrak{g}(a_{i-1})$ in this definition are to be thought of as suitable values of the multi-valued function $\mathfrak{g}(z)$, determined by possibly different branches of $\mathfrak{g}(z)$ (when considered over $F = \mathbb{C}$). Note that if the coefficients of $g(x, y)$ lie in a subfield k of F, over which F is algebraic, then the set of periodic points of $\mathfrak{g}(z)$ in F is invariant under the action of $\mathrm{Gal}(F/k)$. In this part the main focus will be on the multi-valued function $\mathfrak{g}(z)$, whose minimal polynomial is the polynomial

$$g(x, y) = (y^4 + 2y^3 + 4y^2 + 3y + 1)x^5 - y(y^4 - 3y^3 + 4y^2 - 2y + 1)$$

considered in Part II, related to the Rogers-Ramanujan continued fraction $r(\tau)$ (in the notation of [7]). Recall that the function $r(\tau)$ satisfies the modular equation

$$g(r(\tau), r(5\tau)) = 0, \quad \tau \in \mathbb{H},$$

where $\mathbb{H}$ is the upper half-plane. (See [1], [2], [7].)

I will show, that when transported to the $\mathfrak{p}$-adic domain – specifically to $\mathsf{K}_5(\sqrt{5})$, where K_5 is the maximal unramified algebraic extension of the 5-adic field $\mathbb{Q}_5$ – the "multi-valued-ness" disappears, in that the $a_i = T_5^i(a) \in \mathsf{K}_5(\sqrt{5})$ become values of a *single-valued* algebraic function $T_5(x)$, defined on a suitable domain $\mathsf{D}_5 \subset \mathsf{K}_5(\sqrt{5})$. Thus, 5-adically, a and its companions a_i are periodic points of $T_5(x)$ in the usual sense. Setting $\varepsilon = \frac{-1+\sqrt{5}}{2}$, this single-valued algebraic function is given by the 5-adically convergent series

$$T_5(x) = x^5 + 5 + \sqrt{5} \sum_{k=2}^{\infty} a_k \left(\frac{5\sqrt{5}}{x^5 - \varepsilon^5} \right)^{k-1}, \quad a_k = \sum_{j=1}^{4} \binom{j/5}{k}, \tag{1.1}$$

for x in the domain

$$\mathsf{D}_5 = \{x \in \mathsf{K}_5(\sqrt{5}) : |x|_5 \leq 1 \ \wedge \ x \not\equiv 2 \ (\mathrm{mod}\ \sqrt{5})\}.$$

More precisely, half of the periodic points of $\mathfrak{g}(z)$ lie in D_5; namely, those which lie in the unramified extension K_5. The other half are periodic points of the function $T \circ T_5^{-1} \circ T$ and lie in $T(\mathsf{D}_5)$, where

$$T(x) = \frac{-(1+\sqrt{5})x + 2}{2x + 1 + \sqrt{5}}.$$

The function $T_5(x)$ has the property that $y = T_5(x)$ is the unique solution in $\mathsf{K}_5(\sqrt{5})$ of the equation $g(x, y) = 0$, for any $x \in \mathsf{K}_5(\sqrt{5})$ for which $x \not\equiv 2$ $(\mathrm{mod}\ \sqrt{5})$. Thus, $T_5(x)$ is one of the values of $\mathfrak{g}(x)$, for $x \in \mathsf{D}_5$.

In Part II [14] it was shown that the conjugates over $\mathbb{Q}$ of the values $\eta = r(w/5)$ of the Rogers-Ramanujan continued fraction are periodic points of the algebraic function $\mathfrak{g}(z)$, for specific elements w in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$. In this part it will be shown that these values are, together with 0 and $\frac{-1\pm\sqrt{5}}{2}$, the *only* periodic points of $\mathfrak{g}(z)$. Let d_K denote the discriminant of $K = \mathbb{Q}(\sqrt{-d})$, where $\left(\frac{-d}{5}\right) = +1$, and let $\wp_5$ denote a prime divisor of $(5) = \wp_5\wp_5'$ in K. Recall that $p_d(x)$ is the minimal polynomial over $\mathbb{Q}$ of the value $r(w_d/5)$, where w_d is given by equation (2) below.

Theorem 1.1. *(a) The set of periodic points in $\overline{\mathbb{Q}}$ (or $\overline{\mathbb{Q}}_5$ or $\mathbb{C}$) of the multi-valued algebraic function $\mathfrak{g}(z)$ defined by the equation $g(z, \mathfrak{g}(z)) = 0$ consists of $0, \frac{-1\pm\sqrt{5}}{2}$, and the roots of the polynomials $p_d(x)$, for negative quadratic discriminants $-d = d_K f^2$ satisfying $\left(\frac{-d}{5}\right) = +1$.*

(b) Over $\mathbb{C}$ the latter values coincide with the values $\eta = r(w_d/5)$ and their conjugates over $\mathbb{Q}$, where $r(\tau)$ is the Rogers-Ramanujan continued fraction and the argument $w_d \in K = \mathbb{Q}(\sqrt{-d})$ satisfies

$$w_d = \frac{v + \sqrt{-d}}{2} \in R_K, \quad \wp_5^2 \mid w_d, \text{ and } (N(w_d), f) = 1. \tag{1.2}$$

(c) Over $\overline{\mathbb{Q}}_5$, all the periodic points of $\mathfrak{g}(z)$ lie in $\mathsf{K}_5(\sqrt{5})$. Moreover, the periodic points of $\mathfrak{g}(z)$ in K_5 are periodic points in D_5 of the single-valued 5-adic function $T_5(x)$.

From this theorem and the results of Part II we can assert the following. Let F_d denote the abelian extension $F_d = \Sigma_5\Omega_f$ $(d \neq 4f^2)$ or $F_d = \Sigma_5\Omega_{5f}$ $(d = 4f^2 > 4)$ of $K = \mathbb{Q}(\sqrt{-d})$, where Σ_5 is the ray class field of conductor $\mathfrak{f} = (5)$ over K and Ω_f is the ring class field of conductor f over K. Since $(f, 5) = 1$ and $\Omega_{5f} = \Omega_5\Omega_f$ when $d \neq 4f^2$ (see [9, Satz 3]), then $F_d = \Sigma_5\Omega_{5f}$ in either case. Furthermore, F_d coincides with what Cox [4] calls the extended ring class field $L_{\mathcal{O},5}$ for the order $\mathcal{O} = \mathsf{R}_{-d}$ of discriminant $-d$ in K. Cox refers to Cho [3], who denotes this field by $K_{(5),\mathcal{O}}$, but these fields are already discussed in Söhngen [20, see p. 318], who shows they are generated by division values of the τ-function, together with suitable values of the j-function. See also Stevenhagen [21] and the monograph of Schertz [19, p. 108].

Theorem 1.2. *Let $K = \mathbb{Q}(\sqrt{-d})$, with $\left(\frac{-d}{5}\right) = +1$ and $-d = d_K f^2$, as above. If $\mathcal{O} = \mathsf{R}_{-d}$ is the order of discriminant $-d$ in K, the extended ring class field $F_d = \Sigma_5\Omega_{5f}$ over K is generated over $\mathbb{Q}$ by a periodic point $\eta = r(w_d/5)$ of the function $\mathfrak{g}(z)$ (w_d is as in (1.2)), together with a primitive 5-th root of unity ζ_5:*

$$F_d = \Sigma_5\Omega_{5f} = \mathbb{Q}(\eta, \zeta_5). \tag{1.3}$$

Conversely, if $\eta \neq 0, \frac{-1\pm\sqrt{5}}{2}$ is any periodic point of $\mathfrak{g}(z)$, then for some $-d = d_K f^2$ for which $\left(\frac{-d}{5}\right) = +1$, the field $\mathbb{Q}(\eta, \zeta_5) = F_d$. Furthermore, the field $\mathbb{Q}(\eta)$ generated by η alone is the inertia field for the prime divisor $\wp_5$ or for its conjugate $\wp_5'$ in the field F_d.

This theorem provides explicit examples of Satz 22 in Hasse's *Zahlbericht* [8], according to which any abelian extension of K is obtained from $\Sigma = \Omega_f(\zeta_n)$, for some integer $f \geq 1$ and some n-th root of unity ζ_n, by adjoining square-roots of elements of Σ. This holds because $\eta = r(w_d/5)$ satisfies a quadratic equation over $\Omega_f(\zeta_5)$. See [14, Prop. 4.3, Cor. 4.7, Thm. 4.8].

Here the method of Part I [13] and [16], which yielded an interpretation and alternate derivation of special cases of a class number formula of Deuring, leads to the following *new* class number formula.

Theorem 1.3. *Let $\mathfrak{D}_{n,5}$ be the set of discriminants $-d = d_K f^2 \equiv \pm 1 \pmod 5$ of orders in imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ for which the automorphism $\tau_5 = \left(\frac{F_{d,5}/K}{\wp_5}\right)$ has order n in the Galois group $Gal(F_{d,5}/K)$, where $F_{d,5}$ is the inertia field for $\wp_5$ in the abelian extension F_d/K. If $h(-d)$ is the class number of the order $\mathsf{R}_{-d} \subset K$, then for $n > 1$,*

$$\sum_{-d \in \mathfrak{D}_{n,5}} h(-d) = \frac{1}{2} \sum_{k|n} \mu(n/k) 5^k. \tag{1.4}$$

Based on this theorem and numerical calculations, I make the following

Conjecture 1. *Let $q > 5$ be a prime number. Let $L_{\mathcal{O},q} = L_{\mathsf{R}_{-d},q}$ be the extended ring class field over $K = K_d = \mathbb{Q}(\sqrt{-d})$ for the order $\mathcal{O} = \mathsf{R}_{-d}$ of discriminant $-d = d_K f^2$ in K, and let $h(-d)$ denote the class number of the order $\mathcal{O}$. Also, let $F_{d,q}$ be the inertia field for the prime divisor $\wp_q$ (dividing q in K_d) in the abelian extension $L_{\mathcal{O},q}$ of K_d. Then the following class number formula holds:*

$$\sum_{-d \in \mathfrak{D}_{n,q}} h(-d) = \frac{2}{q-1} \sum_{k|n} \mu(n/k) q^k, \quad n > 1,$$

where $\mathfrak{D}_{n,q}$ is the set of discriminants $-d = d_K f^2$ for which $\left(\frac{-d}{q}\right) = +1$ and the Frobenius automorphism $\tau_q = \left(\frac{F_{d,q}/K_d}{\wp_q}\right)$ has order n.

As was shown in [14] for the prime $q = 5$, the extension $L_{\mathsf{R}_{-d},q}$ is equal to $\Sigma_q \Omega_f/K$, if $d \neq 3f^2$ or $4f^2$; and is equal to $\Sigma_q \Omega_{qf}/K$, if $q \equiv 1 \pmod 4$ and $d = 4f^2$; or $q \equiv 1 \pmod 3$ and $d = 3f^2$. The field $F_{d,q}$ has degree $(q-1)/2$ and is cyclic over the ring class field Ω_f of conductor f over K.

One naturally expects that this conjecture describes an aspect of a much more general phenomenon. For example, one could consider families of quadratic fields $K = \mathbb{Q}(\sqrt{-d})$ for which the prime divisors q of a given fixed integer Q all split in K. These are the Q-admissible quadratic fields. Analogous formulas should hold for certain sets of class fields over the family of (imaginary?) abelian extensions of a fixed degree over $\mathbb{Q}$, whose Galois groups belong to a fixed isomorphism type, and in which a given rational prime q splits.

In Section 6 I show that a similar situation exists for the algebraic function $w = \mathfrak{f}(z)$ whose minimal polynomial over $\overline{\mathbb{Q}}(z)$ is $h(z, w)$, where

$$\begin{aligned}h(z,w) = w^5 &- (6 + 5z + 5z^3 + z^5)w^4 + (21 + 5z + 5z^3 + z^5)w^3 \\ &- (56 + 30z + 30z^3 + 6z^5)w^2 + (71 + 30z + 30z^3 + 6z^5)w \\ &- 120 - 55z - 55z^3 - 11z^5.\end{aligned}$$

I showed in Part II (Theorem 5.4) that any ring class field Ω_f over the imaginary quadratic field K, whose conductor is relatively prime to 5, is generated over K by a periodic point υ of $\mathfrak{f}(z)$, which satisfies $\upsilon = \eta - \frac{1}{\eta}$, for a certain periodic point η of $\mathfrak{g}(z)$. In Theorem 6.2 of this paper I show that *any* periodic point $\upsilon \neq -1$ of $\mathfrak{f}(z)$ is related to a periodic point of $\mathfrak{g}(z)$ by $\upsilon = \eta - \frac{1}{\eta} = \phi(\eta)$, and that the 5-adic function

$$\mathsf{T}_5(x) = \phi \circ T_5 \circ \phi^{-1}(x), \quad x \in \widetilde{\mathsf{D}}_5 = \phi(\mathsf{D}_5 \cap \{z \in \mathsf{K}_5 : |z|_5 = 1\}),$$

plays the same role for $\mathfrak{f}(z)$ that $T_5(x)$ plays for $\mathfrak{g}(z)$. In particular, Theorems 6.2 and 6.3 show that Conjecture 1 of Part I is true for the prime $p = 5$. This leads to a proof of Deuring's formula for the prime 5 in Theorem 6.5 and its corollary, analogous to the proof given in Part I and in [16] for the prime 2 and in [12] for the prime 3.

2. Iterated resultants

Set

$$g(X,Y) = (Y^4+2Y^3+4Y^2+3Y+1)X^5 - Y(Y^4-3Y^3+4Y^2-2Y+1). \quad (2.1)$$

In Part II [14] it was shown that $(X, Y) = (\eta, \eta^{\tau_5})$, with $\eta = r(w_d/5)$ and w_d given by (1.2), is a point on the curve $g(X, Y) = 0$. Here $\tau_5 = \left(\frac{\mathbb{Q}(\eta)/K}{\wp_5}\right)$ is the Frobenius automorphism for the prime divisor $\wp_5$ of $K = \mathbb{Q}(\sqrt{-d})$. This fact implies that $r(w_d/5)$ and its conjugates over $\mathbb{Q}$ are periodic points of the function $\mathfrak{g}(z)$ defined by $g(z, \mathfrak{g}(z)) = 0$. (See Part II, Theorem 5.3.) In this section and Sections 3-4 it will be shown that these values, together with the fixed points $0, \frac{-1\pm\sqrt{5}}{2}$, represent *all* the periodic points of the algebraic function $\mathfrak{g}(z)$. To do this we begin by considering a sequence of iterated resultants defined using the polynomial $g(x, y)$, as in Part I, Section 3.

We start by defining $R^{(1)}(x, x_1) := g(x, x_1)$, and note that

$$R^{(1)}(x,x_1) \equiv (x_1 + 3)^4(x^5 - x_1) \pmod 5.$$

Then we define the polynomial $R^{(n)}(x, x_n)$ inductively by

$$R^{(n)}(x,x_n) := \text{Resultant}_{x_{n-1}}(R^{(n-1)}(x,x_{n-1}), g(x_{n-1},x_n)), \quad n \geq 2.$$

It is easily seen using induction that

$$R^{(n)}(x,x_n) \equiv (-1)^{n-1}(x_n + 3)^{5^n-1}(x^{5^n} - x_n) \pmod 5,$$

so that the polynomial $R_n(x) := R^{(n)}(x,x)$ satisfies

$$R_n(x) \equiv (-1)^{n-1}(x+3)^{5^n-1}(x^{5^n}-x) \pmod 5, \quad n \geq 1. \tag{2.2}$$

The roots of $R_n(x)$ are all the periodic points of the multi-valued function $\mathfrak{g}(z)$ in any algebraically closed field containing $\mathbb{Q}$, whose periods are divisors of the integer n. (See Part I, p. 727.)

From this we deduce, by a similar argument as in the Lemma of Part I (pp. 727-728), that

$$\deg(R_n(x)) = 2 \cdot 5^n - 1, \quad n \geq 1.$$

As in Part I, we define the expression $\mathsf{P}_n(x)$ by

$$\mathsf{P}_n(x) = \prod_{k|n} R_k(x)^{\mu(n/k)}, \tag{2.3}$$

and show that $\mathsf{P}_n(x) \in \mathbb{Z}[x]$. From (2.2) it is clear that $R_n(x)$, for $n > 1$, is divisible (mod 5) by the N irreducible (monic) polynomials $\bar{f}_i(x)$ of degree n over $\mathbb{F}_5$, where

$$N = \frac{1}{n}\sum_{k|n} \mu(n/k)5^k,$$

and that these polynomials are simple factors of $R_n(x)$ (mod 5). It follows from Hensel's Lemma that $R_n(x)$ is divisible by distinct irreducible polynomials $f_i(x)$ of degree n over $\mathbb{Z}_5$, the ring of integers in $\mathbb{Q}_5$, for $1 \leq i \leq N$, with $f_i(x) \equiv \bar{f}_i(x)$ (mod 5). In addition, all the roots of $f_i(x)$ are periodic of minimal period n and lie in the unramified extension K_5. Furthermore, n is the smallest index for which $f_i(x) \mid R_n(x)$.

Now we make use of the following identity for $g(x,y)$:

$$\left(x + \frac{1+\sqrt{5}}{2}\right)^5 \left(y + \frac{1+\sqrt{5}}{2}\right)^5 g(T(x),T(y)) = \left(\frac{5+\sqrt{5}}{2}\right)^5 g(y,x),$$

where

$$T(x) = \frac{-(1+\sqrt{5})x + 2}{2x+1+\sqrt{5}}.$$

We have

$$T(x) - 2 = -\left(\frac{5+\sqrt{5}}{2}\right)\frac{2x-1+\sqrt{5}}{2x+1+\sqrt{5}}.$$

If the periodic point a of $\mathfrak{g}(z)$, with minimal period $n > 1$, is a root of one of the polynomials $f_i(x)$, then a is a unit in K_5, and for some $a_1, \dots, a_{n-1}$ we have

$$g(a,a_1) = g(a_1,a_2) = \cdots = g(a_{n-1},a) = 0. \tag{2.4}$$

Furthermore $a \not\equiv 2 \pmod{\sqrt{5}}$, since otherwise $a \equiv 2$ (mod 5) would have degree 1 over $\mathbb{F}_5$ (using that K_5 is unramified over $\mathbb{Q}_5$). Hence, $2a+1+\sqrt{5}$ is a unit and $b = T(a) \equiv 2 \pmod{\sqrt{5}}$. All the a_i satisfy $a_i \not\equiv 2 \pmod{\sqrt{5}}$, as well, since the congruence $g(2,y) \equiv 4(y+3)^5$ (mod 5) has only $y \equiv 2$ as a solution. Hence, if some $a_i \equiv 2$, then $a_j \equiv 2$ for $j > i$, which would imply that $a \equiv 2$, as well. The elements $b_i = T(a_i)$ are distinct and lie in $\mathsf{K}_5(\sqrt{5})$, and the above identity implies that

$$g(b, b_{n-1}) = g(b_{n-1}, b_{n-2}) = \cdots = g(b_1, b) = 0 \tag{2.5}$$

in $\mathsf{K}_5(\sqrt{5})$. Thus, all the $b_i \equiv 2 \pmod{\sqrt{5}}$, and the orbit $\{b, b_{n-1}, \dots, b_1\}$ is distinct from all the orbits in (2.4). Now the map $T(x)$ has order 2, so it is clear that $b = T(a)$ has minimal period n in (2.5), since otherwise $a = T(b)$ would have period smaller than n. It follows that there are at least $2N$ periodic orbits of minimal period $n > 1$. Noting that

$$R_1(x) = g(x, x) = \ x(x^2+1)(x^2+x-1)(x^4+x^3+3x^2-x+1),$$

these distinct orbits and factors account for at least

$$2 \cdot 5 - 1 + \sum_{d|n, d>1} (2 \sum_{k|d} \mu(d/k) 5^k) = -1 + 2 \sum_{d|n} (\sum_{k|d} \mu(d/k) 5^k) = 2 \cdot 5^n - 1$$

roots, and therefore all the roots, of $R_n(x)$. This shows that the roots of $R_n(x)$ are distinct and the expressions $\mathsf{P}_n(x)$ are polynomials. Furthermore, over $\mathsf{K}_5(\sqrt{5})$ we have the factorization

$$\mathsf{P}_n(x) = \pm \prod_{1 \le i \le N} f_i(x) \tilde{f}_i(x), \quad n > 1, \tag{2.6}$$

where $\tilde{f}_i(x) = c_i(2x+1+\sqrt{5})^{deg(f_i)} f_i(T(x))$, and the constant c_i is chosen to make $\tilde{f}_i(x)$ monic. Finally, the periodic points of $\mathfrak{g}(z)$ of minimal period n are the roots of $\mathsf{P}_n(x)$ and

$$\deg(\mathsf{P}_n(x)) = 2 \sum_{k|n} \mu(n/k) 5^k, \quad n > 1. \tag{2.7}$$

This discussion proves the following.

Theorem 2.1. *All the periodic points of $\mathfrak{g}(z)$ in $\overline{\mathbb{Q}}_5$ lie in $\mathsf{K}_5(\sqrt{5})$. The periodic points of minimal period n coincide with the roots of the polynomial $\mathsf{P}_n(x)$ defined by (2.3), and have degree n over $\mathbb{Q}_5(\sqrt{5})$. For $n > 1$, exactly half of the periodic points of $\mathfrak{g}(z)$ of minimal period n lie in K_5.*

The last assertion in this theorem follows from the fact that $T(x)$ is a linear fractional expression in the quantity $\sqrt{5}$:

$$T(x) = \frac{-x\sqrt{5} - x + 2}{\sqrt{5} + 2x + 1},$$

with determinant $-2(x^2+1)$. If it were the case that $a \in \mathsf{K}_5$ and $T(a) \in \mathsf{K}_5$, for $n > 1$, then the last fact would imply that $\sqrt{5} \in \mathsf{K}_5$, which is not the case. Therefore, for $n > 1$, the only roots of $\mathsf{P}_n(x)$ which lie in K_5 are the roots of the factors $f_i(x)$, in the above notation. Furthermore, the factors $f_i(x)$ are irreducible over $\mathbb{Q}_5(\sqrt{5})$, since this field is purely ramified over $\mathbb{Q}_5$, which implies that the factors $\tilde{f}_i(x)$ are irreducible over $\mathbb{Q}_5(\sqrt{5})$, as well.

3. A 5-adic function

Lemma 3.1. *Any root η' of the polynomial $p_d(x)$ which is conjugate to $\eta = r(w_d/5)$ over $K = \mathbb{Q}(\sqrt{-d})$ satisfies $\eta' \not\equiv 2 \pmod{\mathfrak{p}}$, for any prime divisor $\mathfrak{p}$ of $\wp_5$ in $F_1 = \mathbb{Q}(\eta)$.*

Proof. It suffices to prove this for $\eta' = \eta$. Assume $\eta \equiv 2 \pmod{\mathfrak{p}}$, where $\mathfrak{p} \mid \wp_5$ in F_1. Then the element $z = \eta^5 - \frac{1}{\eta^5}$ satisfies $z \equiv 2^5 - 2^{-5} \equiv -1 \pmod{\mathfrak{p}}$. Hence the proof of [14, Theorem 4.6] implies that d can only be one of the values $d = 11, 16, 19$. In these three cases $h(-d) = 1$, so η satisfies a quadratic polynomial over $K = \mathbb{Q}(\sqrt{-d})$. We have

$$\begin{aligned}
p_{11}(x) &= x^4 - x^3 + x^2 + x + 1 \\
&= \left(x^2 + \frac{-1+\sqrt{-11}}{2}x - 1\right)\left(x^2 + \frac{-1-\sqrt{-11}}{2}x - 1\right); \\
p_{16}(x) &= x^4 - 2x^3 + 2x + 1 \\
&= (x^2 + (-1-i)x - 1)(x^2 + (-1+i)x - 1); \\
p_{19}(x) &= x^4 + x^3 + 3x^2 - x + 1 \\
&= \left(x^2 + \frac{1+\sqrt{-19}}{2}x - 1\right)\left(x^2 + \frac{1-\sqrt{-19}}{2}x - 1\right).
\end{aligned}$$

In each case $\eta = r(w_d/5)$, where, respectively:

$$\begin{aligned}
w_{11} &= \frac{33+\sqrt{-11}}{2}, \quad N(w_{11}) = 5^2 \cdot 11, \\
w_{16} &= 11 + 2i, \qquad N(w_{16}) = 5^3, \\
w_{19} &= \frac{41+\sqrt{-19}}{2}, \quad N(w_{19}) = 5^2 \cdot 17.
\end{aligned}$$

Since $F_1 = K(\eta)$ is unramified over $\wp_5$ and ramified over $\wp_5'$, the minimal polynomial $m_d(x)$ over K of η in each case is the first factor listed above. Since $\wp_5^2 \mid w_d$, we conclude that

$$\sqrt{-11} \equiv 2, \; i \equiv 2, \; \sqrt{-19} \equiv 4$$

modulo $\wp_5$ in R_K. Then

$$m_{11}(x) \equiv x^2 + 3x + 4, \; m_{16}(x) \equiv x^2 + 2x + 4, \; m_{19}(x) \equiv (x+1)(x+4)$$

modulo $\wp_5$, where the first two polynomials are irreducible mod 5. It follows that η cannot be congruent to 2 modulo any prime divisor of $\wp_5$. In each case we also have $m_d(x) \equiv (x+3)^2 \pmod{\wp_5'}$. □

Computing the partial derivative

$$\begin{aligned}
\frac{\partial g(x,y)}{\partial y} &= (4y^3 + 6y^2 + 8y + 3)x^5 - 5y^4 + 12y^3 - 12y^2 + 4y - 1 \\
&\equiv 4(x+3)^5(y+3)^3 \pmod{5},
\end{aligned}$$

we see that the points $(x,y)=(\eta,\eta^{\tau_5})$ on the curve $g(x,y)=0$ satisfy the condition

$$\frac{\partial g(x,y)}{\partial y}|_{(x,y)=(\eta,\eta^{\tau_5})} \not\equiv 0 \bmod \mathfrak{p},$$

for any prime divisor $\mathfrak{p}$ of $\wp_5$. Hence, the $\mathfrak{p}$-adic implicit function theorem implies that η^{τ_5} can be written as a single-valued function of η in a suitable neighborhood of $x=\eta$. (See [18, p. 334].) We shall now derive an explicit expression for this single-valued function.

To do this, we consider $g(X,Y)=0$ as a quintic equation in Y. Using Watson's method of solving a quintic equation from the paper [10] of Lavallee, Spearman and Williams, we find that the roots Y of $g(X,Y)=0$ are

$$\begin{aligned} Y =&\frac{Z+3}{5}+\frac{\zeta}{10}(2Z+11+5\sqrt{5})^{4/5}(2Z+11-5\sqrt{5})^{1/5}\\ &+\frac{\zeta^2}{10}(2Z+11+5\sqrt{5})^{3/5}(2Z+11-5\sqrt{5})^{2/5}\\ &+\frac{\zeta^3}{10}(2Z+11+5\sqrt{5})^{2/5}(2Z+11-5\sqrt{5})^{3/5}\\ &+\frac{\zeta^4}{10}(2Z+11+5\sqrt{5})^{1/5}(2Z+11-5\sqrt{5})^{4/5}, \end{aligned}$$

where ζ is any fifth root of unity and $Z=X^5$. This can also be written in the form

$$\begin{aligned} Y =&\frac{Z+3}{5}+\frac{\zeta}{5}(Z-\bar{\varepsilon}^5)^{4/5}(Z-\varepsilon^5)^{1/5}+\frac{\zeta^2}{5}(Z-\bar{\varepsilon}^5)^{3/5}(Z-\varepsilon^5)^{2/5}\\ &+\frac{\zeta^3}{5}(Z-\bar{\varepsilon}^5)^{2/5}(Z-\varepsilon^5)^{3/5}+\frac{\zeta^4}{5}(Z-\bar{\varepsilon}^5)^{1/5}(Z-\varepsilon^5)^{4/5},\\ =&\frac{Z+3}{5}+\frac{1}{5}(Z-\varepsilon^5)(U^4+U^3+U^2+U), \quad U=\zeta^{-1}\left(\frac{Z-\bar{\varepsilon}^5}{Z-\varepsilon^5}\right)^{1/5}. \end{aligned}$$

Now, $\varepsilon^5=\frac{-11+5\sqrt{5}}{2}\equiv\frac{-1}{2}\equiv 2 \pmod 5$, so for $\zeta=1$ and $Z\not\equiv 2 \pmod 5$, the functions U^j can be expanded into a convergent series:

$$U^j=\left(\frac{Z-\bar{\varepsilon}^5}{Z-\varepsilon^5}\right)^{j/5}=\left(1+\frac{\varepsilon^5-\bar{\varepsilon}^5}{Z-\varepsilon^5}\right)^{j/5}=\sum_{k=0}^{\infty}\binom{\frac{j}{5}}{k}\left(\frac{5\sqrt{5}}{Z-\varepsilon^5}\right)^k.$$

This series converges for all $Z\not\equiv 2 \pmod{\sqrt{5}}$ in the field $\mathsf{K}_5(\sqrt{5})$. The terms in this series tend to 0 in the 5-adic valuation, because

$$5^k\binom{\frac{j}{5}}{k}=\frac{j(j-5)(j-10)\cdots(j-5(k-1))}{k!}$$

and because the additive 5-adic valuation of $k!$ satisfies

$$v_5(k!)=\frac{k-s_k}{4}\le\frac{k}{4},$$

where s_k is the sum of the 5-adic digits of k. Thus, for all $x \not\equiv 2 \pmod{\sqrt{5}}$ in $\mathsf{K}_5(\sqrt{5})$ the expression

$$y = T_5(x) = \frac{x^5+3}{5} + \frac{1}{5}(x^5-\varepsilon^5)\sum_{k=0}^{\infty} a_k \left(\frac{5\sqrt{5}}{x^5-\varepsilon^5}\right)^k, \quad a_k = \sum_{j=1}^{4} \binom{\frac{j}{5}}{k}, \tag{3.1}$$

represents a root of the equation $g(x,y) = 0$ in the field $\mathsf{K}_5(\sqrt{5})$. This formula for $T_5(x)$ simplifies to:

$$T_5(x) = \ x^5 + 5 + \sqrt{5}\sum_{k=2}^{\infty} a_k \left(\frac{5\sqrt{5}}{x^5-\varepsilon^5}\right)^{k-1}. \tag{3.2}$$

Note that

$$T_5(x) \equiv x^5 \pmod 5, \quad |x|_5 \le 1. \tag{3.3}$$

This follows from the fact that 5 divides the individual terms

$$b_k = 5^k a_k (\sqrt{5})^{k-2}$$

(ignoring the unit denominators) in the series (3.2), for $2 \le k \le 7$, as can be checked by direct computation, and from the following estimate for $v_5(b_k)$, the normalized additive valuation of b_k in $\mathsf{K}_5(\sqrt{5})$:

$$v_5(5^k a_k (\sqrt{5})^{k-2}) \ge \frac{k}{2} - 1 - \frac{k}{4} = \frac{k}{4} - 1 \ge 1, \text{ for } k \ge 8.$$

It follows from this that the function $T_5(x)$ can be iterated on the set

$$\mathsf{D}_5 = \{x \in \mathsf{K}_5(\sqrt{5}) : |x|_5 \le 1 \ \wedge \ x \not\equiv 2 \pmod{\sqrt{5}}\}. \tag{3.4}$$

I claim now that (3.1) (or (3.2)) gives the *only* root of $g(x,y) = 0$ in the field $\mathsf{K}_5(\sqrt{5})$, for a fixed $x \not\equiv 2 \pmod{\sqrt{5}}$. From the above formulas, a second root of this equation must have the form

$$y_1 = \frac{x^5+3}{5} + \frac{1}{5}(x^5-\varepsilon^5)(U^4+U^3+U^2+U),$$

where

$$U = \zeta^{-1}\left(\frac{x^5-\bar{\varepsilon}^5}{x^5-\varepsilon^5}\right)^{1/5},$$

for some fifth root of unity $\zeta \ne 1$. But then

$$U^4+U^3+U^2+U = \frac{U^5-1}{U-1} - 1 \in \mathsf{K}_5(\sqrt{5}),$$

so $U \in \mathsf{K}_5(\sqrt{5})$; and since ζU is also in $\mathsf{K}_5(\sqrt{5})$, it follows that $\zeta \in \mathsf{K}_5(\sqrt{5})$. This is impossible, since the ramification index of 5 in $\mathsf{K}_5(\zeta)$ is $e = 4$, while the ramification index of 5 in $\mathsf{K}_5(\sqrt{5})$ is only $e = 2$.

Proposition 3.2. *If $x \in \mathsf{D}_5$, the subset of $\mathsf{K}_5(\sqrt{5})$ defined by (3.4), then the series*

$$y = T_5(x) = \ x^5 + 5 + \sqrt{5}\sum_{k=2}^{\infty} a_k \left(\frac{5\sqrt{5}}{x^5 - \varepsilon^5}\right)^{k-1}, \quad a_k = \sum_{j=1}^{4} \binom{\frac{j}{5}}{k}, \tag{3.5}$$

gives the unique solution of the equation $g(x, y) = 0$ in the field $\mathsf{K}_5(\sqrt{5})$. Moreover, the image $T_5(x)$ also lies in D_5, so the map T_5 can be iterated on this set.

Corollary 3.3. *The function $T_5(x)$ satisfies $T_5(\mathsf{D}_5 \cap \mathsf{K}_5) \subseteq \mathsf{D}_5 \cap \mathsf{K}_5$.*

Proof. Let σ denote the non-trivial automorphism of $\mathsf{K}_5(\sqrt{5})/\mathsf{K}_5$. If $x \in \mathsf{D}_5 \cap \mathsf{K}_5$, then $g(x, T_5(x)) = 0$ and $T_5(x) \in \mathsf{K}_5(\sqrt{5})$ imply that $g(x^\sigma, T_5(x)^\sigma) = g(x, T_5(x)^\sigma) = 0$. The theorem gives that $T_5(x)^\sigma = T_5(x)$, implying that $T_5(x) \in \mathsf{K}_5$. ☐

Now the completion $(F_1)_\mathfrak{p}$ of the field $F_1 = \mathbb{Q}(\eta)$ with respect to a prime divisor $\mathfrak{p}$ of R_{F_1} dividing $\wp_5$ is a subfield of $\mathsf{K}_5(\sqrt{5})$. This is because F_1 is unramified at the prime $\mathfrak{p}$ and is abelian over K, so that $(F_1)_\mathfrak{p}$ is unramified and abelian over $K_{\wp_5} = \mathbb{Q}_5$.

By Lemma 3.1, we can substitute $x = \eta$ in (3.5), and since η^{τ_5} is a solution of $g(\eta, Y) = 0$ in K_5, we conclude that $\eta^{\tau_5} = T_5(\eta)$. Letting $\zeta = 1$ and $U = -u$ gives

$$\eta^{\tau_5} = \frac{\eta^5 + 3}{5} + \frac{1}{5}(\eta^5 - \varepsilon^5)(u^4 - u^3 + u^2 - u), \quad u = -\left(\frac{\eta^5 - \bar{\varepsilon}^5}{\eta^5 - \varepsilon^5}\right)^{1/5} = \frac{1}{\varepsilon\xi} \in F;$$

which agrees with the result of [14, Theorem 3.3] (see the second line in the proof of that theorem). The automorphism τ_5 is canonically defined on the unramified extension $\mathbb{Q}_5(\eta)$; defining τ_5 to be trivial on $\mathbb{Q}_5(\sqrt{5})$, we have that $T_5(\eta^{\tau_5}) = T_5(\eta)^{\tau_5}$, and hence that

$$\eta^{\tau_5^n} = T_5^n(\eta), \quad n \geq 1. \tag{3.6}$$

This also follows inductively from

$$g(\eta^{\tau_5^{n-1}}, \eta^{\tau_5^n}) = g(\eta^{\tau_5^{n-1}}, T_5(\eta^{\tau_5^{n-1}})) = g(\eta^{\tau_5^{n-1}}, T_5^n(\eta)) = 0.$$

Therefore, $\eta = r(w/5)$ is a periodic point of T_5 in D_5, and the minimal period of η with respect to T_5 is equal to the order of the automorphism $\tau_5 = \left(\frac{F_1/K}{\wp_5}\right)$.

By Theorem 2.1, the periodic points of $\mathfrak{g}(z)$ lie in $\mathsf{K}_5(\sqrt{5})$. In particular, the minimal period of $\eta = r(w_d/5)$ with respect to $\mathfrak{g}(z)$ is the order n of the automorphism τ_5. This is because any values η_i, for which

$$g(\eta, \eta_1) = g(\eta_1, \eta_2) = \cdots = g(\eta_{m-1}, \eta) = 0,$$

must themselves be periodic points with $\eta_i \not\equiv 2 \pmod{\sqrt{5}}$. This implies that $\eta_i \in \mathsf{D}_5$, and then $\eta_i = T_5^i(\eta)$ follows from Proposition 3.2, so that m must be a multiple of n. Hence, $\eta = r(w_d/5)$ must be a root of the polynomial $\mathsf{P}_n(x)$.

Theorem 3.4. *For any discriminant* $-d \equiv \pm 1$ *(mod 5), for which the automorphism* $\tau_5 = \left(\frac{F_1/K}{\wp_5}\right)$ *has order* n*, the polynomial* $p_d(x)$ *divides* $\mathsf{P}_n(x)$*.*

4. Identifying the factors of $\mathsf{P}_n(x)$

We will now show that the polynomials $p_d(x)$ in Theorem 3.4 are the only irreducible factors of $\mathsf{P}_n(x)$ over $\mathbb{Q}$. The argument is similar to the argument in [12, pp. 877-878], with added complexity due to the nontrivial nature of the points in $E_5[5] - \langle(0,0)\rangle$, plus the necessity of dealing with the action of the icosahedral group in this case.

To motivate the calculation below, we prove the following lemma. As in Part II, F_1 denotes the field $F_1 = \mathbb{Q}(\eta)$, where $\eta = r(w_d/5)$.

Lemma 4.1. *If* $w = w_d$ *is defined as in (1.2), and* $\tau_5 = \left(\frac{F_1/K}{\wp_5}\right)$*, then for some* 5*-th root of unity* ζ^i*, we have*

$$\eta^{\tau_5^{-1}} = r\left(\frac{w}{5}\right)^{\tau_5^{-1}} = \zeta^i r\left(\frac{w}{25}\right).$$

Proof. Define τ_5 on $F_1(\sqrt{5}) = \mathbb{Q}(\eta, \sqrt{5})$ so that it fixes $\sqrt{5}$. This is possible since F_1 and $K(\sqrt{5})$ are disjoint, abelian extensions of K. (See the discussion in Sections 5.2 and 5.3 of [14], where $\tau_5 = \sigma_1\phi|_{F_1}$ and both σ_1 and ϕ fix the field $L = \mathbb{Q}(\zeta)$.) Recall the linear fractional expression from Part II that was denoted

$$\tau(b) = \frac{-b + \varepsilon^5}{\varepsilon^5 b + 1}.$$

From $\tau(\xi^5) = \eta^5$ and $T(\eta^{\tau_5}) = \xi$ (Part II, Thms. 3.3 and 5.1) we then obtain

$$\eta^{5\tau_5^{-1}} = \tau(\xi^5)^{\tau_5^{-1}} = \tau\left((\xi^{\tau_5^{-1}})^5\right) = \tau(T(\eta)^5) = \mathfrak{r}(\eta),$$

where

$$\mathfrak{r}(z) = z\frac{z^4 - 3z^3 + 4z^2 - 2z + 1}{z^4 + 2z^3 + 4z^2 + 3z + 1},$$

as in the Introduction to Part II. On the other hand,

$$\mathfrak{r}(\eta) = \mathfrak{r}\left(r\left(\frac{w}{5}\right)\right) = r^5\left(\frac{w}{25}\right),$$

by Ramanujan's modular equation. Thus, $\eta^{5\tau_5^{-1}} = r^5(w/25)$, and the assertion follows. □

By (3.3), we have $f_i(T_5(x)) \equiv f_i(x^5)$ (mod 5), and since $T_5(a)$ is an "unramified" periodic point in D_5 whenever a is, it follows that $\sigma : x \to T_5(x)$ is a lift of the Frobenius automorphism on the roots of $f_i(x)$, for each i with

$1 \le i \le N$. We may assume that σ fixes $\sqrt{5}$, since K_5 and $\mathbb{Q}_5(\sqrt{5})$ are linearly disjoint over $\mathbb{Q}_5$. In order to apply σ to all the maps occurring in the proof below, we also extend σ to the field $\mathsf{K}_5\left(\sqrt{\frac{-5+\sqrt{5}}{2}}\right)$, so that it fixes elements of the field $\mathbb{Q}_5\left(\sqrt{\frac{-5+\sqrt{5}}{2}}\right)$; this is a cyclic quartic and totally ramified extension of $\mathbb{Q}_5$ (the minimal polynomial of the square-root being the Eisenstein polynomial x^4+5x^2+5).

Theorem 4.2. *For $n>1$ the polynomial $\mathsf{P}_n(x)$ is a product of polynomials $p_d(x)$:*

$$\mathsf{P}_n(x) = \pm \prod_{-d \in \mathfrak{D}_{n,5}} p_d(x), \tag{4.1}$$

where $\mathfrak{D}_{n,5}$ is the set of discriminants $-d = d_K f^2$ of imaginary quadratic orders $\mathsf{R}_{-d} \subset K = \mathbb{Q}(\sqrt{-d})$ for which $\left(\frac{-d}{5}\right) = +1$ and the corresponding automorphism $\tau_5 = \left(\frac{F_1/K}{\wp_5}\right)$ has order n in $Gal(F_1/K)$. Here $F_1 = \mathbb{Q}(r(w_d/5))$ is the inertia field for the prime divisor $\wp_5 = (5, w_d)$ in the abelian extension $\Sigma_5\Omega_f$ ($d \neq 4f^2$) or $\Sigma_5\Omega_{5f}$ ($d = 4f^2 > 4$) of K; and $p_d(x)$ is the minimal polynomial of the value $r(w_d/5)$ over $\mathbb{Q}$.

Proof. Let $\{\eta = \eta_0, \eta_1, \dots, \eta_{n-1}\}$, $n \ge 2$, be a periodic orbit of $T_5(x)$ contained in D_5, where $T_5^n(\eta) = \eta$, and let

$$\xi = T(\eta_1) = T(T_5(\eta)) = T(\eta^\sigma).$$

Then the relation $g(\eta,\eta_1) = g(\eta, T(\xi)) = 0$ implies that (η,ξ) is a point on the curve

$$\mathcal{C}_5: \ X^5 + Y^5 = \varepsilon^5(1 - X^5Y^5).$$

Rewrite this relation as

$$\xi^5 = \frac{-\eta^5+\varepsilon^5}{\varepsilon^5\eta^5+1} = \tau(\eta^5), \quad \tau(b) = \frac{-b+\varepsilon^5}{\varepsilon^5 b+1}, \quad b = \eta^5.$$

Let

$$E_5(b): \ Y^2 + (1+b)XY + bY = X^3 + bX^2$$

be the Tate normal form for a point of order 5; and let $E_{5,5}(b)$ be the isogenous curve

$$\begin{aligned} E_{5,5}(b): \ Y^2 + (1+b)XY + 5bY = X^3 &+ 7bX^2 + 6(b^3+b^2-b)X \\ &+ b^5 + b^4 - 10b^3 - 29b^2 - b. \end{aligned}$$

The X-coordinate of the map $\psi: E_5(b) \to E_{5,5}(b)$ is given by

$$X(\psi(P)) = \frac{b^4 + (3b^3+b^4)x + (3b^2+b^3)x^2 + (b-b^2-b^3)x^3 + x^5}{x^2(x+b)^2}, \quad b = \eta^5,$$

with $x = X(P)$. Note that $\ker(\psi) = \langle (0,0) \rangle$, and ψ is defined over $\mathbb{Q}(b)$. (See [11, p. 259].)

The relation $\xi^5 = \tau(\eta^5)$ implies that there is an isogeny $\phi : E_5(\eta^5) \to E_5(\tau(\eta^5)) = E_5(\xi^5)$. This is because the j-invariant of $E_5(\xi^5)$ is

$$\begin{aligned} j_\xi &= \frac{(1-12\xi^5+14\xi^{10}+12\xi^{15}+\xi^{20})^3}{\xi^{25}(1-11\xi^5-\xi^{10})} \\ &= \frac{(1+228\eta^5+494\eta^{10}-228\eta^{15}+\eta^{20})^3}{\eta^5(1-11\eta^5-\eta^{10})^5}, \end{aligned}$$

where the latter value is $j(E_{5,5}(\eta^5))$. Thus, $E_{5,5}(\eta^5) \cong E_5(\xi^5)$ by an isomorphism ι_1. Composing ψ (for $b = \eta^5$) with this isomorphism gives the isogeny $\phi = \iota_1 \circ \psi$. Furthermore, $j(E_{5,5}(\eta^5))$ is invariant under the substitution $\eta \to T(\eta) = \xi^{\sigma^{-1}}$, so

$$\begin{aligned} j_\xi &= \left(\frac{(1+228\xi^5+494\xi^{10}-228\xi^{15}+\xi^{20})^3}{\xi^5(1-11\xi^5-\xi^{10})^5}\right)^{\sigma^{-1}} \\ &= \left(\frac{(1-12\eta^5+14\eta^{10}+12\eta^{15}+\eta^{20})^3}{\eta^{25}(1-11\eta^5-\eta^{10})}\right)^{\sigma^{-1}} \\ &= j_{\eta^{\sigma^{-1}}}. \end{aligned}$$

It follows that $E_5(\xi^5) \cong E_5((\eta^{\sigma^{-1}})^5)$ by an isomorphism ι_2. Composing ι_2 with ϕ gives an isogeny $\iota_2 \circ \phi = \phi_1 : E_5(\eta^5) \to E_5(\eta^5)^{\sigma^{-1}}$ of degree 5. Applying σ^{-i+1} to the coefficients of ϕ_1 gives an isogeny

$$\phi_i : \ E_5(\eta^5)^{\sigma^{-(i-1)}} \to E_5(\eta^5)^{\sigma^{-i}}, \quad 1 \le i \le n,$$

which also has degree 5. Hence, $\iota = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$ is an isogeny from $E_5(\eta^5)$ to $E_5(\eta^5)^{\sigma^{-n}}$ of degree 5^n. But σ^n is trivial on $\mathbb{Q}_5(\eta, \sqrt{5})$, since $T_5^n(\eta) = \eta$. Hence, $\iota : E_5(\eta^5) \to E_5(\eta^5)$.

We will show that ι is a cyclic isogeny by showing that some point $P \in E_5(\eta^5)[5]$ is not in $\ker(\iota)$. The following formula from [15] gives the X-coordinate on $E_5(b)$ for a point P of order 5, which does not lie in $\langle (0,0) \rangle$:

$$X(P) = \frac{-\varepsilon^4}{2} \frac{(-2u^2+(1+\sqrt{5})u-3\sqrt{5}-7)(2u^2+(2\sqrt{5}+4)u+3\sqrt{5}+7)}{(-2u^2+(\sqrt{5}+1)u-2)(u+1)^2},$$

where

$$u^5 = -\frac{b-\bar{\varepsilon}^5}{b-\varepsilon^5}, \quad b = \eta^5, \ \bar{\varepsilon} = -\frac{1+\sqrt{5}}{2}.$$

A calculation on Maple shows that

$$X_1 = X(\psi(P)) = \frac{-5+\sqrt{5}}{10}(b^2+\varepsilon^4 b+\bar{\varepsilon}^2), \quad b = \eta^5.$$

This is the X-coordinate of the point $P' = \psi(P)$ on $E_{5,5}(b)$. On the other hand, an isomorphism $\iota_1 : E_{5,5}(b) \to E_5(\tau(b))$ is given by $\iota_1(X_1, Y_1) = (X_2, Y_2)$, where

$$X_2 = \lambda_1^2 X_1 + \lambda_1^2 \frac{b^2+30b+1}{12} - \frac{\tau(b)^2+6\tau(b)+1}{12},$$

and

$$\lambda_1^2 = \frac{\sqrt{5}\bar{\varepsilon}^5}{(b-\bar{\varepsilon}^5)^2} = \frac{\sqrt{5}\bar{\varepsilon}^5}{(\eta^5-\bar{\varepsilon}^5)^2}.$$

Under this isomorphism, $X_1 = X(\psi(P))$ maps to $X_2 = 0$, whence $\phi(P) = \iota_1 \circ \psi(P) = \pm(0,0)$ on $E_5(\tau(b)) = E_5(\xi^5)$. Note that the map ϕ is defined over $\Lambda = \mathbb{Q}\left(\eta, \sqrt{\sqrt{5}\bar{\varepsilon}}\right) = \mathbb{Q}\left(\eta, \sqrt{\frac{-5-\sqrt{5}}{2}}\right)$, since λ_1 lies in this field.

Now we find an explicit formula for the isomorphism ι_2 between $E_5(\xi^5)$ and $E_5(\eta^{5\sigma^{-1}})$. The Weierstrass normal form $Y^2 = 4X^3 - g_2X - g_3$ of $E_5(b)$ has coefficients

$$g_2(b) = \frac{1}{12}(b^4 + 12b^3 + 14b^2 - 12b + 1),$$
$$g_3(b) = \frac{-1}{216}(b^2+1)(b^4 + 18b^3 + 74b^2 - 18b + 1).$$

An isomorphism $\iota_2 : \; E_5(\xi^5) \to E_5(\eta^{5\sigma^{-1}})$ is determined by a number λ_2 satisfying the equations

$$g_2(\eta^{5\sigma^{-1}}) = \lambda_2^4 \cdot g_2(\xi^5), \quad g_3(\eta^{5\sigma^{-1}}) = \lambda_2^6 \cdot g_3(\xi^5).$$

We now use computations analogous to those in Lemma 4.1, obtaining

$$\eta^{5\sigma^{-1}} = \tau(\xi^5)^{\sigma^{-1}} = \tau\left((\xi^{\sigma^{-1}})^5\right) = \tau(T(\eta)^5) = \mathfrak{r}(\eta).$$

Then we solve for λ_2^2 from

$$\lambda_2^2 = \frac{g_3(\mathfrak{r}(\eta))g_2(\tau(\eta^5))}{g_2(\mathfrak{r}(\eta))g_3(\tau(\eta^5))}$$

and find that

$$\lambda_2^2 = \frac{(11\sqrt{5}-25)(2\eta+1+\sqrt{5})^2(-2\eta^2+(3+\sqrt{5})\eta-3-\sqrt{5})^2}{40(-2\eta^2-2\eta-3+\sqrt{5})^2}.$$

Here, λ_2 lies in the field $\mathbb{Q}\left(\eta, \sqrt{-\sqrt{5}\varepsilon}\right) = \mathbb{Q}\left(\eta, \sqrt{\frac{-5+\sqrt{5}}{2}}\right)$, which coincides with the field Λ above. Hence, the desired isomorphism is given on X-coordinates by

$$X_3 = \iota_2(X_2) = \lambda_2^2 X_2 + \lambda_2^2 \frac{\tau(\eta^5)^2 + 6\tau(\eta^5) + 1}{12} - \frac{\mathfrak{r}(\eta)^2 + 6\mathfrak{r}(\eta) + 1}{12},$$

if (X_2, Y_2) are the coordinates on $E_5(\xi^5)$ and (X_3, Y_3) are the coordinates on $E_5(\eta^{5\sigma^{-1}})$. Therefore, the points with $X_2 = 0$ map to points with

$$X_3 = \frac{(-5+\sqrt{5})(\eta\sqrt{5}+2\eta^2-\sqrt{5}-3\eta+3)(\eta\sqrt{5}-2\eta^2-\sqrt{5}+3\eta-3)}{20(-2\eta^2+\sqrt{5}-2\eta-3)}.$$

Finally, we choose $u = \frac{1}{\varepsilon\xi} \in \mathsf{K}_5(\sqrt{5})$, so that

$$u^5 = \frac{1}{\varepsilon^5\xi^5} = -\bar{\varepsilon}^5 \frac{\varepsilon^5\eta^5+1}{-\eta^5+\varepsilon^5} = -\frac{\eta^5-\bar{\varepsilon}^5}{\eta^5-\varepsilon^5},$$

as required above for the formula $X(P)$. Then we compute that

$$u^{\sigma^{-1}} = \frac{1}{\varepsilon \xi^{\sigma^{-1}}} = \frac{1}{\varepsilon T(\eta)},$$

which implies that $\eta = T\left(\varepsilon^{-1} u^{-\sigma^{-1}}\right)$. Substituting this expression for η in X_3 gives

$$X_3 = \frac{-\varepsilon^4}{2} \frac{(-2u_1^2 + (1+\sqrt{5})u_1 - 3\sqrt{5} - 7)(2u_1^2 + (2\sqrt{5}+4)u_1 + 3\sqrt{5} + 7)}{(-2u_1^2 + (\sqrt{5}+1)u_1 - 2)(u_1+1)^2},$$

with $u_1 = u^{\sigma^{-1}}$. Comparing with the above formula for $X(P)$ shows that $X_3 = X(P)^{\sigma^{-1}}$ and therefore the points $\pm(0,0)$ on $E_5(\xi^5)$ map to $\pm P^{\sigma^{-1}}$ on $E_5(\eta^{5\sigma^{-1}})$.

This discussion shows that the isogeny $\phi_1 = \iota_2 \circ \iota_1 \circ \psi$ from $E_5(\eta^5)$ to $E_5(\eta^5)^{\sigma^{-1}}$ satisfies

$$\phi_1(P) = \pm P^{\sigma^{-1}}.$$

Applying σ^{-i+1} to this gives $\phi_i(P^{\sigma^{-i+1}}) = \pm P^{\sigma^{-i}}$, and therefore

$$\iota(P) = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1(P) = \pm P^{\sigma^{-n}} = \pm P.$$

Since P is a point of order 5 on $E_5(\eta^5)$, and P does not lie in $\ker(\iota)$, we see that ι is indeed a cyclic isogeny.

From this and the fact that $\deg(\iota) = 5^n$ we conclude that the j-invariant $j_\eta = j(E_5(\eta^5))$ satisfies the modular equation

$$\Phi_{5^n}(j_\eta, j_\eta) = 0.$$

On the other hand, from [4, p. 263],

$$\Phi_{5^n}(X, X) = c_n \prod_{-d} H_{-d}(X)^{r(d,5^n)},$$

where the product is over the discriminants of orders R_{-d} of imaginary quadratic fields and

$$r(d, 5^n) = |\{\alpha \in \mathsf{R}_{-d} : \ \alpha \text{ primitive}, \ N(\alpha) = 5^n\}/\mathsf{R}_{-d}^\times|.$$

Thus, $r(d, 5^n)$ is nonzero only when the equation $4^k \cdot 5^n = x^2 + dy^2$, $(k = 0, 1)$, has a primitive solution. Now the polynomial $\mathsf{P}_n(x) \in \mathbb{Z}[x]$ splits completely in $\mathsf{K}_5(\sqrt{5})$, and its "unramified" roots all lie in K_5. Furthermore the "ramified" roots all have the form $\xi = T(\eta^\sigma)$ for some unramified root η, and the corresponding j-invariants have the form

$$j_\xi = \frac{(1 - 12\xi^5 + 14\xi^{10} + 12\xi^{15} + \xi^{20})^3}{\xi^{25}(1 - 11\xi^5 - \xi^{10})},$$

which equals

$$j_\xi = \frac{(1 + 228\eta^5 + 494\eta^{10} - 228\eta^{15} + \eta^{20})^3}{\eta^5(1 - 11\eta^5 - \eta^{10})^5}.$$

It follows that all the j-invariants j_η, j_ξ lie in K_5. Hence, the value d for which $H_{-d}(j_\eta) = 0$ is not divisible by 5. Thus, $(5, xyd) = 1$, and therefore $\left(\frac{-d}{5}\right) = +1$.

From $H_{-d}(j_\eta) = H_{-d}((j_\eta)^{\sigma^{-1}}) = H_{-d}(j_\xi) = 0$ we see that the periodic point η is a root of both polynomials $F_d(x^5), G_d(x^5)$, where

$$F_d(x) = x^{5h(-d)}(1-11x-x^2)^{h(-d)}H_{-d}\left[\frac{(x^4+12x^3+14x^2-12x+1)^3}{x^5(1-11x-x^2)}\right]$$

and

$$G_d(x) = x^{h(-d)}(1-11x-x^2)^{5h(-d)}H_{-d}\left[\frac{(x^4-228x^3+494x^2+228x+1)^3}{x(1-11x-x^2)^5}\right].$$

Now the roots of the polynomial $G_d(x^5)$ are invariant under the action of the icosahedral group $G_{60} = \langle S, T\rangle$, where T is as before and $S(z) = \zeta z$, with $\zeta = e^{2\pi i/5}$. (See [11], [17].) Since $H_{-d}(X)$ is irreducible over the field $L = \mathbb{Q}(\zeta)$, containing the coefficients of all the maps in G_{60}, the polynomial $G_d(x^5)$ factors over L into a product of irreducible polynomials of the same degree. (See the similar argument in [12, p. 864].) By the results of [14, pp. 1193, 1202], one of these irreducible factors is $p_d(x)$, whose degree is $4h(-d)$, and $p_d(x)$ is invariant under the action of the subgroup

$$H = \langle U, T\rangle, \quad U(z) = \frac{-1}{z},$$

a Klein group of order 4. The normalizer of H in G_{60} is $N = \langle A, H\rangle \cong A_4$, where $A = STS^{-2}$ is the map

$$A(z) = \zeta^3\frac{(1+\zeta)z+1}{z-1-\zeta^4}$$

of order 3, and $ATA^{-1} = U, AUA^{-1} = T_2 = TU$. The distinct left cosets of H in G_{60} are represented by the elements

$$M_{ij} = S^jA^i, \quad 0 \le i \le 2,\ 0 \le j \le 4.$$

(See [17, Prop. 3.3].) We would like to show that η is a root of the factor $p_d(x)$.

Since all the roots of $G_d(x^5)$ have the form $M_{ij}(\alpha)$, for some root α of $p_d(x)$ ([14, p. 1203]), the factors of $G_d(x^5)$ over L have the form

$$p_{i,j}(x) = (cx+d)^{4h(-d)}p_d(A^iS^j(x)),$$

where $A^iS^j(x) = \frac{ax+b}{cx+d}$. The stabilizer of this polynomial in G_{60} is

$$(A^iS^j)^{-1}HA^iS^j = S^{-j}HS^j,$$

which contains the map $S^{-j}US^j(x) = \frac{-\zeta^{-2j}}{x}$. If $p_{i,j}(\eta) = 0$, where $j \neq 0$, then both η and $\frac{-\zeta^{-2j}}{\eta}$ are roots of $p_{i,j}(x)$, which would imply that ζ^{-2j} is contained in the splitting field of $\mathsf{P}_n(x)$ over $\mathbb{Q}$, and is therefore contained in $\mathsf{K}_5(\sqrt{5})$, which is not the case. Hence, η can only be a root of $p_{i,0}(x) =$

$(c_i x + d_i)^{4h(-d)} p_d(A^i(x))$, for some i. But then the elements in $HA^i(\eta)$ are roots of $p_d(x)$. Assume $i = 1$. Since $A(\eta)$ is a root of $p_d(x)$, so is $A^{\rho^j}(\eta)$, where ρ is the automorphism of $\mathsf{K}_5(\zeta)/\mathsf{K}_5$ for which $\zeta^\rho = \zeta^2$. But $A^\rho = A^{-1}U$, so that $A^{\rho^2} = A^{-\rho}U = UAU$ and $A^{\rho^3} = UA^\rho U = UA^{-1}$. Thus, $A^{\rho^3}(\eta)$ being a root of $p_d(x)$ and $U \in H$ imply that $A^{-1}(\eta)$ is also a root of $p_d(x)$. But then η is a common root of $p_{1,0}(x) = (c_1 x + d_1)^{4h(-d)} p_d(A(x))$ and $p_{2,0}(x) = (c_2 x + d_2)^{4h(-d)} p_d(A^{-1}(x))$, which is impossible, since these are two of the irreducible factors of $G_d(x^5)$ over L, and the latter polynomial has no multiple roots, for $d \neq 4$. (See [17, §2.2].) A similar argument works if $i = 2$, since $A^2 = A^{-1}$ and $A = UA^{-\rho}$. For $d = 4$, we have

$$\begin{aligned} G_4(x^5) &= (x^{20} - 228x^{15} + 494x^{10} + 228x^5 + 1)^3 - 1728x^5(1 - 11x^5 - x^{10})^5 \\ &= (x^2+1)^2(x^4 + 2x^3 - 6x^2 - 2x + 1)^2(x^8 - x^6 + x^4 - x^2 + 1)^2 \\ &\quad \times\ (x^8 + 4x^7 + 17x^6 + 22x^5 + 5x^4 - 22x^3 + 17x^2 - 4x + 1)^2 \\ &\quad \times\ (x^8 - 6x^7 + 17x^6 - 18x^5 + 25x^4 + 18x^3 + 17x^2 + 6x + 1)^2, \end{aligned}$$

and the only periodic point $\eta \in \mathsf{D}_5$ which is a root of $G_4(x^5)$ is the fixed point

$$\eta = i = 3 + 3 \cdot 5 + 2 \cdot 5^2 + 3 \cdot 5^3 + 5^4 + \cdots \quad \in \mathbb{Q}_5.$$

Thus, $d = 4$ does not occur when $n \geq 2$. (Except for the primitive 20-th roots of unity, which do not lie in $\mathsf{K}_5(\sqrt{5})$, the other roots of $G_4(x^5) = 0$ satisfy $x \equiv 2 \bmod 5$, and so do not lie in D_5.)

Hence, the only possibility is that $p_d(\eta) = 0$. This shows that all periodic points of $T_5(x)$ in D_5 are roots of some $p_d(x)$ for which $(-d/5) = +1$. Since $T_5(\eta) = \eta^{\tau_5}$ for such a root by (3.6), it is clear that τ_5 has order n in the corresponding Galois group $\mathrm{Gal}(F_1/\mathbb{Q})$, as well. All the roots of $\mathsf{P}_n(x)$ which do not lie in D_5 have the form $T(\eta)$, for $\eta \in \mathsf{D}_5$, by the discussion in Section 2, and are also roots of $p_d(x)$ for one of these integers d, since $T(x)$ stabilizes the roots of $p_d(x)$.

Thus, if $n \geq 2$, the only irreducible factors of $\mathsf{P}_n(x)$ over $\mathbb{Q}$ are the polynomials $p_d(x)$ for which $(-d/5) = +1$ and $\tau_5 \in \mathrm{Gal}(F_1/\mathbb{Q})$ has order n. This proves (4.1). □

For use in the following corollary, note that the substitution $(X, Y) \to \left(\frac{-1}{X}, \frac{-1}{Y}\right)$ represents an automorphism of the curve $g(X, Y) = 0$, since

$$X^5 Y^5 g\left(\frac{-1}{X}, \frac{-1}{Y}\right) = g(X, Y). \tag{4.2}$$

As in [14], put

$$g_1(X, Y) = Y^5 g\left(X, \frac{-1}{Y}\right). \tag{4.3}$$

In the following corollary, we prove the claim stated in the last paragraph of [14, p. 1212]. In that paragraph, the polynomial $x^2 + x - 1$ should have also been listed along with x, x^2+1 and $p_d(x)$ as factors of the resultants $R_n(x)$. As we will see below, however, x^2+x-1 never divides $\tilde{R}_n(x)$.

Corollary 4.3. *Let $\tilde{R}_n(x)$ be the $(n-1)$-fold iterated resultant*

$$Res_{x_{n-1}}(...(Res_{x_2}(Res_{x_1}(g(x,x_1),g(x_1,x_2)),g(x_2,x_3)),...,g_1(x_{n-1},x))$$

for $n \geq 2$. If $\alpha \neq 0$ is a root of $\tilde{R}_n(x)$, then α is either $\pm i$ or a root of some polynomial $p_d(x)$, where $p_d(x) \mid R_{2n}(x)$.

Proof. A root $\alpha \neq 0$ of $\tilde{R}_n(x)$ satisfies the simultaneous equations

$$g(\alpha,\alpha_1) = g(\alpha_1,\alpha_2) = \cdots = g(\alpha_{n-2},\alpha_{n-1}) = g_1(\alpha_{n-1},\alpha) = 0,$$

for some elements α_i in $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Note that $\alpha_i \neq 0$, for $1 \leq i \leq n-1$, because $g(X,0) = X^5$, so that $\alpha_i = 0$ implies $\alpha_{i-1} = 0$. But the definition of $g_1(X,Y)$ and the final equation in the above chain give that $g\left(\alpha_{n-1}, \frac{-1}{\alpha}\right) = 0$. Now the identity (4.2) implies, using the above simultaneous equations, that

$$g\left(\frac{-1}{\alpha}, \frac{-1}{\alpha_1}\right) = g\left(\frac{-1}{\alpha_1}, \frac{-1}{\alpha_2}\right) = \cdots = g\left(\frac{-1}{\alpha_{n-1}}, \alpha\right) = 0.$$

Tacking this chain of equations onto the first chain following the equation $g\left(\alpha_{n-1}, \frac{-1}{\alpha}\right) = 0$ shows that α is a root of $R_{2n}(x) = 0$. Setting $p_4(x) = x^2+1$ (see below), we only have to verify that α is not a root of x^2+x-1 to conclude that α is a root of some polynomial $p_d(x)$, because

$$\mathsf{P}_1(x) = x(x^2+1)(x^2+x-1)(x^4+x^3+3x^2-x+1) = x(x^2+x-1)p_4(x)p_{19}(x).$$

For in that case α is either a root of $p_4(x)p_{19}(x)$ or a root of some $\mathsf{P}_m(x)$, for $m > 1$. But if $\alpha = \frac{-1\pm\sqrt{5}}{2}$, then α is a fixed point, $g(\alpha,y) = 0 \Rightarrow y = \alpha$, but

$$g_1(\alpha,\alpha) = \alpha^5 g(\alpha,\overline{\alpha}) = \frac{625-275\sqrt{5}}{2} \neq 0.$$

Thus, α cannot be a root of $\tilde{R}_n(x)$ for any $n \geq 1$. □

Remark. This justifies the claims made in Section 5 of Part II about the resultant $\tilde{R}_n(x)$. In particular, all its irreducible factors are x^2+1 and polynomials of the form $p_d(x)$. This shows also that the polynomial in Example 2 of that section (pp. 1210-1211) is indeed $p_{491}(x)$. The computation of the degree $\tilde{R}_3(x)$ was in error, however, at the beginning of that example. In fact the degree is 250, and there are five factors of degree 12, not three, as was claimed before: these factors are the polynomials $p_d(x)$ for $d = 31, 44, 124, 211, 331$.

Note that the root $-i = r\left(\frac{-7+i}{5}\right)$, so $p_4(x)$ is the minimal polynomial of a value $r(w_4/5)$, with $w_4 = -7+i \in \mathbb{Q}(\sqrt{-4})$ and $\wp_5^2 = (-2+i)^2 \mid w_4$. This justifies the notation $p_4(x)$. See [7, p. 139].

The following theorem is immediate from Theorem 4.2 and the computations of Section 2.

Theorem 4.4. *The set of periodic points in* $\overline{\mathbb{Q}}$ *(or* $\overline{\mathbb{Q}}_5$ *or* $\mathbb{C}$*) of the multi-valued algebraic function* $\mathfrak{g}(z)$ *defined by the equation* $g(z, \mathfrak{g}(z)) = 0$ *consists of* $0, \frac{-1\pm\sqrt{5}}{2}$*, and the roots of the polynomials* $p_d(x)$*, for negative discriminants* $-d$ *satisfying* $\left(\frac{-d}{5}\right) = +1$*. Over* $\overline{\mathbb{Q}}$ *or* $\mathbb{C}$ *the latter values coincide with the values* $\eta = r(w_d/5)$ *and their conjugates over* $\mathbb{Q}$*, where* $r(\tau)$ *is the Rogers-Ramanujan continued fraction and the argument* $w_d \in K = \mathbb{Q}(\sqrt{-d})$ *satisfies*

$$w_d = \frac{v + \sqrt{-d}}{2} \in R_K, \quad \wp_5^2 \mid w_d, \quad \text{and } (N(w_d), f) = 1.$$

The fixed points $0, \frac{-1\pm\sqrt{5}}{2}$ come from the factors $x, x^2 + x - 1$ of the polynomial $\mathsf{P}_1(x)$.

Equating degrees in the formula (4.1) yields

$$\deg(\mathsf{P}_n(x)) = \sum_{-d\in\mathfrak{D}_{n,5}} 4h(-d), \quad n > 1.$$

From (2.7) we get the following class number formula.

Theorem 4.5. *For* $n > 1$ *we have*

$$\sum_{-d\in\mathfrak{D}_{n,5}} h(-d) = \frac{1}{2}\sum_{k|n} \mu(n/k)5^k,$$

where $\mathfrak{D}_{n,5}$ *has the meaning given in Theorem 1.3.*

This proves Theorem 1.3, where the field F_1 has been denoted as $F_{d,5}$, to indicate its dependence on d. Note that the corresponding formula for $n = 1$ reads

$$\sum_{-d\in\mathfrak{D}_{1,5}} h(-d) = h(-4) + h(-19) = 2 = \frac{1}{2}(5 - 1).$$

5. Ramanujan's modular equations for $r(\tau)$

In this section we take a slight detour to show how the polynomials $p_{4d}(x), p_{9d}(x)$ and $p_{49d}(x)$ can be computed, if the polynomial $p_d(x)$ is known.

From Berndt's book [2, p. 17] we take the following identity relating $u = r(\tau)$ and $v = r(3\tau)$:

$$(v - u^3)(1 + uv^3) = 3u^2v^2. \tag{5.1}$$

Let

$$P_3(u, v) = (v - u^3)(1 + uv^3) - 3u^2v^2.$$

This polynomial satisfies the identity

$$v^4 P_3\left(u, \frac{-1}{v}\right) = P_3(v, u).$$

The following theorem gives a simple method of calculating $p_{9d}(x)$ from $p_d(x)$.

Theorem 5.1. *For any negative discriminant $-d \equiv \pm 1$ (mod 5), the polynomial $p_{9d}(x)$ divides the resultant*

$$Res_y(P_3(y,x), p_d(y)).$$

Proof. Let $-d = d_K f^2$, where d_K is the discriminant of $K = \mathbb{Q}(\sqrt{-d})$. One of the roots of $p_{9d}(x)$ is $\eta' = r(w_{9d}/5)$, where $w_{9d} = \frac{v+\sqrt{-9d}}{2} \in \mathsf{R}_{-9d}$, $\wp_5^2 \mid w_{9d}$ and $N(w_{9d}) = \frac{v^2+9d}{4}$ is prime to $3f$. Let $f = 3^s f'$, with $(f',3) = 1$. For some integer k, $w_{9d} + 25f'k = \frac{v+50f'k+\sqrt{-9d}}{2}$ satisfies $v + 50f'k \equiv v - 4f'k \equiv 3$ mod 9. Furthermore,

$$\eta' = r\left(\frac{w_{9d} + 25f'k}{5}\right) = r\left(\frac{w_{9d}}{5} + 5f'k\right) = r\left(\frac{w_{9d}}{5}\right).$$

Thus, we may assume $3 \mid\mid v$, and then $9 \mid N(w_{9d})$. In that case $w_d = \frac{w_{9d}}{3} \in \mathsf{R}_{-d}$, where $(N(w_d), f) = 1$, even when $3 \mid f$. Furthermore, $\wp_5^2 \mid w_d$. Hence, $\eta = r(w_d/5)$ is a root of $p_d(x)$. From (5.1) we have

$$P_3(\eta,\eta') = P_3(r(w_d/5), r(w_{9d}/5)) = P_3(r(w_d/5), r(3w_d/5)) = 0.$$

Hence, η' is a root of the resultant, which therefore has its minimal polynomial $p_{9d}(x)$ as a factor. □

Example 1. We compute

$$\mathrm{Res}_y(P_3(y,x), p_4(y)) = \mathrm{Res}_y(P_3(y,x), y^2+1) = x^8+x^6-6x^5+9x^4+6x^3+x^2+1.$$

Since the latter polynomial is irreducible, the theorem shows that it equals $p_{36}(x)$:

$$p_{36}(x) = x^8 + x^6 - 6x^5 + 9x^4 + 6x^3 + x^2 + 1.$$

This verifies once again the entry for $d = 36$ in Table 1 of [14], which we used in Example 1 of that paper (p. 1208). In the same way, we compute

$$\begin{aligned}\mathrm{Res}_y(P_3(y,x), p_{36}(y)) &= (x^2+1)^4(x^{24} - 18x^{23} + 81x^{22} - 60x^{21} + 594x^{20} \\ &+ 1074x^{19} + 118x^{18} - 1002x^{17} - 261x^{16} + 6882x^{15} + 12078x^{14} \\ &+ 1014x^{13} - 18585x^{12} - 1014x^{11} + 12078x^{10} - 6882x^9 - 261x^8 \\ &+ 1002x^7 + 118x^6 - 1074x^5 + 594x^4 + 60x^3 + 81x^2 + 18x + 1) \\ &= p_4(x)^4 p_{324}(x).\end{aligned}$$

There is also the identity from [2, p. 12] relating $u = r(\tau)$ and $v = r(2\tau)$:

$$(v - u^2) = (v + u^2) \cdot uv^2. \tag{5.2}$$

Setting

$$P_2(u,v) = (v + u^2) \cdot uv^2 - (v - u^2),$$

we have the following identity, analogous to the identity for $P_3(u,v)$.

$$v^3 P_2\left(u, \frac{-1}{v}\right) = P_2(v,u).$$

An argument similar to the proof of Theorem 5.1 yields

Theorem 5.2. *For any negative discriminant* $-d \equiv \pm 1$ *(mod* 5*), the polynomial* $p_{4d}(x)$ *divides the resultant*

$$Res_y(P_2(y,x), p_d(y)).$$

Proof. Again, let $-d = d_K f^2$, where d_K is the discriminant of $K = \mathbb{Q}(\sqrt{-d})$. One of the roots of $p_{4d}(x)$ is $\eta' = r(w_{4d}/5)$, where $w_{4d} = \frac{v+\sqrt{-4d}}{2} \in \mathsf{R}_{-4d}$, $\wp_5^2 \mid w_{4d}$ and $N(w_{4d}) = \frac{v^2+4d}{4}$ is prime to $2f$. Thus, $v \equiv 2d+2$ (mod 4). If f is odd, we set

$$w' = w_{4d} + 25f = \left(\frac{v}{2} + 25f\right) + \sqrt{-d} = v' + \sqrt{-d}.$$

Then,

$$r\left(\frac{w'}{5}\right) = r\left(\frac{w_{4d}}{5} + 5f\right) = r\left(\frac{w_{4d}}{5}\right) = \eta'.$$

Moreover, $v' \equiv \frac{v}{2} + 1 \equiv d$ (mod 2). Now let $w_d = \frac{w'}{2} = \frac{v'+\sqrt{-d}}{2} \in \mathsf{R}_{-d}$, where $(N(w_d), f) = 1$. Then $\wp_5^2 \mid w_d$ and $\eta = r(w_d/5)$ is a root of $p_d(x)$. From (5.2) we have

$$P_2(\eta,\eta') = P_2(r(w_d/5), r(w_{4d}/5)) = P_2(r(w_d/5), r(2w_d/5)) = 0.$$

Hence, η' is a root of the resultant, which therefore has its minimal polynomial $p_{4d}(x)$ as a factor.

On the other hand, if f is even, let $f = 2^s f'$, with f' odd. Then d is even, so $v/2$ is odd. In this case we choose k so that

$$v' = \frac{v}{2} + 25f'k \equiv \begin{cases} 0 \text{ (mod 4)}, & \text{if } 4 \mid\mid d; \\ 2 \text{ (mod 4)}, & \text{if } 8 \mid d. \end{cases}$$

With this choice of k we have $v' \equiv d$ (mod 2), so letting $w' = v' + \sqrt{-d} = w_{4d} + 25f'k$ and $w_d = \frac{w'}{2}$, we have $w_d \in \mathsf{R}_{-d}$ and

$$N(w_d) = \frac{v'^2 + d}{4} \equiv \begin{cases} \frac{d}{4} \equiv 1 \text{ (mod 2)}, & \text{if } 4 \mid\mid d; \\ \frac{v'^2}{4} \equiv 1 \text{ (mod 2)}, & \text{if } 8 \mid d. \end{cases}$$

In either case, we get that $(N(w_d), f) = 1$. We have $r(w'/5) = r(w_{4d}/5)$, as before, and letting $\eta = r(w_d/5)$ be a root of $p_d(x)$, we obtain $P_2(\eta,\eta') = 0$ as above, and the assertion of the theorem follows. ∎

Example 2. We have

$$\begin{aligned}\mathrm{Res}_y(P_2(y,x),p_{36}(y)) &= (x^8+x^6-6x^5+9x^4+6x^3+x^2+1)\\ &\times (x^{16}-2x^{15}+18x^{14}+24x^{13}+83x^{12}+78x^{11}+74x^{10}+40x^9\\ &\quad +9x^8-40x^7+74x^6-78x^5+83x^4-24x^3+18x^2+2x+1)\\ &= p_{36}(x)p_{144}(x)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}_y(P_2(y,x),\ p_{144}(y)) &= (x^8+x^6-6x^5+9x^4+6x^3+x^2+1)^2\\ &\times\ (x^{32}-32x^{31}+586x^{30}-2856x^{29}+5818x^{28}-160x^{27}-23408x^{26}\\ &+41964x^{25}-6573x^{24}-63520x^{23}+64426x^{22}+12736x^{21}-38746x^{20}\\ &-11464x^{19}+55416x^{18}-38148x^{17}-5743x^{16}+38148x^{15}+55416x^{14}\\ &+11464x^{13}-38746x^{12}-12736x^{11}+64426x^{10}+63520x^9-6573x^8\\ &-41964x^7-23408x^6+160x^5+5818x^4+2856x^3+586x^2+32x+1)\\ &= p_{36}(x)^2p_{576}(x).\end{aligned}$$

We can use Theorems 5.1 and 5.2 to construct polynomials $p_d(x)$ for which the Conjecture (1) in [14, p. 1199] does not hold. For example, starting with

$$p_{51}(x) = x^8+x^7+x^6-7x^5+12x^4+7x^3+x^2-x+1,$$

applying Theorem 5.2 once gives that

$$\begin{aligned}p_{204}(x) &= x^{24}-x^{23}+38x^{22}+36x^{21}+166x^{20}+33x^{19}+57x^{18}+22x^{17}\\ &+573x^{16}+1603x^{15}+2465x^{14}+1225x^{13}+1768x^{12}-1225x^{11}\\ &+2465x^{10}-1603x^9+573x^8-22x^7+57x^6-33x^5+166x^4-36x^3\\ &+38x^2+x+1,\end{aligned}$$

whose discriminant is exactly divisible by 17^{12}, in accordance with Conjecture (1). Applying Theorem 5.2 to this polynomial yields the polynomial $p_{816}(x)$, of degree 48, whose discriminant is exactly divisible by 17^{40}:

$$\mathrm{disc}(p_{816}(x)) = 2^{160}3^{120}5^{276}7^{40}17^{40}31^{24}47^8 79^8 179^4 191^{12} 241^8 491^8 541^8 691^8;$$

whereas Conjecture (1) predicts that 17^{24} should be the power of 17 dividing $\mathrm{disc}(p_{816}(x))$.

Note that the period of the roots of $p_{51}(x)$ is 4, whereas the period of the roots of $p_{204}(x)$ and $p_{816}(x)$ is 12.

We modify the statement of Conjecture (1) in [14, p. 1199] as follows.

Conjecture 2. *If $q > 5$ is a prime which divides the field discriminant d_K of $K = \mathbb{Q}(\sqrt{-d})$, then $q^{2h(-d_K)}$ exactly divides disc$(p_{d_K}(x))$.*

Now define the polynomial $P_7(u,v)$ by

$$P_7(u,v) = u^8v^7 + (-7v^5+1)u^7 + 7u^6v^3 + 7(-v^6+v)u^5 + 35u^4v^4 + 7(v^7+v^2)u^3 - 7u^2v^5 - (v^8+7v^3)u - v.$$

Note that $P_7(u,v)$ satisfies the polynomial identity

$$v^8 P_7\left(u, \frac{-1}{v}\right) = P_7(v,u).$$

From [22, Thm. 3.3] we have the following fact.

Proposition (Yi)**.** The Rogers-Ramanujan continued fraction $r(\tau)$ satisfies the equation $P_7(r(\tau), r(7\tau)) = 0$.

Theorem 5.3. *For any negative discriminant $-d \equiv \pm 1$ (mod 5), the polynomial $p_{49d}(x)$ divides the resultant*

$$Res_y(P_7(y,x), p_d(y)).$$

The proof is the same, mutatis mutandis, as the proof of Theorem 5.1, on replacing the prime 3 by 7.

Example 3. We compute that

$$\begin{aligned} \mathrm{Res}_y(P_7(y,x), p_4(y)) &= p_{196}(x) \\ &= x^{16} + 14x^{15} + 64x^{14} + 84x^{13} - 35x^{12} - 14x^{11} + 196x^{10} \\ &\quad + 672x^9 + 1029x^8 - 672x^7 + 196x^6 + 14x^5 - 35x^4 \\ &\quad - 84x^3 + 64x^2 - 14x + 1. \end{aligned}$$

As a check, note that $h(-4 \cdot 7^2) = 4$ and the discriminant of $p_{196}(x)$ is

$$\mathrm{disc}(p_{196}(x)) = 2^{32} \cdot 3^{12} \cdot 5^{28} \cdot 7^{14} \cdot 19^4 \cdot 71^8,$$

all of whose prime factors are less than $d = 196 = 4 \cdot 7^2$.

6. Periodic points for $h(t,u)$

6.1. Reduction to periodic points of $g(x,y)$. From [14] the equation connecting $t = X - \frac{1}{X}$ and $u = Y - \frac{1}{Y}$ in the function field of the curve $g(X,Y) = 0$ is

$$\begin{aligned} h(t,u) = u^5 &- (6+5t+5t^3+t^5)u^4 + (21+5t+5t^3+t^5)u^3 \\ &- (56+30t+30t^3+6t^5)u^2 + (71+30t+30t^3+6t^5)u \\ &- 120 - 55t - 55t^3 - 11t^5. \end{aligned}$$

On this curve $\upsilon = \eta - \frac{1}{\eta} \in \Omega_f$, with $\eta = r(w_d/5)$, satisfies

$$h(\upsilon, \upsilon^{\tau_5}) = 0, \quad \tau_5 = \left(\frac{\Omega_f/\mathbb{Q}(\sqrt{-d})}{\wp_5}\right).$$

This yielded the following theorem.

Theorem 6.1. *If Ω_f is the ring class field of conductor f (relatively prime to 5) over the field $K = \mathbb{Q}(\sqrt{-d})$, where $-d = d_K f^2$ and $\left(\frac{-d}{5}\right) = +1$, then $\Omega_f = K(\upsilon)$, where $\upsilon = \eta - \frac{1}{\eta}$ is a periodic point of the algebraic function $\mathfrak{f}(z)$ defined by $h(z, \mathfrak{f}(z)) = 0$.*

Note the identity

$$X^5Y^5 h\left(X - \frac{1}{X}, Y - \frac{1}{Y}\right) = -g(X,Y)g_1(X,Y), \qquad (6.1)$$

where $g(X,Y)$ is given by (2.1) and $g_1(X,Y)$ is defined in (4.3). Also, recall that

$$X^5Y^5 g\left(\frac{-1}{X}, \frac{-1}{Y}\right) = g(X,Y), \quad X^5Y^5 g_1\left(\frac{-1}{X}, \frac{-1}{Y}\right) = g_1(X,Y), \qquad (6.2)$$

where the second identity is an easy consequence of the first. Using these facts we can prove the following.

Theorem 6.2. *If $\upsilon \neq -1$ is any periodic point of the algebraic function $\mathfrak{f}(z)$ in Theorem 6.1, then*

$$\upsilon = \eta - \frac{1}{\eta},$$

for some periodic point η of $\mathfrak{g}(z)$, and υ generates a ring class field Ω_f over some field $K = \mathbb{Q}(\sqrt{-d})$, where $-d = d_K f^2$ and $\left(\frac{-d}{5}\right) = +1$.

Proof. Assume that there exist elements υ_i for which

$$h(\upsilon, \upsilon_1) = h(\upsilon_1, \upsilon_2) = \cdots = h(\upsilon_{n-1}, \upsilon) = 0. \qquad (6.3)$$

Since the substitution $x = y - \frac{1}{y}$ transforms the polynomial

$$h(x,x) = -(x+1)(x^2+4)(x^2-x+3)(x^2-2x+2)(x^2+x+5),$$

(after multiplying by y^9) into the product

$$\begin{aligned} -(y^2+y-1)(y^2+1)^2(y^4-y^3+y^2+y+1)(y^4-2y^3+2y+1) \\ \times (y^4+y^3+3y^2-y+1) \\ = -(y^2+y-1)p_4(y)^2 p_{11}(y)p_{16}(y)p_{19}(y), \end{aligned}$$

we may assume $n \geq 2$. Set $g_0(X,Y) = g(X,Y)$ and write $\upsilon = \eta - \frac{1}{\eta}$ and $\upsilon_i = \eta_i - \frac{1}{\eta_i}$. By (6.1), equation (6.3) is equivalent to a set of simultaneous equations

$$g_{i_1}(\eta, \eta_1) = g_{i_2}(\eta_1, \eta_2) = \cdots = g_{i_n}(\eta_{n-1}, \eta) = 0, \qquad (6.4)$$

where each $i_k = 0$ or 1. Using the same idea as in the proof of Corollary 4.3, we will transform this set of equations into a set of equations which only involve the polynomial $g = g_0$. Assume first that $i_1 = 1$. Then

$$0 = g_1(\eta, \eta_1) = g\left(\eta, \frac{-1}{\eta_1}\right).$$

Now we use (6.2) to rewrite the remaining equations, so that we have

$$0 = g\left(\eta, \frac{-1}{\eta_1}\right) = g_{i_2}\left(\frac{-1}{\eta_1}, \frac{-1}{\eta_2}\right) = \cdots = g_{i_n}\left(\frac{-1}{\eta_{n-1}}, \frac{-1}{\eta}\right),$$

with the same subscripts i_r, for $r \geq 2$, as before. Now assume we have transformed the first $k-1$ equations so that only the polynomial $g(X, Y)$ appears. Then, on renaming the elements $\pm\eta_i^{\pm 1}$ as η_i, we have the simultaneous equations

$$0 = g(\eta, \eta_1) = \cdots = g(\eta_{k-2}, \eta_{k-1}) = g_{i_k}(\eta_{k-1}, \eta_k) = \cdots = g_{i_n}(\eta_{n-1}, \pm\eta^{\pm 1}).$$

If $i_k = 0$ we replace k by $k+1$ and continue. If $i_k = 1$ we replace $g_{i_k}(\eta_{k-1}, \eta_k)$ by $g(\eta_{k-1}, -1/\eta_k)$ and use (6.2) to replace η_r in the remaining equations by $-1/\eta_r, r \geq k$. Then, on renaming the η's again, we get a chain of equations

$$0 = g(\eta, \eta_1) = \cdots = g(\eta_{k-1}, \eta_k) = \cdots = g_{i_n}(\eta_{n-1}, \pm\eta^{\pm 1}).$$

Thus, by induction, we see that (6.4) is equivalent to a chain of equations

$$0 = g(\eta, \eta_1) = \cdots = g(\eta_{n-1}, \pm\eta^{\pm 1})$$

only involving the polynomial g. If the final η is simply η, then η is a periodic point of g having period n. On the other hand, if the final η appearing in these equations is $-\eta^{-1}$, then we use the same argument as in Corollary 4.3 to show that η is a periodic point of period $2n$. Then we know η is not 0 or a root of x^2+x-1, and therefore must be a root of some $p_d(x)$. By Theorem 6.1, this implies that $K(v) = \Omega_f$, for $K = \mathbb{Q}(\sqrt{-d})$ and $-d = d_K f^2$. This proves the theorem. □

Taken together, Theorems 6.1 and 6.2 verify Conjecture 1(b) of Part I for the case $p = 5$. To verify Conjecture 1(a), we define the function

$$\mathsf{T}_5(z) = T_5(\eta) - \frac{1}{T_5(\eta)}, \quad \eta = \frac{z \pm \sqrt{z^2+4}}{2}.$$

We can also write

$$\mathsf{T}_5(z) = \phi \circ T_5 \circ \phi^{-1}(z), \quad \phi(z) = z - \frac{1}{z},$$

where $\phi^{-1}(z) \in \{\frac{z\pm\sqrt{z^2+4}}{2}\}$ is two-valued. Since

$$g(z, T_5(z)) = 0 \ \Rightarrow \ g\left(\frac{-1}{z}, \frac{-1}{T_5(z)}\right) = 0,$$

it follows from Proposition 3.2 that

$$T_5\left(\frac{-1}{z}\right) = \frac{-1}{T_5(z)}, \ \text{for } z \in \mathsf{D}_5 \cap \{z : |z|_5 = 1\}.$$

Since the two solutions $\eta^{(+)}, \eta^{(-)}$ of $\phi(\eta^{(\pm)}) = z$ satisfy $\eta^{(+)}\eta^{(-)} = -1$, the value taken for $\phi^{-1}(z)$ does not affect the value of $\mathsf{T}_5(z)$. In other words, we have the symmetric formula

$$\mathsf{T}_5(z) = T_5(\eta^{(+)}) + T_5(\eta^{(-)}), \quad \eta^{(\pm)} = \frac{z \pm \sqrt{z^2+4}}{2}.$$

Then from $T_5(\eta^{(+)}) \cdot T_5(\eta^{(-)}) = -1$ and (3.3) it follows that $\mathsf{T}_5(z) \in \phi(\mathsf{D}_5 \cap \{z : |z|_5 = 1\})$, which implies that

$$\mathsf{T}_5^n(z) = T_5^n(\eta^{(+)}) + T_5^n(\eta^{(-)}),\ n \geq 1,\ \eta^{(\pm)} = \frac{z \pm \sqrt{z^2+4}}{2}.$$

Furthermore, $g(z, T_5(z)) = 0$ implies that

$$h(z - 1/z, \mathsf{T}_5(z - 1/z)) = -g(z, T_5(z))g_1(z, T_5(z)) = 0.$$

We deduce the following.

Theorem 6.3. *For any negative discriminant $-d = d_K f^2$ with $\left(\frac{-d}{5}\right) = +1$, and for $\eta = r(w_d/5)$, as in Part II, the $h(-d)$ distinct conjugate values*

$$\upsilon^\tau = \eta^\tau - \frac{1}{\eta^\tau}, \quad \tau \in Gal(F_1/K),$$

lying in the ring class field Ω_f of $K = \mathbb{Q}(\sqrt{-d})$, are periodic points of the 5-adic algebraic function $\mathsf{T}_5(z)$ in the 5-adic domain

$$\widetilde{D}_5 = \phi(D_5 \cap \{z \in K_5 : |z|_5 = 1\}).$$

The period of υ^τ is equal to the order of the automorphism $\tilde{\tau}_5 = \left(\frac{\Omega_f/K}{\wp_5}\right)$.

Proof. This is immediate from

$$\mathsf{T}_5(\upsilon^\tau) = \mathsf{T}_5\left(\eta^\tau - \frac{1}{\eta^\tau}\right) = T_5(\eta^\tau) - \frac{1}{T_5(\eta^\tau)} = \eta^{\tau\tau_5} - \frac{1}{\eta^{\tau\tau_5}} = \upsilon^{\tau\tau_5},$$

where the third equality above follows from $g(\eta^\tau, \eta^{\tau\tau_5}) = 0$. The fact that the period is the order of $\tilde{\tau}$ is a consequence of the fact that $\mathbb{Q}(\upsilon) = \Omega_f$ and that

$$\tilde{\tau}_5 = \tau_5|_{\Omega_f}, \quad \tau_5 = \left(\frac{F_1/K}{\wp_5}\right).$$

□

Corollary 6.4. *Conjecture 1(a) of [13] holds for the prime $p = 5$: Every ring class field Ω_f over $K = \mathbb{Q}(\sqrt{-d})$, with $\left(\frac{-d}{5}\right) = +1$ and $(f, 5) = 1$, is generated over $\mathbb{Q}$ by a periodic point of the 5-adic algebraic function $\mathsf{T}_5(z)$ which is contained in the domain $\widetilde{D}_5 = \phi(D_5 \cap \{z \in K_5 : |z|_5 = 1\}) \subset K_5$.*

Note: it is clear that $\mathsf{T}_5(\widetilde{\mathsf{D}}_5) \subseteq \widetilde{\mathsf{D}}_5$, since $T_5(x)$ maps the set $\mathsf{D}_5 \cap \{z \in \mathsf{K}_5 : |z|_5 = 1\}$ into itself, by Corollary 3.3 and equation (3.3).

The values υ^τ and their complex conjugates coincide with the roots of the polynomial $t_d(x)$, for which

$$x^{2h(-d)} t_d\left(x - \frac{1}{x}\right) = p_d(x), \quad d > 4. \tag{6.5}$$

Theorem 6.2 shows that every periodic point $v \neq -1, \pm 2i$ of $\mathfrak{f}(z)$ is a root of some polynomial $t_d(x)$ with $d > 4$.

6.2. Deuring's class number formula. Let

$$S^{(1)}(t, t_1) := h(t, t_1) \equiv 4(t_1 + 1)^4 (t^5 - t_1) \pmod 5$$

and

$$S^{(n)}(t, t_n) := \text{Resultant}_{t_{n-1}}(S^{(n-1)}(t, t_{n-1}), h(t_{n-1}, t_n)), \quad n \geq 2.$$

Then it follows by induction that

$$S^{(n)}(t, t_n) \equiv 4(t_n + 1)^{5^n - 1}(t^{5^n} - t_n) \pmod 5, \quad n \geq 1.$$

Hence, the polynomial $S_n(t) := S^{(n)}(t, t)$ satisfies the congruence

$$S_n(t) \equiv 4(t + 1)^{5^n - 1}(t^{5^n} - t) \pmod 5. \tag{6.6}$$

It follows that

$$\deg(S_n(t)) = 2 \cdot 5^n - 1, \quad n \geq 1.$$

(See the Lemma on pp. 727-728 of Part I, [13].)

Let $L(z) = \frac{-z+4}{z+1}$. Then

$$L\left(x - \frac{1}{x}\right) = \frac{-x^2 + 4x + 1}{x^2 + x - 1} = T(x) - \frac{1}{T(x)},$$

and we have the identity

$$(x + 1)^5 (y + 1)^5 h(L(x), L(y)) = 5^5 h(y, x). \tag{6.7}$$

Moreover,

$$L(z) + 1 = \frac{5}{z + 1}. \tag{6.8}$$

Using (6.6), (6.7) and (6.8), it follows by the same reasoning as in Section 2 that $S_n(x)$ has distinct roots and that

$$\mathsf{Q}_n(x) = \prod_{k|n} S_k(x)^{\mu(n/k)} \tag{6.9}$$

is a polynomial. Furthermore, all of the roots of $\mathsf{Q}_n(x)$ lie in K_5. From Theorem 6.3 we see that the polynomial $t_d(x)$ divides $\mathsf{Q}_n(x)$ whenever the automorphism $\tilde{\tau}_5$ has order n, and from Theorem 6.2, we see that these are the only irreducible factors of $\mathsf{Q}_n(x)$ over $\mathbb{Q}$. This gives

Theorem 6.5. *For $n > 1$, the polynomial $\mathsf{Q}_n(x)$ is given by the product*

$$\mathsf{Q}_n(x) = \pm \prod_{-d \in \mathfrak{D}_n^{(5)}} t_d(x),$$

where $t_d(x)$ is defined by (6.5) and $\mathfrak{D}_n^{(5)}$ is the set of negative quadratic discriminants $-d$ with $\left(\frac{-d}{5}\right) = +1$, for which the automorphism $\tilde{\tau}_{5,d} = \tilde{\tau}_5 = \left(\frac{\Omega_f/K}{\wp_5}\right)$ has order n in $Gal(\Omega_f/K)$, the Galois group of the ring class field Ω_f over $K = \mathbb{Q}(\sqrt{-d})$.

For $\mathsf{Q}_1(x)$ we have the factorization

$$\begin{aligned}\mathsf{Q}_1(x) &= -(x+1)(x^2+4)(x^2-x+3)(x^2-2x+2)(x^2+x+5)\\ &= -(x+1)t_4(x)t_{11}(x)t_{16}(x)t_{19}(x),\end{aligned}$$

where $t_4(x)$ satisfies

$$x^2 t_4\left(x-\frac{1}{x}\right) = (x^2+1)^2 = p_4(x)^2.$$

Since $\deg(t_d(x)) = 2h(-d)$, Theorem 6.3 shows that half of the roots of $t_d(x)$ lie in the domain $\widetilde{\mathsf{D}}_5$, while the other roots ξ satisfy $\xi \equiv -1 \pmod 5$ in K_5, a fact which follows from (6.7) and (6.8). Also see eq. (32) in [14].

The fact that $\deg(t_d(x)) = 2h(-d)$ now implies the following class number formula.

Corollary 6.6. *For $n > 1$ we have*

$$\sum_{-d\in\mathfrak{D}_n^{(5)}} h(-d) = \sum_{k|n} \mu(n/k)5^k.$$

This formula is equivalent to Deuring's formula for the prime $p = 5$ from [5], [6], as in [16].

References

[1] Andrews, George E.; Berndt, Bruce C. Ramanujan's lost notebook. Part I. *Springer, New York*, 2005. xiv+437 pp. ISBN: 978-0387-25529-3; 0-387-25529-X. MR2135178, Zbl 1075.11001. 788

[2] Berndt, Bruce C. Ramanujan's notebooks. Part V. *Springer-Verlag, New York*, 1998. xiv+624 pp. ISBN: 0-387-94941-0. MR1486573, Zbl 0886.11001. 788, 806, 807

[3] Cho, Bumkyu. Primes of the form $x^2 + ny^2$ with conditions $x \equiv 1 \bmod N$, $y \equiv 0 \bmod N$. *J. Number Theory* **130** (2010), no. 4, 852–861. MR2600406, Zbl 1211.11103, doi: 10.1016/j.jnt.2009.07.013. 789

[4] Cox, David A. Primes of the form $x^2 + ny^2$. Fermat, class field theory and complex multiplication. Second edition. Pure and Applied Mathematics. *John Wiley & Sons, Inc., Hoboken, NJ*, 2013. xviii+356 pp. ISBN: 978-1-118-39018-4. MR3236783, Zbl 1275.11002, doi: 10.1002/9781118400722. 789, 802

[5] Deuring, Max. Die Typen der Multiplikatorenringe elliptischer Funktionenkörper. *Abh. Math. Sem. Hansischen Univ.* **14** (1941), 197–272. MR0005125, Zbl 0025.02003, doi: 10.1007/BF02940746. 815

[6] Deuring, Max. Die Anzahl der Typen von Maximalordnungen einer definiten Quaternionenalgebra mit primer Grundzahl. *Jber. Deutsch. Math.-Verein.* **54** (1950), 24-41. MR0036777, Zbl 0039.02902. 815

[7] Duke, William. Continued fractions and modular functions. *Bull. Amer. Math. Soc.* **42** (2005), no. 2, 137–162. MR2133308, Zbl 1109.11026, doi: 10.1090/S0273-0979-05-01047-5. 788, 805

[8] Hasse, Helmut. Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper. Teil I: Klassenkörpertheorie. Jahresbericht

der Deutschen Mathematiker-Vereinigung **35** (1926), 1-55; reprinted by *Physica-Verlag, Würzburg-Vienna*, 1965, iv + 135 pp. ISBN 3-7908-0010-4. MR0195847, JFM 52.1050.19, doi: 10.1007/978-3-662-39429-8. 789

[9] Hasse, Helmut. Ein Satz über die Ringklassenkörper der komplexen Multiplikation, *Monatshefte für Mathematik und Physik* **38** (1931), 323-330. Reprinted in Helmut Hasse Mathematische Abhandlungen, Bd. 2, paper 36, pp. 61-68, Walter de Gruyter, Berlin, 1975, ISBN 3-11-005931-2. MR1549921, Zbl 0002.33101, doi: 10.1007/BF01700703. 789

[10] Lavallee, Melisa J.; Spearman, Blair K.; Williams, Kenneth S. Watson's method of solving a quintic equation. *JP J. Algebra, Number Theory Appl.* **5** (2005), no. 1, 49–73. MR2140318, Zbl 1102.12001. 795

[11] Morton, Patrick. Explicit identities for invariants of elliptic curves. *J. Number Theory* **120** (2006), no. 2, 234–271. MR2257546, Zbl 1193.11062, doi: 10.1016/j.jnt.2005.12.008. 799, 803

[12] Morton, Patrick. Solutions of the cubic Fermat equation in ring class fields of imaginary quadratic fields (as periodic points of a 3-adic algebraic function). *Int. J. Number Theory* **12** (2016), no. 4, 853–902. MR3484288, Zbl 06580489, arXiv:1410.6798, doi: 10.1142/S179304211650055X. 791, 798, 803

[13] Morton, Patrick. Solutions of diophantine equations as periodic points of p-adic algebraic functions. I. *New York J. Math.* **22** (2016), 715–740. MR3548120, Zbl 1419.11059. 790, 813, 814

[14] Morton, Patrick. Solutions of diophantine equations as periodic points of p-adic algebraic functions, II: The Rogers-Ramanujan continued fraction. *New York J. Math.* **25** (2019), 1178–1213. MR4028831, Zbl 1441.11064. 788, 789, 790, 791, 794, 797, 798, 803, 804, 807, 809, 810, 815

[15] Morton, Patrick. Product formulas for the 5-division points on the Tate normal form and the Rogers-Ramanujan continued fraction. *J. Number Theory* **200** (2019), 380–396. MR3944443, Zbl 07038702, doi: 10.1016/j.jnt.2018.12.013. 800

[16] Morton, Patrick. Periodic points of algebraic functions and Deuring's class number formula. *Ramanujan J.* **50** (2019), no. 2, 323–354. MR4022234, Zbl 07140967, doi: 10.1007/s11139-018-0120-x. 790, 791, 815

[17] Morton, Patrick. On the Hasse invariants of the Tate normal forms E_5 and E_7. *J. Number Theory* **218** (2021), 234–271. MR4157698, Zbl 07257228, doi: 10.1016/j.jnt.2020.07.008. 803, 804

[18] Prestel, Alexander; Ziegler, Martin. Model-theoretic methods in the theory of topological fields. *J. Reine Angew. Math.* **299(300)** (1978), 318–341. MR0491852, Zbl 0367.12014. 795

[19] Schertz, Reinhard. Complex multiplication. New Mathematical Monographs, 15. *Cambridge University Press, Cambridge*, 2010. xiv+361 pp. ISBN: 978-0-521-76668-5. MR2641876, Zbl 1203.11001. 789

[20] Söhngen, Heinz. Zur komplexen Multiplikation. *Math. Ann.* **111** (1935), no. 1, 302–328. MR1512998, Zbl 0012.00902, JFM 61.0169.02, doi: 10.1007/BF01472223. 789

[21] Stevenhagen, Peter. Hilbert's 12th problem, complex multiplication and Shimura reciprocity. *Class field theory–its centenary and prospect* (Tokyo, 1998), 161–176, Adv. Stud. Pure Math., 30. *Math. Soc. Japan, Tokyo*, 2001. MR1846457, Zbl 1097.11535. 789

[22] Yi, Jinhee. Modular Equations for the Rogers-Ramanujan Continued Fraction and the Dedekind Eta-Function, *Ramanujan J. Math.* **5** (2001), 377–384. MR1891418, Zbl 1043.11041, doi: 10.1023/A:1013991704758. 810

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